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On ordered Γ-hypersemigroups and their relation to lattice ordered semigroups

Niovi KEHAYOPULU*
Nikomidias 18, 16122 Kesariani, Greece

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Abstract: The concept of Γ-hypersemigroup has been introduced in Turk J Math 2020; 44 (5): 1835-1851 in which it has in which it has been shown that various results on Γ-hypersemigroups can be obtained directly as corollaries of more general results from the theory of *le*-semigroups (i.e. lattice ordered semigroups having a greatest element) or *poe*-semigroups. As a continuation of the paper mentioned above, in the present paper, the concept of ordered Γ-hypersemigroups has been introduced, and their relation to lattice ordered semigroups is given. It has been shown that although the results on ordered Γ-hypersemigroups cannot be obtained as corollaries to the corresponding results of *le* or *poe*-semigroups, still the main idea comes from the *le*-semigroups or *poe*-semigroups, and the proofs go along the lines of the *le* or *poe*-semigroups.

Key words: Lattice ordered semigroup, ordered Γ-hypersemigroup, regular, intra-regular, left (right) regular

1. Introduction

As we have already seen in [4], many results on hypersemigroups do not need any proof as they can be obtained from results in the lattice ordered semigroup or *poe*-semigroup setting. Later in [5], the concept of Γ-hypersemigroup has been introduced and it has been shown that many results on Γ-hypersemigroups as well can be obtained from more general results on lattice ordered semigroups or *poe*-semigroups. It may be instructive to prove them directly just to show how an independent proof works, but this direct, independent proof will follow along the lines of *le* or *poe*-semigroups. It has been set, as a future work, in [5] the examination of what happened in case of an ordered Γ-hypersemigroup. As a continuation of [5], in the present paper, the concept of an ordered Γ-hypersemigroup has been introduced, and the aim is to show that, although this is not exactly the case for ordered Γ-hypersemigroups, the idea of having various results comes from *le* or *poe*-semigroups and direct proofs derived along the line of those in the *le* or *poe*-semigroups setting. In this respect, we introduce the concepts of regular, intra-regular, left (right) regular ordered Γ-hypersemigroups as well, and we prove the results on ordered Γ-hypersemigroups that correspond to the results on lattice ordered semigroups in section 2 in [5]. Considering that every Γ-hypersemigroup with the order $\leq := \{(a,b) \mid a = b\}$ is an ordered Γ-hypersemigroup, the results stated without proof in section 3 in [5], follow as application. For definitions, notations, and results not given in the present paper, we refer to [5].

*Correspondence: nkehayop@math.uoa.gr
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2. Main results

If M is a Γ-hypergroupoid and "$\leq$" is an order relation on M, denote by "$\preceq$" the relation on the set of all nonempty subsets $\mathcal{P}^*(M)$ of M defined by: $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. This is a transitive and reflexive relation on $\mathcal{P}^*(M)$; that is a preorder on $\mathcal{P}^*(M)$.

Definition 2.1 *A Γ-hypergroupoid M is called ordered Γ-hypergroupoid if there exists an order relation "$\leq$" on M such that*

$$a \leq b \text{ implies } a\gamma c \preceq b\gamma c \text{ and } c\gamma a \preceq c\gamma b \text{ for every } \gamma \in \Gamma \text{ and every } c \in M.$$

Lemma 2.2 *If M is an ordered Γ-hypergroupoid, $a \leq b$, $c \leq d$ and $\gamma \in \Gamma$, then $a\gamma c \preceq b\gamma d$.*

Proof Let $a \leq b$, $c \leq d$ and $\gamma \in \Gamma$. Since $a \leq b$ and $\gamma \in \Gamma$, we have $a\gamma c \preceq b\gamma c$. Since $c \leq d$ and $\gamma \in \Gamma$, we have $b\gamma c \preceq b\gamma d$. Since the relation "$\preceq$" is a transitive relation on $\mathcal{P}^*(M)$, we have $a\gamma c \preceq b\gamma d$. □

For a Γ-hypergroupoid M and a nonempty subset A of M, denote by $(A]$ the subset of M defined by $(A] = \{t \in M \mid t \leq a \text{ for some } a \in A\}$, and we have the following:

(1) If $A \subseteq B$, then $(A] \subseteq (B]$.

(2) If A is a left (right) ideal of M, then $(A] = A$.

(3) $M = (M]$.

(4) $\big((A]\big] = (A]$.

(5) $(A \cup B] = (A] \cup (B]$.

(See, for example [3] -as the operation Γ does not play any role in them).

When is convenient and no confusion is possible, we identify the singleton $\{a\}$ by the element a and write, for example, $M\Gamma a$ instead of $M\Gamma\{a\}$, $a\Gamma a\Gamma M$ instead of $\{a\}\Gamma M\Gamma\{a\}$.

We will give the theorems on ordered Γ-hypersemigroups that correspond to lattice ordered semigroups in [5; Section 2] in the row appeared in [5]. So, we begin with the theorem on ordered Γ-hypersemigroup that corresponds to [5; Theorem 2.2].

A natural extension of the concept of regular ordered semigroup [2] to regular ordered Γ-hypersemigroup is given by the following definition.

Definition 2.3 *An ordered Γ-hypersemigroup M is called regular if for every $a \in M$ there exist $x \in M$ and $\gamma, \mu \in \Gamma$ such that $\{a\} \preceq (a\gamma x)\overline{\mu}\{a\}$; in other words, there exist $x, t \in M$ and $\gamma, \mu \in \Gamma$ such that*

$$t \in (a\gamma x)\overline{\mu}\{a\} \text{ and } a \leq t.$$

Proposition 2.4 *Let M be an ordered Γ-hypersemigroup. The following are equivalent:*

(1) M is regular.

(2) For any nonempty subset A of M, we have $A \subseteq (A\Gamma M\Gamma A]$.

(3) For any $a \in M$, we have $a \in (a\Gamma M\Gamma a]$.

Proof $(1) \Longrightarrow (2)$. Let A be a nonempty subset of M and $a \in A$. Since M is regular, there exist $x, t \in M$ and $\gamma, \mu \in \Gamma$ such that $t \in (a\gamma x)\overline{\mu}\{a\}$ and $a \leq t$. Since $a \in A$, $\gamma \in \Gamma$, $x \in M$, by [5; Lemma 3.7(2)], we have $a\gamma x \subseteq A\Gamma M$. Since $a\gamma x \subseteq A\Gamma M$ and $\{a\} \subseteq A$, by [5; Lemma 3.6], $(a\gamma x)\overline{\mu}\{a\} \subseteq (A\Gamma M)\overline{\mu}\{a\}$. By [5; Def. 3.3], $(A\Gamma M)\overline{\mu}\{a\} \subseteq (A\Gamma M)\Gamma\{a\}$. By [5; Lemma 3.8], $(A\Gamma M)\Gamma\{a\} \subseteq (A\Gamma M)\Gamma A$. Thus, we have $a \leq t \in A\Gamma M\Gamma A$ and so $a \in (A\Gamma M\Gamma A]$ and (2) holds.

The implication (2) $\Rightarrow$ (3) is obvious.
(3) $\Longrightarrow$ (1). Let $a \in M$. By hypothesis, we have $a \in (a\Gamma M\Gamma a]$, that is $a \le t$ for some $t \in (a\Gamma M)\Gamma a$. By [5; Lemma 3.7(1)], $t \in u\mu a$ for some $u \in a\Gamma M$, $\mu \in \Gamma$ and $u \in a\gamma x$ for some $\gamma \in \Gamma$, $x \in M$. By [5; Lemmas 3.5 and 3.6], $t \in u\mu a = \{u\}\overline{\mu}\{a\} \subseteq (a\gamma x)\overline{\mu}\{a\}$. We have $x, t \in M$, $\gamma, \mu \in \Gamma$, $t \in (a\gamma x)\overline{\mu}\{a\}$ and $a \le t$, thus M is regular. □

Definition 2.5 *If $(M, \Gamma, \le)$ is an ordered Γ-hypergroupoid, a nonempty subset A of M is called a right (resp. left) ideal of M if it is a right (resp. left) ideal of the Γ-hypergroupoid (M, Γ) (that is, if $A\Gamma M \subseteq A$ (resp. $M\Gamma A \subseteq A$)[5] and, in addition,*

$$\textit{if } a \in A \textit{ and } M \ni b \le a, \textit{ then } b \in A; \textit{ that is if } (A] = A.$$

For a nonempty subset A of M, denote by $R(A)$, $L(A)$ and $I(A)$ the right ideal, left ideal, and ideal of M, respectively, generated by A. For $A = \{a\}$, we write $R(a)$ instead of $R(\{a\})$; similarly, we write $L(a)$, $I(a)$.

Lemma 2.6 *If M is an ordered hypergroupoid, then, for any nonempty subsets A, B of M, we have*

$$(A]\Gamma(B] \subseteq (A\Gamma B].$$

Proof Let $x \in (A]\Gamma(B]$. By [5; Lemma 3.7(1)], $x \in u\gamma v$ for some $u \in (A]$, $\gamma \in \Gamma$, $v \in (B]$. We have $u \le a$ for some $a \in A$, $v \le b$ for some $b \in B$ and $\gamma \in \Gamma$. By Lemma 2.2, we have $u\gamma v \preceq a\gamma b$. Since $x \in u\gamma v$, there exists $y \in a\gamma b$ such that $x \le y$. Since $x \le y \in a\gamma b$, we have $x \in (a\gamma b]$. Since $a \in A$, $\gamma \in \Gamma$, $b \in B$, by [5; Lemma 3.7(2)], we have $a\gamma b \subseteq A\Gamma B$. Then, we have $(a\gamma b] \subseteq (A\Gamma B]$ and so $x \in (A\Gamma B]$. □

Lemma 2.7 *If M is an ordered Γ-hypersemigroup, then, for any nonempty subset A of M, we have*

(1) $R(A) = (A \cup A\Gamma M]$.

(2) $L(A) = (A \cup M\Gamma A]$.

(3) $I(A) = (A \cup M\Gamma A \cup A\Gamma M \cup M\Gamma A\Gamma A\Gamma M]$.

Proof (1) The set $(A \cup A\Gamma M]$ is a right ideal of M containing A. In fact, we have

$$\begin{aligned}(A \cup A\Gamma M]\Gamma M &= (A \cup A\Gamma M]\Gamma(M] \subseteq \Big((A \cup A\Gamma M)\Gamma M\Big] \text{ (by Lemma 2.6)}\\ &= \Big((A\Gamma M \cup A\Gamma(M\Gamma M)\Big] = (A\Gamma M]\\ &\subseteq (A \cup A\Gamma M]\end{aligned}$$

and $\big((A\Gamma M]\big] = (A\Gamma M]$ as it holds for any $\emptyset \ne X \subseteq M$. If T is a right ideal of M such that $T \supseteq A$, then $(A \cup A\Gamma M] \subseteq (T \cup T\Gamma M] = (T] = T$, and property (1) is satisfied.
The proof of properties (2) and (3) is similar. □

It might be mentioned that $I(A) = R\big(L(A)\big) = L\big(R(A)\big)$.

Lemma 2.8 *If M is an ordered hypergroupoid, then, for any nonempty subsets A, B of M, we have*

$$(A\Gamma B] = \Big(A\Gamma(B]\Big] = \Big((A]\Gamma B\Big] = \Big((A]\Gamma(B]\Big].$$

Proof Since $A \subseteq (A]$ and $B \subseteq (B]$, we have $A\Gamma B \subseteq (A]\Gamma(B]$ and so $(A\Gamma B] \subseteq \Big((A]\Gamma(B]\Big]$. On the other hand, by Lemma 2.6, we have $\Big((A]\Gamma(B]\Big] \subseteq \Big((A\Gamma B]\Big] = (A\Gamma B]$ and so $(A\Gamma B] = \Big((A]\Gamma(B]\Big]$.

Clearly, $(A\Gamma B] \subseteq \Big(A\Gamma(B]\Big]$. Let now $x \in \Big(A\Gamma(B]\Big]$. Then, $x \le t$ for some $t \in A\Gamma(B]$, $t \in a\gamma u$ for some $a \in A$, $\gamma \in \Gamma$, $u \in (B]$ and $u \le b$ for some $b \in B$. By Lemma 2.2, $a\gamma u \preceq a\gamma b$ and since $t \in a\gamma u$, there exists $v \in a\gamma b$ such that $t \le v$. We have $x \le v \in a\gamma b \in A\Gamma B$ and so $x \in (A\Gamma B]$; and $(A\Gamma B] = \Big(A\Gamma(B]\Big]$. The remainder equality can be proved at a similar way. □

The theorem on regular ordered Γ-hypersemigroups that corresponds to Theorem 2.2 in [5] is the following.

Theorem 2.9 *Let M be an ordered* Γ*-hypersemigroup. The following are equivalent:*

(1) M is regular.

(2) $A \cap B = (A\Gamma B]$ *for every right ideal A and every left ideal B of M.*

(3) $A \cap B \subseteq (A\Gamma B]$ *for every right ideal A and every left ideal B of M.*

Proof $(1) \Longrightarrow (2)$. Let A be a right ideal and B be a left ideal of *M*. By [5; Proposition 3.12], the set $A \cap B$ is nonempty. Since M is regular, by Proposition 2.4, we have $A \cap B \subseteq \Big((A\cap B)\Gamma M\Gamma(A\cap B)\Big]$. Since $A \cap B \subseteq A, B$, by [5; Lemma 3.8], $(A\cap B)\Gamma M\Gamma(A\cap B) \subseteq A\Gamma M\Gamma B$. Thus we have

$$\begin{aligned} A \cap B &\subseteq \Big((A\Gamma M)\Gamma B\Big] \subseteq (A\Gamma B] \subseteq (A\Gamma M] \cap (M\Gamma B] \\ &\subseteq (A] \cap (B] = A \cap B. \end{aligned}$$

Then we have $A \cap B = (A\Gamma B]$ and property (2) is satisfied.
The implication $(2) \Longrightarrow (3)$ is obvious.
$(3) \Longrightarrow (1)$. Let A be a nonempty subset of *M*. By hypothesis, we have

$$\begin{aligned} A &\subseteq R(A) \cap L(A) \subseteq \Big(R(A)\Gamma L(A)\Big] = \Big((A \cup A\Gamma M]\Gamma(A \cup M\Gamma A]\Big] \text{ (by Lemma 2.7)} \\ &= \Big((A \cup A\Gamma M)\Gamma(A \cup M\Gamma A)\Big] \text{ (by Lemma 2.8)} \\ &= \Big(A\Gamma A \cup A\Gamma M\Gamma A \cup A\Gamma(M\Gamma M)\Gamma A\Big] \\ &= (A\Gamma A \cup A\Gamma M\Gamma A]. \end{aligned}$$

Then we have

$$\begin{aligned} A\Gamma A &\subseteq (A\Gamma A \cup A\Gamma M\Gamma A]\Gamma(A] \subseteq \Big((A\Gamma A \cup A\Gamma M\Gamma A)\Gamma A\Big] \text{ (by Lemma 2.6)} \\ &= \Big(A\Gamma A\Gamma A \cup A\Gamma(M\Gamma A)\Gamma A\Big] \subseteq (A\Gamma M\Gamma A]. \end{aligned}$$

Then $A \subseteq \Big((A\Gamma M\Gamma A]\Big] = (A\Gamma M\Gamma A]$ and, by Proposition 2.4, M is regular. □

A natural extension of the concept of intra-regular ordered semigroup [3] to intra-regular ordered Γ-hypersemigroup is given by the following definition.

Definition 2.10 *An ordered* Γ*-hypersemigroup* M *is called intra-regular if, for every* $a \in M$*, there exist* $x, y \in M$ *and* $\gamma, \mu, \rho \in \Gamma$ *such that* $\{a\} \preceq (x\gamma a)\overline{\mu}(a\rho y)$*; in other words, there exist* $x, y, t \in M$ *and* $\gamma, \mu, \rho \in \Gamma$ *such that*

$$t \in (x\gamma a)\overline{\mu}(a\rho y) \text{ and } a \leq t.$$

Proposition 2.11 *Let M be an ordered* Γ*-hypersemigroup. The following are equivalent:*

(1) M is intra-regular.

(2) For any nonempty subset A of M, we have $A \subseteq (M\Gamma A\Gamma A\Gamma M]$.

(3) For every $a \in M$*, we have* $a \in (M\Gamma a\Gamma a\Gamma M]$.

Proof (1) $\Longrightarrow$ (2). Let A be a nonempty subset of M and $a \in A$. Since M is intra-regular, there exist $x, y, t \in M$ and $\gamma, \mu, \rho \in \Gamma$ such that $t \in (x\gamma a)\overline{\mu}(a\rho y)$ and $a \leq t$. Since $x \in M$, $\gamma \in \Gamma$ and $a \in A$, by [5; Lemma 3.7(2)], we have $x\gamma a \subseteq M\Gamma A$; and since $a \in A$, $\rho \in \Gamma$ and $y \in M$, we have $a\rho y \subseteq A\Gamma M$. Since $x\gamma a \subseteq M\Gamma A$, $a\rho y \subseteq A\Gamma M$ and $\mu \in \Gamma$, by [5; Lemma 3.6], we have $(x\gamma a)\overline{\mu}(a\rho y) \subseteq (M\Gamma A)\overline{\mu}(A\Gamma M)$. By [5; Definition 3.3], $(M\Gamma A)\overline{\mu}\Gamma(A\Gamma M) \subseteq (M\Gamma A)\Gamma(A\Gamma M)$. By [5; Proposition 3.17], $(M\Gamma A)\Gamma(A\Gamma M) = M\Gamma A\Gamma A\Gamma M$. Hence we obtain $a \leq t \in M\Gamma A\Gamma A\Gamma M$, that is $a \in (M\Gamma A\Gamma A\Gamma M]$ and property (2) holds.
The implication (2) $\Rightarrow$ (3) is obvious.
(3) $\Longrightarrow$ (1). Let $a \in M$. By hypothesis, $a \in (M\Gamma a\Gamma a\Gamma M]$, then $a \leq t$ for some $t \in (M\Gamma a)\Gamma(a\Gamma M)$. By [5; Lemma 3.7(1)], $t \in u\mu v$ for some $u \in M\Gamma a$, $\mu \in \Gamma$, $v \in a\Gamma M$, $u \in x\gamma a$ for some $x \in M$, $\gamma \in \Gamma$ and $v \in a\rho y$ for some $\rho \in \Gamma$, $y \in M$. By [5; Lemmas 3.5 and 3.6], we have $t \in u\mu v = \{u\}\overline{\mu}\{v\} \subseteq (x\gamma a)\overline{\mu}(a\rho y)$. We have $x, y, t \in M$, $\gamma, \mu, \rho \in \Gamma$, $t \in (x\gamma a)\overline{\mu}(a\rho y)$ and $a \leq t$ and so M is intra-regular. □

The theorem on intra-regular ordered Γ-hypersemigroups that corresponds to Theorem 2.4 in [5], is the following.

Theorem 2.12 *An ordered* Γ*-hypersemigroup M is intra-regular if and only if for every right ideal A and every left ideal B of M, we have*

$$A \cap B \subseteq (B\Gamma A].$$

Proof $\Longrightarrow$. Let A be a right ideal and B be a left ideal of M. By [5; Proposition 3.12], the set $A \cap B$ is nonempty. Since M is intra-regular, by Proposition 2.11, we have

$$A \cap B \subseteq \Big(\Big(M\Gamma(A \cap B)\Big)\Gamma\Big((A \cap B)\Gamma M\Big)\Big].$$

Since $A \cap B \subseteq B, A$, by [5; Lemma 3.8], we have $M\Gamma(A \cap B)\Gamma(A \cap B)\Gamma M \subseteq (M\Gamma B)\Gamma(A\Gamma M) \subseteq B\Gamma A$ and so $A \cap B \subseteq (B\Gamma A]$.
$\Longleftarrow$. Let $a \in M$. By hypothesis, we have

$$\begin{aligned}
a &\in R(a) \cap L(a) \subseteq \Big(L(a)\Gamma R(a)\Big] = \Big((a \cup M\Gamma a]\Gamma(a \cup a\Gamma M]\Big] \text{ (by Lemma 2.7)}\\
&= \Big((a \cup M\Gamma a)\Gamma(a \cup a\Gamma M)\Big] \text{ (by Lemma 2.8)}\\
&= (a\Gamma a \cup M\Gamma a\Gamma a \cup a\Gamma a\Gamma M \cup M\Gamma a\Gamma a\Gamma M]\\
&= (a\Gamma a] \cup (M\Gamma a\Gamma a] \cup (a\Gamma a\Gamma M] \cup (M\Gamma a\Gamma a\Gamma M].
\end{aligned}$$

If $a \in (a\Gamma a]$, then we have

$$\begin{aligned} a\Gamma a &\subseteq (a\Gamma a]\Gamma(a\Gamma a] \subseteq (a\Gamma a\Gamma a\Gamma a] \text{ (by Lemma 2.6)} \\ &\subseteq (M\Gamma a\Gamma a\Gamma M], \end{aligned}$$

so $a \in (a\Gamma a] \subseteq \Big((M\Gamma a\Gamma a\Gamma M]\Big] = (M\Gamma a\Gamma a\Gamma M]$.

If $a \in (M\Gamma a\Gamma a]$, then we have

$$\begin{aligned} M\Gamma a\Gamma a &\subseteq M\Gamma(M\Gamma a\Gamma a]\Gamma a \subseteq (M]\Gamma(M\Gamma a\Gamma a]\Gamma(a] \\ &\subseteq (M\Gamma M\Gamma a\Gamma a\Gamma a] \text{ (by Lemma 2.6)} \\ &= \Big((M\Gamma M)\Gamma(a\Gamma a\Gamma a)\Big] \\ &\subseteq \Big(M\Gamma(a\Gamma a\Gamma M)\Big]. \end{aligned}$$

Then $a \in (M\Gamma a\Gamma a] \subseteq \Big((M\Gamma a\Gamma a\Gamma M]\Big] = (M\Gamma a\Gamma a\Gamma M]$.

If $(a\Gamma a\Gamma M]$, then in a similar way we prove that $a \in (M\Gamma a\Gamma a\Gamma M]$. In each case, we have $a \in (M\Gamma a\Gamma a\Gamma M]$ and, by Proposition 2.11, M is intra-regular.

The natural extension of the notion of right (left) regular ordered semigroup [1] to right (left) regular ordered Γ-hypersemigroup is given by the following definition.

Definition 2.13 *An ordered Γ-hypersemigroup M is called right regular if, for every $a \in M$, there exist $x \in M$ and $\gamma, \mu \in \Gamma$ such that $\{a\} \preceq (a\gamma a)\overline{\mu}\{x\}$; in other words, there exist $x, t \in M$ and $\gamma, \mu \in \Gamma$ such that*

$$t \in (a\gamma a)\overline{\mu}\{x\} \text{ and } a \leq t.$$

It is called left regular if for every $a \in M$ there exist $x \in M$ and $\gamma, \mu \in \Gamma$ such that $\{a\} \preceq \{x\}\overline{\gamma}(a\mu a)$; in other words, there exist $x, t \in M$ and $\gamma, \mu \in \Gamma$ such that

$$t \in \{x\}\overline{\gamma}(a\mu a) \text{ and } a \leq t.$$

Proposition 2.14 *Let M be an ordered Γ-hypersemigroup. The following are equivalent:*

(1) M is right regular.

(2) For every nonempty subset A of M, we have $A \subseteq (A\Gamma A\Gamma M]$.

(3) For every $a \in M$, we have $a \in (a\Gamma a\Gamma M]$.

Proof $(1) \Longrightarrow (2)$. Let A be a nonempty subset of M and $a \in A$. Since M is right regular, there exist $x, t \in M$ and $\gamma, \mu \in \Gamma$ such that $t \in (a\gamma a)\overline{\mu}\{x\}$ and $a \leq t$. By [5; Definition 3.3], $(a\gamma a)\overline{\mu}\{x\} \subseteq (a\gamma a)\Gamma\{x\}$. By [5; Lemma 3.7(2)], $a\gamma a \subseteq a\Gamma a$ and, by [5; Lemma 3.8 and Prop. 3.17], $(a\gamma a)\Gamma\{x\} \subseteq (A\Gamma A)\Gamma M = A\Gamma A\Gamma M$. We have $a \leq t \in A\Gamma A\Gamma M$ and so $a \in (A\Gamma A\Gamma M]$.

The implication $(2) \Rightarrow (3)$ is obvious.

$(3) \Longrightarrow (1)$. Let $a \in M$. By hypothesis, we have $a \in (a\Gamma a\Gamma M]$. By [5; Prop. 3.17], $a \leq t$ for some $t \in (a\Gamma a)\Gamma M$. By [5; Lemma 3.7(1)], $t \in u\mu x$ for some $u \in a\Gamma a$, $\mu \in \Gamma$, $x \in M$ and $u \in a\gamma a$ for some $\gamma \in \Gamma$. By [5; Lemmas 3.5 and 3.6], we get $t \in u\mu x = \{u\}\overline{\mu}\{x\} \subseteq (a\gamma a)\overline{\mu}\{x\}$. We have $x, t \in M$, $\gamma, \mu \in \Gamma$, $t \in (a\gamma a)\overline{\mu}\{x\}$, $a \leq t$ and so M is right regular. □

In a similar way the following proposition holds.

Proposition 2.15 *An ordered* Γ*-hypersemigroup* M *is left regular if and only if, for every nonempty subset* A *of* M*, we have* $A \subseteq (M\Gamma A\Gamma A]$*, equivalently, for every* $a \in M$*, we have* $a \in (M\Gamma a\Gamma a]$*.*

Definition 2.16 *An ordered* Γ*-hypergroupoid* M *is called right duo if the right ideals of* M *are at the same time left ideals of* M*; that is, ideals of* M*. It is called left duo if the left ideals of* M *are ideals of* M*.*

Lemma 2.17 *Let* M *be an ordered* Γ*-hypersemigroup. Then, for every nonempty subsets* A, B, C *of* M*, we have*

$$(A\Gamma B\Gamma C] = \Big(A\Gamma(B]\Gamma C\Big].$$

Proof Since $B \subseteq (B]$, we have $A\Gamma B\Gamma C \subseteq A\Gamma(B]\Gamma C$ and so $(A\Gamma B\Gamma C] \subseteq \Big(A\Gamma(B]\Gamma C\Big]$. On the other hand,

$$\begin{aligned} A\Gamma(B]\Gamma C &\subseteq A\Gamma\Big((B]\Gamma(C]\Big) \subseteq A\Gamma(B\Gamma C] \text{ (by Lemma 2.6)} \\ &\subseteq (A]\Gamma(B\Gamma C] \subseteq \Big(A\Gamma(B\Gamma C)\Big] \text{ (by Lemma 2.6)} \\ &= (A\Gamma B\Gamma C] \end{aligned}$$

and so $\Big(A\Gamma(B]\Gamma C\Big] \subseteq \Big((A\Gamma B\Gamma C]\Big] = (A\Gamma B\Gamma C]$.

The theorem on right regular and right duo ordered Γ-hypersemigroups that corresponds to Theorem 2.7 in [5], is the following.

Theorem 2.18 *An ordered* Γ*-hypersemigroup* M *is right regular and right duo if and only if, for every right ideals* A *and* B *of* M*, we have*

$$A \cap B = (A\Gamma B].$$

Proof $\Longrightarrow$. Let A, B be right ideals of M. Then $A\Gamma B \subseteq A\Gamma M \subseteq A$; since M is right duo, B is a left ideal of M as well, that is $A\Gamma B \subseteq M\Gamma B \subseteq B$. Thus we have $(A\Gamma B] \subseteq (A] = A$ and $(A\Gamma B] \subseteq (B] = B$ and so $(A\Gamma B] \subseteq A \cap B$. Since A is a right ideal and B is a left ideal of M, by [5; Proposition 3.12], the set $A \cap B$ is nonempty. Since M is right regular and $A \cap B \neq \emptyset$, by Proposition 2.14, we have

$$A \cap B \subseteq \Big((A \cap B)\Gamma(A \cap B)\Gamma M\Big].$$

Since $A \cap B \subseteq A, B$, by [5; Lemma 3.8], $(A \cap B)\Gamma M \subseteq A\Gamma M \cap B\Gamma M \subseteq A \cap B$. Then we have

$$A \cap B \subseteq \Big((A \cap B)\Gamma(A \cap B)\Big] \subseteq (A\Gamma B],$$

and so $A \cap B = (A\Gamma B]$.

$\Longleftarrow$. Let A be a right ideal of M. Since M is a right ideal of M, by hypothesis, we have $A = M \cap A = (M\Gamma A]$, so A is a left ideal of M and M is right duo.

Let now $a \in M$. By hypothesis, we have

$$\begin{aligned} a \in R(a) \cap R(a) &= \Big(R(a)\Gamma R(a)\Big] = \Big((a \cup a\Gamma M]\Gamma(a \cup a\Gamma M]\Big] \text{ (by Lemma 2.7)} \\ &= \Big((a \cup a\Gamma M)\Gamma(a \cup a\Gamma M)\Big] \text{ (by Lemma 2.8)} \\ &= (a\Gamma a \cup a\Gamma M\Gamma a \cup a\Gamma a\Gamma M \cup a\Gamma M\Gamma a\Gamma M] \\ &= (a\Gamma a] \cup (a\Gamma M\Gamma a] \cup (a\Gamma a\Gamma M] \cup (a\Gamma M\Gamma a\Gamma M]. \end{aligned}$$

If $a \in (a\Gamma a]$, then $a\Gamma a \subseteq (a\Gamma a]\Gamma a \subseteq (a\Gamma a]\Gamma(a] \subseteq (a\Gamma a\Gamma a] \subseteq (a\Gamma a\Gamma M]$. Then

$$a \in (a\Gamma a] \subseteq \Big((a\Gamma a\Gamma M]\Big] = (a\Gamma a\Gamma M].$$

Let $a \in (a\Gamma M\Gamma a]$. Then

$$\begin{aligned} a \in \Big(a\Gamma M\Gamma(a\Gamma M\Gamma a]\Big] &= \Big((a\Gamma M)\Gamma(a\Gamma M\Gamma a)\Big] \text{ (by Lemma 2.8)} \\ &= \Big((a\Gamma M)\Gamma(a\Gamma M)\Gamma a)\Big] \\ &= \Big((a\Gamma M)\Gamma(a\Gamma M]\Gamma a\Big] \text{ (by Lemma 2.17)} \\ &= \Big(a\Gamma M\Gamma(a\Gamma M]\Gamma a\Big]. \end{aligned}$$

The set $(a\Gamma M]$ is a right ideal of M. Since M is right duo, it is a left ideal of M as well, that is, $M\Gamma(a\Gamma M] \subseteq (a\Gamma M]$. Thus we have

$$\begin{aligned} a \in \Big(a\Gamma(a\Gamma M]\Gamma a\Big] &= \Big(a\Gamma(a\Gamma M)\Gamma a\Big] \text{ (by Lemma 2.17)} \\ &= \Big(a\Gamma a\Gamma(M\Gamma a)\Big] \subseteq (a\Gamma a\Gamma M], \end{aligned}$$

and so $a \in (a\Gamma a\Gamma M]$.

Let $a \in (a\Gamma M\Gamma a\Gamma M]$. By Lemma 2.8, $a \in \Big((a\Gamma M)\Gamma(a\Gamma M]\Big]$. Since $(a\Gamma M]$ is a right ideal of M and M is right duo, it is a left ideal of M as well and so $M\Gamma(a\Gamma M] \subseteq (a\Gamma M]$. Thus we have $a \in \Big(a\Gamma(a\Gamma M]\Big] = (a\Gamma a\Gamma M]$ by Lemma 2.8.

In each case, we have $a \in (a\Gamma a\Gamma M]$ and, by Proposition 2.14, M is right regular. □

In a similar way, we prove the following theorem that corresponds to [5; Theorem 2.8].

Theorem 2.19 *An ordered Γ-hypersemigroup M is left regular and left duo if and only if, for every left ideals A and B of M, we have*

$$A \cap B = (B\Gamma A].$$

Definition 2.20 *Let M be a Γ-hypersemigroup. A nonempty subset T of M is called semiprime if for any nonempty subset A of T such that $A\Gamma A \subseteq T$, we have $A \subseteq T$.*
Equivalent Definition: For every $a \in M$ such that $a\Gamma a \subseteq T$, we have $a \in T$.

The theorem on ordered Γ-hypersemigroups that corresponds to [5; Theorem 2.10] is the following:

Theorem 2.21 *An ordered Γ-hypersemigroup M is intra-regular if and only if the ideals of M are semiprime.*

Proof $\Longrightarrow$. Let T be an ideal of M and A be a nonempty subset of T such that $A\Gamma A \subseteq T$. Since M is intra-regular, by Proposition 2.11, we have $A \subseteq (M\Gamma A\Gamma A\Gamma M] = \Big(M\Gamma(A\Gamma A)\Gamma M\Big]$. Since $A\Gamma A \subseteq T$, we have $M\Gamma(A\Gamma A)\Gamma M \subseteq M\Gamma T\Gamma M \subseteq T$. Thus we have $A \subseteq (T] = T$ and so M is semiprime.
$\Longleftarrow$. Let $a \in M$. The set $I(a\Gamma a)$ is an ideal of M such that $a\Gamma a \subseteq I(a\Gamma a)$. Since $I(a\Gamma a)$ is semiprime, we have

$$a \in I(a\Gamma a) = (a\Gamma a \cup M\Gamma a\Gamma a \cup a\Gamma a\Gamma M \cup M\Gamma a\Gamma a\Gamma M].$$

Then M is intra-regular (see the proof of the "$\Leftarrow$"-part of Theorem 2.12). □

The proposition on ordered Γ-hypersemigroups that corresponds to [5; Proposition 2.11] is the following.

Proposition 2.22 *If an ordered Γ-hypersemigroup M is right (or left) regular, then it is intra-regular.*

Proof Let M be right regular and A a nonempty subset of M. By Proposition 2.14, we have $A \subseteq (A\Gamma A\Gamma M]$. Moreover,

$$\begin{aligned}
A\Gamma A\Gamma M &\subseteq A\Gamma(A\Gamma A\Gamma M]\Gamma M \subseteq (A]\Gamma(A\Gamma A\Gamma M]\Gamma(M] \\
&\subseteq \Big(A\Gamma(A\Gamma A\Gamma M)\Gamma M\Big] \text{ (by Lemma 2.6)} \\
&= \Big((A\Gamma A)\Gamma A\Gamma(M\Gamma M)\Big] \\
&\subseteq (M\Gamma A\Gamma M] \subseteq \Big(M\Gamma(A\Gamma A\Gamma M]\Gamma M\Big] \\
&= \Big(M\Gamma(A\Gamma A\Gamma M)\Gamma M\Big] \text{ (by Lemma 2.17)} \\
&= \Big(M\Gamma(A\Gamma A)\Gamma(M\Gamma M)\Big] \subseteq (M\Gamma A\Gamma A\Gamma M].
\end{aligned}$$

Thus we have $A \subseteq \Big((M\Gamma A\Gamma A\Gamma M]\Big] = (M\Gamma A\Gamma A\Gamma M]$ and, by Proposition 2.11, M is intra-regular. □

The following proposition, corresponds to [5; Proposition 2.12].

Proposition 2.23 *Let M be an ordered Γ-hypersemigroup. The following are equivalent:*

(1) M is right regular.

(2) $R(A) = R(A\Gamma A)$ for every nonempty subset A of M.

(3) $R(A) \subseteq R(A\Gamma A)$ for every nonempty subset A of M.

Proof $(1) \Longrightarrow (2)$. Let A be a nonempty subset of M. We have $R(A) = (A \cup A\Gamma M]$. Since M is right regular, by Proposition 2.14, we have $A \subseteq (A\Gamma A\Gamma M]$. Then we have

$$A \cup A\Gamma M \subseteq (A\Gamma A\Gamma M] \cup (A\Gamma A\Gamma M]\Gamma M = (A\Gamma A\Gamma M] \cup (A\Gamma A\Gamma M]\Gamma(M].$$

Since $(A\Gamma A\Gamma M]\Gamma(M] \subseteq \Big(A\Gamma A\Gamma(M\Gamma M)\Big] \subseteq (A\Gamma A\Gamma M]$, we have $A \cup A\Gamma M = (A\Gamma A\Gamma M]$. Then we have

$$\begin{aligned}
R(A) &= (A \cup A\Gamma M] = \Big((A\Gamma A\Gamma M]\Big] = \Big((A\Gamma A)\Gamma M\Big] \\
&\subseteq \Big(A\Gamma A \cup (A\Gamma A)\Gamma M\Big] = R(A\Gamma A).
\end{aligned}$$

On the other hand,

$$\begin{aligned} R(A\Gamma A) &= \Big(A\Gamma A \cup (A\Gamma A)\Gamma M\Big] = (A\Gamma A] \cup \Big(A\Gamma(A\Gamma M)\Big] \subseteq (A\Gamma M] \\ &\subseteq (A \cup A\Gamma M] = R(A). \end{aligned}$$

Thus we have $R(A) = R(A\Gamma A)$.
The implication $(2) \Longrightarrow (3)$ is obvious.
$(3) \Longrightarrow (1)$. let A be a nonempty subset of M. By hypothesis, we have

$$\begin{aligned} A &\subseteq R(A) \subseteq R(A\Gamma A) = \Big(A\Gamma A \cup (A\Gamma A)\Gamma M\Big] \\ &= (A\Gamma A] \cup \Big(A\Gamma(A\Gamma M)\Big] \subseteq (A\Gamma M], \end{aligned}$$

from which $A\Gamma A \subseteq A\Gamma(A\Gamma M]$. Then we have $(A\Gamma A] \subseteq \Big((A\Gamma(A\Gamma M]\Big] = (A\Gamma A\Gamma M]$ and so $A \subseteq (A\Gamma A\Gamma M]$. By Proposition 2.14, M is right regular. □

The following corresponds to [5; Proposition 2.13].

Proposition 2.24 *An ordered Γ-hypersemigroup M is left regular if and only if, for any nonempty subset A of M, we have*

$$L(A) = L(A\Gamma A), \text{ equivalently, } L(A) \subseteq L(A\Gamma A).$$

The following corresponds to [5; Theorem 2.14].

Proposition 2.25 *An ordered Γ-hypersemigroup M is right regular if and only if the right ideals of M are semiprime.*

Proof $\Longrightarrow$. Let T be a right ideal of M and A a nonempty subset of M such that $A\Gamma A \subseteq T$. Since M is right regular, by Proposition 2.14, we have $A \subseteq \Big((A\Gamma A)\Gamma M\Big] \subseteq (T\Gamma M] \subseteq T$, then $A \subseteq T$ and so T is semiprime.

$\Longleftarrow$. Let A be a nonempty subset of M. Since $R(A\Gamma A)$ is a right ideal of M, by hypothesis, it is semiprime. Since $A\Gamma A \subseteq R(A\Gamma A)$ and $R(A\Gamma A)$ is semiprime, we have

$$A \subseteq R(A\Gamma A) = \Big(A\Gamma A \cup (A\Gamma A)\Gamma M\Big] = (A\Gamma A] \cup \Big(A\Gamma(A\Gamma M)\Big] \subseteq (A\Gamma M].$$

Then $A\Gamma A \subseteq A\Gamma(A\Gamma M] \subseteq (A]\Gamma(A\Gamma M] \subseteq (A\Gamma A\Gamma M]$ and so $A \subseteq (A\Gamma A\Gamma M]$. By Proposition 2.14, M is right regular. □

In a similar way, we get the following proposition that corresponds to [5; Theorem 2.15].

Proposition 2.26 *An ordered Γ-hypersemigroup M is left regular if and only if the left ideals of M are semiprime.*

Definition 2.27 *An ordered Γ-hypergroupoid M is called right (resp. left) simple if M is the only right (resp. left) ideal of M. That is, if A is a right (resp. left) ideal of M, then $A = M$.*

The following corresponds to [5; Proposition 2.17].

Proposition 2.28 *Let M be an ordered Γ-hypersemigroup. The following are equivalent:*

(1) M is right (resp. left) simple.

(2) $(A\Gamma M]=M$ (resp. $(M\Gamma A]=M$) for every nonempty subset A of M.

(3) $(a\Gamma M]=M$ (resp. $(M\Gamma a]=M$) for every $a\in M$.

Proof $(1)\Longrightarrow(2)$. Assuming M is right simple, let A be a nonempty subset of M. Since $(A\Gamma M]$ is a right ideal of M and M is right simple, we have $(A\Gamma M]=M$.
The implication $(2)\Rightarrow(3)$ is obvious.
$(3)\Longrightarrow(1)$. Suppose $(a\Gamma M]=M$ for every $a\in M$ and let T be a left ideal of M. Then $T=M$. Indeed: Let $a\in M$. Take an element $b\in T$ $(T\neq\emptyset)$. By hypothesis, we have $(b\Gamma M]=M$. Then, $a\in(b\Gamma M]\subseteq(T\Gamma M]\subseteq(T]=T$ and so $a\in T$. □

The following proposition corresponds to [5; Proposition 2.19].

Proposition 2.29 *If an ordered Γ-hypersemigroup M is both right and left simple, then it is regular.*

Proof Let A be a nonempty subset of M. Since M is right simple, by Proposition 2.28, we have $(A\Gamma M]=M$; since M is left simple, we have $(M\Gamma A]=M$. Then we have

$$A\ \subseteq\ (A\Gamma M]=\Big(A\Gamma(M\Gamma A]\Big]=\Big(A\Gamma(M\Gamma A)\Big]=(A\Gamma M\Gamma A],$$

and by Proposition 2.4, M is regular. □

Definition 2.30 *A nonempty subset A of an ordered Γ-hypersemigroup M is called a bi-ideal of M if we have the following:*

(1) $B\Gamma M\Gamma B\subseteq B$ and

(2) if $a\in B$ and $M\ni b\leq a$, then $b\in B$.

By a subidempotent bi-ideal of M we mean a bi-ideal A of M such that $A\Gamma A\subseteq A$ (in other words a bi-ideal of M that is at the same time a Γ-subsemihypergroup of M).

The theorem on ordered Γ-hypersemigroups that corresponds to [5; Theorem 2.20] is the following.

Theorem 2.31 *An ordered Γ-hypersemigroup M is both left and right simple if and only if M does not contain proper bi-ideals; equivalently, if M does not contain proper subidempotent bi-ideals.*

Proof $\Longrightarrow$. Let B be a bi-ideal of M. Since S is left simple, by Proposition 2.28, we have $(M\Gamma B]=M$; since M is right simple, we have $(B\Gamma M]=M$. Thus we have

$$M=(M\Gamma B]=\Big((B\Gamma M]\Gamma B\Big]=(B\Gamma M\Gamma B]\subseteq(B]=B,$$

and so $B=M$.
$\Longleftarrow$. Let A be a left ideal of M. Then, A is a subidempotent bi-ideal of M. By hypothesis, we have $A=M$, so M does not contain proper left ideals and so it is left simple. Similarly, M is right simple. □

Theorem 2.31, in case of an ordered semigroup, has been proved in [5]. Using the methodology given in Theorem 2.31, the proof in [5] can be simplified. However, based on [5], we can give a second proof of the "$\Rightarrow$"-part of Theorem 2.31 which, though more technical, is interesting giving further detailed information about the techniques in ordered Γ-hypersemigroups.

For this proof, we need the following lemma.

Lemma 2.32 *If M is a regular Γ-hypersemigroup and A a bi-ideal of M, then $A\Gamma A \subseteq A$.*

Proof Since A is a bi-ideal of M, we have $A\Gamma M\Gamma A \subseteq A$, and then $(A\Gamma M\Gamma A] \subseteq (A] = A$. Since M is regular, we have $A \subseteq (A\Gamma M\Gamma A]$ and so $A = (A\Gamma M\Gamma A]$. Then we get

$$A\Gamma A \quad \subseteq \quad (A\Gamma M\Gamma A]\Gamma(A] \subseteq \Big(A\Gamma(M\Gamma A)\Gamma A\Big] \subseteq (A\Gamma M\Gamma A] = A,$$

and so $A\Gamma A \subseteq A$. □

Second proof of the "$\Rightarrow$"-part of Theorem 2.31

$\Longrightarrow$. Let A be a bi-ideal of M. Then $A = M$. In fact: Let $a \in M$. Take an element $b \in A$ $(A \neq \emptyset)$. Consider the left ideal $L(b)$ of M generated by b, that is the set $L(b) = (b \cup M\Gamma b]$. Since M is left simple, we have $L(b) = M$. Since $a \in L(b)$, we have $a \le t$ for some $t \in b \cup M\Gamma b$.

(A) If $t = b$, then $t \in A$. Since $M \ni a \le t \in A$ and A is a bi-ideal of M, we have $a \in A$ and the proof is complete.

(B) If $t \in M\Gamma b$, then $t \in x\gamma b$ for some $x \in M$, $\gamma \in \Gamma$. We consider the right ideal of M generated by b, that is the set $R(b) = (b \cup b\Gamma M]$. Since M is right simple, we have $R(b) = M$. Since $x \in R(b)$, we have $x \le k$ for some $k \in b \cup b\Gamma M$.

(B_1) If $k = b$, then $x \le b$, so $x\gamma b \preceq b\gamma b$ and since $t \in x\gamma b$, there exists $u \in b\gamma b$ such that $t \le u$. Since M is right and left simple, by Proposition 2.29, it is regular. Since M is regular and A is a bi-ideal of M, by Lemma 2.32, have $A\Gamma A \subseteq A$. Since $u \in b\gamma b \subseteq A\Gamma A \subseteq A$, we have $u \in A$. Since $M \ni a \le t \le u \in A$ and A is a bi-ideal of M, we have $a \in A$ and the proof is complete.

(B_2) Let $k \in b\Gamma M$. Then $k \in b\mu y$ for some $\mu \in \Gamma$, $y \in M$. Since $x \le k$, we have $x\gamma b \preceq k\gamma b$ and since $t \in x\gamma b$, there exists $u \in k\gamma b$ such that $t \le u$. We have

$$u \in k\gamma b = \{k\}\overline{\gamma}\{b\} \subseteq (b\mu y)\overline{\gamma}\{b\} \subseteq A\Gamma M\Gamma A \subseteq A,$$

therefore $u \in A$. Since $M \ni a \le t \le u \in A$ and A is a bi-ideal of M, we have $a \in A$ and the proof is complete.

Note It might be mentioned that the Γ-hypersemigroup given in Example 3.24 in [5], endowed with the order relation $\le := \{(a,a),(a,c),(b,b),(b,c),(c,c)\}$ is an example of an ordered Γ-hypersemigroup that is regular, right (resp. left) regular, and, by Proposition 2.22, intra-regular as well. Moreover, it is right simple and left simple. It is duo as well. So, the results of the paper can be applied.

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