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# Certain subclasses of spirallike univalent functions related to Poisson distribution series

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**Abstract:** The aim of the present study is to find the essential properties for some subclasses of analytic functions which are related to Poisson distribution that are member of the classes of spiral-like univalent functions. Further, we studied inclusion relations for such subclasses, and also we determined some properties of an integral operator related to Poisson distribution series. Several corollaries and consequences of the main results are also considered.

**Key words:** Univalent function, spirallike function, starlike and convex functions, Hadamard (convolution) product, Alexander integral, Poisson distribution series

## 1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$, and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Also, denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions of the form given by (1.1) that are univalent in $\mathbb{U}$.

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we denote the Hadamard (or convolution) product of $f$ and $g$ by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n, \ z \in \mathbb{U}.$$

For two analytic functions $f$ and $g$ analytic in $\mathbb{U}$ we say that $f$ is subordinate to $g$, denoted by $f(z) \prec g(z)$, if there exists a function $\omega$ analytic in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. Note that if the function $g$ is univalent in $\mathbb{U}$ (see also Miller and Mocanu [5] and Bulboacă [2]) we have

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

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The two well-known subclasses of $\mathcal{S}$ are mentioned as the class of starlike functions of order $\lambda$ and convex function of order $\lambda$ which were introduced by Robertson [15]. Thus, a function $f \in \mathcal{A}$ is said to be starlike of order $\lambda$ $(0 \leq \lambda < 1)$, if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \lambda, \ z \in \mathbb{U},$$

or equivalently,

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\lambda)z}{1-z},$$

where the symbol "$\prec$" represents the *subordination*.

Such class of functions is denoted by $\mathcal{S}^*(\lambda)$, and $\mathcal{S}^*(0) =: \mathcal{S}^*$ is the well-known class of starlike functions. The concept of starlike functions was introduced by Alexander [1] according to the property that $f$ maps conformally the open unit disk $\mathbb{U}$ onto a starlike domain with respect to the origin.

A function $f \in S$ is said to be convex of order $\lambda$ $(0 \leq \lambda < 1)$, if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \lambda, \ z \in \mathbb{U},$$

that is

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\lambda)z}{1-z}$$

and the class of all convex functions of order $\lambda$ is denoted by $\mathcal{C}(\lambda)$.

Clearly, $\mathcal{C}(0) =: \mathcal{C}$ is the class of convex functions with the property that $f$ maps conformally the open unit disk $\mathbb{U}$ onto a convex domain. The classes $\mathcal{S}^*(\lambda)$ and $\mathcal{C}(\lambda)$ satisfies Alexander's duality relation:

$$f \in \mathcal{C}(\lambda) \Leftrightarrow zf' \in \mathcal{S}^*(\lambda), \ 0 \leq \lambda < 1.$$

A function $f \in \mathcal{A}$ is said to be a spirallike function if

$$\operatorname{Re}\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) > 0, \ z \in \mathbb{U},$$

for some $\alpha \in \mathbb{R}$ with $|\alpha| < \frac{\pi}{2}$. The class of spirallike function was introduced by Spaček [16].

Motivated by the recent remarkable results of various researches that are available in [6–14], hereby we state the two subclasses of $\mathcal{A}$ as follows:

**Definition 1.1** *For* $0 \leq \delta < 1$, $0 \leq \beta < 1$ *and* $\alpha \in \mathbb{R}$ *with* $|\alpha| < \frac{\pi}{2}$, *let* $\mathcal{S}(\alpha, \beta, \delta)$ *be the subclass of* $\mathcal{A}$ *defined as follows:*

$$\mathcal{S}(\alpha,\beta,\delta) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{(1-\delta)f(z)+\delta zf'(z)}\right) > \beta\cos\alpha, \ z \in \mathbb{U} \right\}.$$

Following the Alexander's duality relation, we define the next subclass of $\mathcal{A}$:

**Definition 1.2** *For* $0 \leq \delta < 1$, $0 \leq \beta < 1$ *and* $\alpha \in \mathbb{R}$ *with* $|\alpha| < \frac{\pi}{2}$, *let* $\mathcal{C}(\alpha,\beta,\delta)$ *be the subclass of* $\mathcal{A}$ *defined by*

$$\mathcal{C}(\alpha,\beta,\delta) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left( e^{i\alpha} \frac{zf''(z) + f'(z)}{f'(z) + \delta z f''(z)} \right) > \beta \cos\alpha,\ z \in \mathbb{U} \right\}.$$

**Remark 1.3** *1. As we mentioned above, the subclasses* $\mathcal{S}(\alpha,\beta,\delta)$ *and* $\mathcal{C}(\alpha,\beta,\delta)$ *are connected by the Alexander's type duality relation*

$$f \in \mathcal{C}(\alpha,\beta,\delta) \Leftrightarrow zf' \in \mathcal{S}(\alpha,\beta,\delta).$$

*2. For the special value* $\delta = 0$, *the above two subclasses reduces to the class of* $\alpha$*-spirallike functions of order* $\beta$ *studied in [3, 4]*

$$\mathcal{S}(\alpha,\beta) := \mathcal{S}(\alpha,\beta,0) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left( e^{i\alpha} \frac{zf'(z)}{f(z)} \right) > \beta \cos\alpha,\ z \in \mathbb{U} \right\},$$

*and to the class of* $\alpha$*-convex spirallike functions of order* $\beta$

$$\mathcal{C}(\alpha,\beta) := \mathcal{C}(\alpha,\beta,0) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left[ e^{i\alpha} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] > \beta \cos\alpha,\ z \in \mathbb{U} \right\},$$

*respectively.*

The next subclass $\mathcal{R}^{\tau}(\gamma,\eta)$ was initially introduced by Swaminathan in [17, p. 3] as follows:

**Definition 1.4** *For* $\tau \in \mathbb{C} \setminus \{0\}$, $0 \leq \eta < 1$, *and* $\gamma < 1$, *the subclass* $\mathcal{R}^{\tau}(\gamma,\eta)$ *is defined by*

$$\mathcal{R}^{\tau}(\gamma,\eta) := \left\{ f \in \mathcal{A} : \left| \frac{(1-\eta)\frac{f(z)}{z} + \eta f'(z) - 1}{2\tau(1-\gamma) + (1-\eta)\frac{f(z)}{z} + \eta f'(z) - 1} \right| < 1,\ z \in \mathbb{U} \right\}.$$

Now we will give a few the necessary and sufficient conditions for the functions $f$ given by (1.1) to be in the above subclasses.

**Lemma 1.5** *[6, Corollary 2.1.] A function* $f$ *of the form* (1.1) *belongs to* $\mathcal{S}(\alpha,\beta,\delta)$ *if*

$$\sum_{k=2}^{\infty} \left[ (1-\delta)(k-1)\sec\alpha + (1-\beta)(1+k\delta-\delta) \right] |a_k| \leq 1-\beta,$$

*where* $|\alpha| < \frac{\pi}{2}$, $0 \leq \delta < 1$ *and* $0 \leq \beta < 1$.

**Lemma 1.6** *[12, Lemma 1.5.] A function* $f$ *of the form* (1.1) *belongs to* $\mathcal{C}(\alpha,\beta,\delta)$ *if*

$$\sum_{k=2}^{\infty} k \left[ (1-\delta)(k-1)\sec\alpha + (1-\beta)(1+k\delta-\delta) \right] |a_k| \leq 1-\beta,$$

*where* $|\alpha| < \frac{\pi}{2}$, $0 \leq \delta < 1$ *and* $0 \leq \beta < 1$.

**Lemma 1.7** *[17, Theorem 2.3.] Let $f \in \mathcal{S}$ be of the form (1.1). If $f \in \mathcal{R}^{\tau}(\eta,\gamma)$, then*

$$|a_k| \leq \frac{2\,|\tau|\,(1-\gamma)}{1+\eta(k-1)},\ k \in \mathbb{N}\setminus\{1\}. \tag{1.2}$$

*The bounds given in (1.2) are sharp.*

A random variable $X$ is said to be a Poisson distribution with parameter $m \in \mathbb{R}$ if it takes the values $0,\ 1,\ 2,\ 3,$ ...with the probabilities $e^{-m}$, $\frac{me^{-m}}{1!}$, $\frac{m^2e^{-m}}{2!}$, $\frac{m^3e^{-m}}{3!}$, ...respectively, thus

$$P\big(X=x\big)=\frac{m^x e^{-m}}{x!},\ x=0,\,1,\,2,\,3,\ldots.$$

In 2014 Porwal [13] introduced a power series whose coefficients are probabilities of Poisson distribution, that is

$$\mathcal{P}_m(z) := z + \sum_{k=2}^{\infty} \frac{m^{k-1}\,e^{-m}}{(k-1)!} z^k,\ z \in \mathbb{U}, \tag{1.3}$$

where $m>0$. From the ratio test it follows that the radius of convergence of the above power series is infinity.

Using the Hadamard product, a new linear operator $\mathcal{L}_m : \mathcal{A} \to \mathcal{A}$ defined by a convolution product was introduced and defined by Porwal and Kumar [14] by

$$\mathcal{L}_m f(z) := \mathcal{P}_m(z) * f(z) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}\,e^{-m}}{(k-1)!}\,a_k z^k,\ z \in \mathbb{U}, \tag{1.4}$$

where $f \in \mathcal{A}$ is given by (1.1).

In the present study we will determine sufficient conditions such that the power series $\mathcal{P}_m$ be member of the classes $\mathcal{S}(\alpha,\beta,\delta)$ and $\mathcal{C}(\alpha,\beta,\delta)$. Also, sufficient conditions such that the images of $\mathrm{R}^{\tau}(\eta,\gamma)\cap\mathcal{S}$ by the operator $\mathcal{L}_m$ belongs to the classes $\mathcal{S}(\alpha,\beta,\delta)$ and $\mathcal{C}(\alpha,\beta,\delta)$ were obtained. Finally, sufficient conditions for the function $\mathcal{F}_m$ which is defined by the Alexander integral operator $\mathcal{F}_m(z) := \int_0^z \frac{\mathcal{P}_m(t)}{t}\,dt$ belongs to the classes $\mathcal{S}(\alpha,\beta,\delta)$ and $\mathcal{C}(\alpha,\beta,\delta)$ were discussed.

## 2. Sufficient conditions for $\mathcal{P}_m$ to belong to $\mathcal{S}(\alpha,\beta,\delta)$ and $\mathcal{C}(\alpha,\beta,\delta)$

The two results of this section represent sufficient conditions such that the function $\mathcal{P}_m$ defined by (1.3) belongs to the classes $\mathcal{S}(\alpha,\beta,\delta)$ and $\mathcal{C}(\alpha,\beta,\delta)$, respectively.

**Theorem 2.1** *Let $m>0$, $0\leq\delta<1$, $0\leq\beta<1$, and $|\alpha|<\frac{\pi}{2}$. If*

$$m\left[\sec\alpha+\delta(1-\beta-\sec\alpha)\right] \leq (1-\beta)e^{-m}, \tag{2.1}$$

*then $\mathcal{P}_m \in \mathcal{S}(\alpha,\beta,\delta)$.*

**Proof** Since $\mathcal{P}_m$ has the power series expansion (1.3), according to Lemma 1.5 it is sufficient to prove that

$$\sum_{k=2}^{\infty}[(1-\delta)(k-1)\sec\alpha+(1-\beta)(1+k\delta-\delta)]\,\frac{m^{k-1}}{(k-1)!}e^{-m}\leq 1-\beta. \tag{2.2}$$

From the assumption (2.1) a simple computation yields to

$$\begin{aligned}
&\sum_{k=2}^{\infty}[(1-\delta)(k-1)\sec\alpha+(1-\beta)(1+k\delta-\delta)]\,\frac{m^{k-1}}{(k-1)!}e^{-m}\\
&=\sum_{k=2}^{\infty}\Big\{\big[\sec\alpha+\delta(1-\beta-\sec\alpha)\big]k+(1-\delta)(1-\beta-\sec\alpha)\Big\}\frac{m^{k-1}}{(k-1)!}e^{-m}\\
&=\sum_{k=2}^{\infty}[\sec\alpha+\delta(1-\beta-\sec\alpha)]\,(k-1)\frac{m^{k-1}}{(k-1)!}e^{-m}+\sum_{k=2}^{\infty}(1-\beta)\frac{m^{k-1}}{(k-1)!}e^{-m}\\
&=m\,[\sec\alpha+\delta(1-\beta-\sec\alpha)]+(1-\beta)(1-e^{-m})\leq 1-\beta.
\end{aligned}$$

It follows that the inequality (2.2) holds, therefore $\mathcal{P}_m\in\mathcal{S}(\alpha,\beta,\delta)$. □

**Theorem 2.2** *Let* $m>0$, $0\leq\delta<1$, $0\leq\beta<1$, *and* $|\alpha|<\dfrac{\pi}{2}$. *If*

$$m^2\,[\sec\alpha+\delta(1-\beta-\sec\alpha)]+m\,[(1+2\delta)\,(1-\beta-\sec\alpha)+3\sec\alpha]\leq(1-\beta)e^{-m}, \tag{2.3}$$

*then* $\mathcal{P}_m\in\mathcal{C}(\alpha,\beta,\delta)$.

**Proof** Since $\mathcal{P}_m$ is given by (1.3), in virtue of Lemma 1.6 it is sufficient to show that

$$\sum_{k=2}^{\infty}k\,[(1-\delta)(k-1)\sec\alpha+(1-\beta)(1+k\delta-\delta)]\,\frac{m^{k-1}}{(k-1)!}e^{-m}\leq 1-\beta. \tag{2.4}$$

We easily get that

$$\begin{aligned}
&\sum_{k=2}^{\infty}k\,[(1-\delta)(k-1)\sec\alpha+(1-\beta)(1+k\delta-\delta)]\,\frac{m^{k-1}}{(k-1)!}e^{-m}\\
&=\sum_{k=2}^{\infty}\Big\{\big[\sec\alpha+\delta(1-\beta-\sec\alpha)\big]k^2+(1-\delta)(1-\beta-\sec\alpha)k\Big\}\frac{m^{k-1}}{(k-1)!}e^{-m}\\
&=\sum_{k=2}^{\infty}\big[\sec\alpha+\delta(1-\beta-\sec\alpha)\big](k-1)(k-2)\frac{m^{k-1}}{(k-1)!}e^{-m}\\
&+\sum_{k=2}^{\infty}[(1+2\delta)(1-\beta-\sec\alpha)+3\sec\alpha]\,(k-1)\frac{m^{k-1}}{(k-1)!}e^{-m}\\
&+\sum_{k=2}^{\infty}(1-\beta)\frac{m^{k-1}}{(k-1)!}e^{-m}=m^2\,[\sec\alpha+\delta(1-\beta-\sec\alpha)]\\
&+m\big[(1+2\delta)\,(1-\beta-\sec\alpha)+3\sec\alpha\big]+(1-\beta)(1-e^{-m})\\
&\leq 1-\beta,
\end{aligned}$$

whenever the assumption inequality (2.3) holds. Thus, (2.4) is satisfied that leads to $\mathcal{P}_m \in \mathcal{C}(\alpha,\beta,\delta)$. □

## 3. Images of the $\mathcal{R}^\tau(\eta,\gamma)$ classes by the $\mathcal{L}_m$ convolution operator

Next, we will give as sufficient condition such that the images of the functions from $\mathcal{R}^\tau(\eta,\gamma)\cap\mathcal{S}$ by the operator $\mathcal{L}_m$ defined by (1.4) belong to the classes $\mathcal{S}(\alpha,\beta,\delta)$ and $\mathcal{C}(\alpha,\beta,\delta)$, respectively.

**Theorem 3.1** *Let $\tau \in \mathbb{C}\setminus\{0\}$, $0<\gamma\le 1$, and $\eta<1$. If*

$$\frac{2|\tau|(1-\gamma)}{\eta}\Big\{\big[\sec\alpha+\delta(1-\beta-\sec\alpha)\big]\left(1-e^{-m}\right) + \frac{(1-\delta)(1-\beta-\sec\alpha)\left[1-e^{-m}(1+m)\right]}{m}\Big\} \le 1-\beta, \tag{3.1}$$

*then*

$$\mathcal{L}_m\left(\mathcal{R}^\tau(\eta,\gamma)\cap\mathcal{S}\right)\subset\mathcal{S}(\alpha,\beta,\delta).$$

**Proof** Let $f\in\mathcal{R}^\tau(\eta,\gamma)\cap\mathcal{S}$ be given by (1.1). In order to prove our result, according to Lemma 1.5 it is sufficient to show that

$$\sum_{k=2}^{\infty}\left[(1-\delta)(k-1)\sec\alpha+(1-\beta)(1+k\delta-\delta)\right]\frac{m^{k-1}}{(k-1)!}e^{-m}|a_k|\le 1-\beta. \tag{3.2}$$

Since $f\in\mathcal{R}^\tau(\eta,\gamma)\cap\mathcal{S}$, from Lemma 1.7 we have

$$|a_k|\le\frac{2|\tau|(1-\gamma)}{1+\eta(k-1)},\ k\in\mathbb{N}\setminus\{1\},$$

and using the fact that $1+\eta(k-1)\ge k\eta$ together with the assumption (3.1) we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left[(1-\delta)(k-1)\sec\alpha+(1-\beta)(1+k\delta-\delta)\right]\frac{m^{k-1}}{(k-1)!}e^{-m}|a_k|\\
&\le\frac{2|\tau|(1-\gamma)}{\eta}\left\{\sum_{k=2}^{\infty}\frac{1}{k}\big[(1-\delta)(k-1)\sec\alpha+(1-\beta)(1+k\delta-\delta)\big]\frac{m^{k-1}}{(k-1)!}e^{-m}\right\}\\
&=\frac{2|\tau|(1-\gamma)}{\eta}\left\{\sum_{k=2}^{\infty}\big[\sec\alpha+\delta(1-\beta-\sec\alpha)\big]\frac{m^{k-1}}{(k-1)!}e^{-m}\right.\\
&\left.+\sum_{k=2}^{\infty}(1-\delta)(1-\beta-\sec\alpha)\frac{m^{k-1}}{k!}e^{-m}\right\}\\
&=\frac{2|\tau|(1-\gamma)}{\eta}\Big\{\big[\sec\alpha+\delta(1-\beta-\sec\alpha)\big]\left(1-e^{-m}\right)\\
&+\frac{(1-\delta)(1-\beta-\sec\alpha)\left[1-e^{-m}(1+m)\right]}{m}\Big\}\le 1-\beta.
\end{aligned}$$

Hence, the inequality (3.2) holds and therefore $\mathcal{L}_m f\in\mathcal{S}(\alpha,\beta,\delta)$. □

Using the same technique as in the proof of Theorem 2.2, we similarly get the following result:

**Theorem 3.2** *Let* $\tau \in \mathbb{C} \setminus \{0\}$, $0 < \gamma \leq 1$, *and* $\eta < 1$. *If*

$$\frac{2|\tau|(1-\gamma)}{\eta}\Big\{m^2\big[\sec\alpha + \delta(1-\beta-\sec\alpha)\big] + m\big[(1-\beta-\sec\alpha)(1+2\delta)+3\sec\alpha\big] + (1-\beta)\left(1-e^{-m}\right)\Big\} \leq 1-\beta,$$

*then*

$$\mathcal{L}_m\left(\mathcal{R}^\tau(\eta,\gamma)\cap\mathcal{S}\right) \subset \mathcal{C}(\alpha,\beta,\delta).$$

## 4. Images of the $\mathcal{P}_m$ functions by the Alexander integral operator

Next we will give two sufficient conditions such that the Alexander integral of the $\mathcal{P}_m$ belongs to the classes $\mathcal{S}(\alpha,\beta,\delta)$ and $\mathcal{C}(\alpha,\beta,\delta)$, respectively.

**Theorem 4.1** *If the function* $\mathcal{F}_m$ *is defined by*

$$\mathcal{F}_m(z) := \int_0^z \frac{\mathcal{P}_m(t)}{t}\,dt,\ z \in \mathbb{U}, \tag{4.1}$$

*where* $\mathcal{P}_m$ *is given by* (1.3), *and*

$$[\sec\alpha + \delta(1-\beta-\sec\alpha)]\left(1-e^{-m}\right) + \frac{(1-\delta)(1-\beta-\sec\alpha)\left[1-e^{-m}(1+m)\right]}{m} \leq 1-\beta, \tag{4.2}$$

*then* $\mathcal{F}_m \in \mathcal{S}(\alpha,\beta,\delta)$.

**Proof** Since

$$\mathcal{F}_m(z) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!k} e^{-m} z^k,\ z \in \mathbb{U}, \tag{4.3}$$

according to Lemma 1.5 it is sufficient to verify that

$$\sum_{k=2}^{\infty} \frac{1}{k}\left[(1-\delta)(k-1)\sec\alpha + (1-\beta)(1+k\delta-\delta)\right]\frac{m^{k-1}}{(k-1)!}e^{-m} \leq 1-\beta. \tag{4.4}$$

From the assumption (4.1) we get

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{1}{k}\left[(1-\delta)(k-1)\sec\alpha + (1-\beta)(1+k\delta-\delta)\right]\frac{m^{k-1}}{(k-1)!}e^{-m} \\
&= \sum_{k=2}^{\infty}\left\{[\sec\alpha+\delta(1-\beta-\sec\alpha)] + \frac{1}{k}(1-\delta)(1-\beta-\sec\alpha)\right\}\frac{m^{k-1}}{(k-1)!}e^{-m} \\
&= \sum_{k=2}^{\infty}[\sec\alpha+\delta(1-\beta-\sec\alpha)]\frac{m^{k-1}}{(k-1)!}e^{-m} + \sum_{k=2}^{\infty}(1-\delta)(1-\beta-\sec\alpha)\frac{m^{k-1}}{k!}e^{-m} \\
&= [\sec\alpha+\delta(1-\beta-\sec\alpha)]\left(1-e^{-m}\right) + \frac{(1-\delta)(1-\beta-\sec\alpha)\left[1-e^{-m}(1+m)\right]}{m} \\
&\leq 1-\beta.
\end{aligned}$$

Therefore, the inequality (4.4) holds and thus $\mathcal{F}_m \in \mathcal{S}(\alpha, \beta, \delta)$. □

**Theorem 4.2** *If the function* $\mathcal{F}_m$ *is given by* (4.1) *and*

$$m\left[\sec\alpha + \delta(1-\beta\sec\alpha)\right] \leq (1-\beta)e^{-m},$$

*then* $\mathcal{F}_m \in \mathcal{C}(\alpha, \beta, \delta)$.

**Proof** Since $\mathcal{F}_m$ was the power series expansion (4.3), according to Lemma 1.6 we need only to verify that

$$\sum_{k=2}^{\infty} k\left[(1-\delta)(k-1)\sec\alpha + (1-\beta)(1+k\delta-\delta)\right]\frac{1}{k}\frac{m^{k-1}}{(k-1)!}e^{-m} \leq 1-\beta,$$

or, equivalently

$$\sum_{k=2}^{\infty} \left[(1-\delta)(k-1)\sec\alpha + (1-\beta)(1+k\delta-\delta)\right]\frac{m^{k-1}}{(k-1)!}e^{-m} \leq 1-\beta.$$

The remaining part of the proof is similar to that of Theorem 2.1, and so we will omit the details. □

## 5. Special cases

Considering the special value $\delta = 0$ in the previous six theorems we will obtain the next results connected to the classes $\mathcal{S}(\alpha, \beta)$ and $\mathcal{C}(\alpha, \beta)$, respectively:

**Corollary 5.1** *Let* $m > 0$, $0 \leq \beta < 1$, *and* $|\alpha| < \dfrac{\pi}{2}$. *If*

$$m\sec\alpha \leq (1-\beta)e^{-m},$$

*then* $\mathcal{P}_m \in \mathcal{S}(\alpha, \beta)$.

**Corollary 5.2** *Let* $m > 0$, $0 \leq \beta < 1$, *and* $|\alpha| < \dfrac{\pi}{2}$. *If*

$$m^2\sec\alpha + m\left(1-\beta+2\sec\alpha\right) \leq (1-\beta)e^{-m},$$

*then* $\mathcal{P}_m \in \mathcal{C}(\alpha, \beta)$.

**Corollary 5.3** *Let* $\tau \in \mathbb{C}\setminus\{0\}$, $0 < \gamma \leq 1$, *and* $\eta < 1$. *If*

$$\frac{2\,|\tau|\,(1-\gamma)}{\eta}\left\{\left(1-e^{-m}\right)\sec\alpha + \frac{(1-\beta-\sec\alpha)\left[1-e^{-m}(1+m)\right]}{m}\right\} \leq 1-\beta,$$

*then*

$$\mathcal{L}_m\left(\mathcal{R}^{\tau}(\eta,\gamma)\cap\mathcal{S}\right) \subset \mathcal{S}(\alpha, \beta).$$

**Corollary 5.4** *Let $\tau \in \mathbb{C} \setminus \{0\}$, $0 < \gamma \leq 1$, and $\eta < 1$. If*

$$\frac{2|\tau|(1-\gamma)}{\eta}\Big[m^2 \sec\alpha + m(1-\beta+2\sec\alpha) + (1-\beta)\left(1-e^{-m}\right)\Big] \leq 1-\beta,$$

*then*

$$\mathcal{L}_m\left(\mathcal{R}^\tau(\eta,\gamma)\cap\mathcal{S}\right) \subset \mathcal{C}(\alpha,\beta).$$

**Corollary 5.5** *If the function $\mathcal{F}_m$ is given by (4.1) and*

$$\left(1-e^{-m}\right)\sec\alpha + \frac{(1-\beta-\sec\alpha)\left[1-e^{-m}(1+m)\right]}{m} \leq 1-\beta,$$

*then $\mathcal{F}_m \in \mathcal{S}(\alpha,\beta)$.*

**Corollary 5.6** *If the function $\mathcal{F}_m$ is given by (4.1) and*

$$m\sec\alpha \leq (1-\beta)e^{-m},$$

*then $\mathcal{F}_m \in \mathcal{C}(\alpha,\beta)$.*

## 6. Conclusion

The novelty of the above results consists in the fact that using some recent results we found sufficient conditions such that the function $\mathcal{P}_m$ defined by (1.3) belongs to the classes $\mathcal{S}(\alpha,\beta,\delta)$ and $\mathcal{C}(\alpha,\beta,\delta)$, respectively.

Moreover, with the aid of these new results we gave sufficient condition such that the images of the functions from $\mathcal{R}^\tau(\eta,\gamma)$ by the operator $\mathcal{L}_m$ defined by (1.4) belong to the classes $\mathcal{S}(\alpha,\beta,\delta)$ and $\mathcal{C}(\alpha,\beta,\delta)$.

Finally, in the same way we gave two sufficient conditions such that the Alexander integral of the $\mathcal{P}_m$ belongs to the classes $\mathcal{S}(\alpha,\beta,\delta)$ and $\mathcal{C}(\alpha,\beta,\delta)$.

Moreover, for appropriate choices of the parameters we found a few interesting special cases of the above main results.

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