Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 1444 – 1448
© TÜBİTAK
doi:10.3906/mat-2008-106

A short note on generic initial ideals

Bekir DANIŞ*

Department of Mathematics, Faculty of Science, Bilkent University, Ankara, Turkey

Received: 31.08.2020 • **Accepted/Published Online:** 19.04.2021 • **Final Version:** 20.05.2021

Abstract: The definition of a generic initial ideal includes the assumption $x_1 > x_2 > \cdots > x_n$. A natural question is how generic initial ideals change when we permute the variables. In the article [1, §2], it is shown that the generic initial ideals are permuted in the same way when the variables in the monomial order are permuted. We give a different proof of this theorem. Along the way, we study the Zariski open sets which play an essential role in the definition of a generic initial ideal and also prove a result on how the Zariski open set changes after a permutation of the variables.

Key words: Generic initial ideals, Zariski open sets, permutation of variables

1. Introduction

Let $S = F[x_1, \ldots, x_n]$ be a polynomial ring in n variables where F is an infinite field. For a homogeneous ideal I in S, its initial ideal depends on the choice of coordinates. By a linear and invertible substitution, the initial ideal can be made coordinate independent. The resulting initial ideal in generic coordinates is called the generic initial ideal of I. Generic initial ideals contain a lot of information on the combinatorial, geometrical and homological properties of I. For example, some of the properties of generic initial ideals were used by Hartshorne to prove the connectedness of Hilbert schemes. As another supportive example, we may consider the fact that generic initial ideals were exploited to bound the invariants of projective varieties. We refer the reader to [2, §15.9], [3] and [4, §4] for more background on this matter.

Before we continue, we include some preliminaries concerning generic initial ideals. Let $<$ be a monomial order on S satisfying $x_1 > x_2 > \cdots > x_n$ and $in_<(I)$ denotes the initial ideal of I. Then, we have the following theorem ([4, Theorem 4.1.2]).

Theorem 1.1 *For a graded ideal $I \subset S$, there exists a Zariski open set $V \subseteq GL_n(F)$ such that*

$$in_<(\alpha I) = in_<(\beta I) \text{ for all } \alpha, \beta \in V.$$

After this theorem, we can give the definition of generic initial ideals.

Definition 1.2 *Assume the notation of the previous theorem. Then, the ideal $in_<(\alpha I)$ with $\alpha \in V$ is called the generic initial ideal of I with respect to $<$ and written $gin_<(I)$.*

*Correspondence: bekirdanis@hotmail.com

2010 *AMS Mathematics Subject Classification:* 13A50, 13A02, 13A15.

In the proof of Theorem 1.1, there are two significant lemmas, see [4, Lemma 4.1.4] and [4, Lemma 4.1.5]. In particular, Lemma 4.1.5 [4, 4.1.5] is a crucial tool to construct the Zariski open set V as mentioned in Theorem 1.1. We investigate how these lemmas change when the variables are permuted and focus on finding a connection between the new and old Zariski open set. Note that the new Zariski open set is the Zariski open set which determines the generic initial ideal after a permutation of the variables.

The goal of this article is to discuss how a permutation of the variables affects the generic initial ideal. In [1, Theorem 3], it is proven that generic initial ideals respect a permutation of the variables in the monomial order without considering Zariski open sets. We present a different proof of this theorem by paying particular attention to how the Zariski open set V is affected by changing the ordering of the variables in the monomial order.

2. Main Results

Let $<$ be a monomial order on the set of monomials in S. Let π be a permutation of the variables i.e., $\pi(x_i) = x_{\pi(i)}$. Notice that π induces an isomorphism of S which sends monomials to monomials. We let $<_\pi$ denote the monomial order such that

$$\pi(M_1) <_\pi \pi(M_2) \text{ if and only if } M_1 < M_2.$$

Let S_d denote the d-th homogeneous component of S and we consider the t-th exterior power $\wedge^t S_d$ of the S_d. An element $m_1 \wedge m_2 \wedge \cdots \wedge m_t$, where m_i a monomial of degree d with $m_1 > m_2 > \cdots > m_t$ is called a standard exterior monomial of $\wedge^t S_d$ with respect to $<$. Standard exterior monomials of $\wedge^t S_d$ with respect to $<_\pi$ are defined similarly. If $m_1 \wedge m_2 \wedge \cdots \wedge m_t$ and $w_1 \wedge w_2 \wedge \cdots \wedge w_t$ two standard exterior monomials with respect to $<$, then we set

$$m_1 \wedge m_2 \wedge \cdots \wedge m_t > w_1 \wedge w_2 \wedge \cdots \wedge w_t$$

if $m_i > w_i$ for the smallest index i with $m_i \neq w_i$. This allows us to define $in_<(f)$ for $f \in \wedge^t S_d$ as the largest standart exterior monomial. Standard exterior monomials with respect to $<_\pi$ are ordered similarly and the order is denoted by

$$w_1 \wedge w_2 \wedge \cdots \wedge w_t <_\pi m_1 \wedge m_2 \wedge \cdots \wedge m_t.$$

Let $\alpha \in GL_n(F)$. Then, $\alpha = (\alpha_{ij})$ induces an automorphism on S by

$$x_j \to \sum_{i=1}^{n} \alpha_{ij} x_i \text{ for } 1 \leq j \leq n.$$

Let V be a t-dimensional subspace of S_d and $\{f_1, f_2, \ldots, f_t\}$ is a F-basis of V. Note that $\{\alpha(f_1), \ldots, \alpha(f_t)\}$ forms a F-basis of $\alpha(V)$. Let $w_1 \wedge w_2 \wedge \cdots \wedge w_t$ be as in Lemma 4.1.5 [4, 4.1.5], i.e. the largest standart exterior monomial of $\wedge^t S_d$ with $\exists \alpha \in GL_n(F)$ such that

$$in_<(\alpha(f_1) \wedge \cdots \wedge \alpha(f_t)) = w_1 \wedge w_2 \wedge \cdots \wedge w_t.$$

By Lemma 4.1.4 [4, Lemma 4.1.4] and its proof, we get the following lemma.

Lemma 2.1 *There exists a F-basis $\{\alpha(g_1), \ldots, \alpha(g_t)\}$ of $\alpha(V)$ such that*

$$in_<(\alpha(g_1) \wedge \cdots \wedge \alpha(g_t)) = in_<(\alpha(g_1)) \wedge in_<(\alpha(g_2)) \wedge \cdots \wedge in_<(\alpha(g_t))$$

with $in_<(\alpha(g_1)) > in_<(\alpha(g_2)) > \cdots > in_<(\alpha(g_t))$. *Indeed, we have* $in_<(\alpha(g_i)) = w_i$ *for* $i = 1, \ldots, t$.

Proof Since $in_<(\alpha(f_1)\wedge\cdots\wedge\alpha(f_t)) = in_<(\alpha(g_1)\wedge\cdots\wedge\alpha(g_t))$ for any other basis $\{\alpha(g_1), \ldots, \alpha(g_t)\}$ of $\alpha(V)$, the assertion of the lemma follows. □

Then, we have

$$U = \{\alpha \in GL_n(F) : in_<(\alpha(g_1) \wedge \cdots \wedge \alpha(g_t)) = w_1 \wedge w_2 \wedge \cdots \wedge w_t\}$$

is a nonempty Zariski open subset of $GL_n(F)$ by [4, 4.1.5].

Now, we define the new Zariski open subset U' of $GL_n(F)$ with respect to the new order $<_\pi$ in the same manner.

Let $w_1' \wedge w_2' \wedge \cdots \wedge w_t'$ be the largest standart exterior monomial of $\wedge^t S_d$ with respect to $<_\pi$ with $\exists\beta \in GL_n(F)$ such that

$$in_{<_\pi}(\beta(f_1) \wedge \cdots \wedge \beta(f_t)) = w_1' \wedge w_2' \wedge \cdots \wedge w_t'.$$

By Lemma 2.1, there exists a F-basis $\{\beta(h_1), \ldots, \beta(h_t)\}$ of $\beta(V)$ such that

$$in_{<_\pi}(\beta(h_1) \wedge \cdots \wedge \beta(h_t)) = in_{<_\pi}(\beta(h_1)) \wedge in_{<_\pi}(\beta(h_2)) \wedge \cdots \wedge in_{<_\pi}(\beta(h_t))$$

with $in_{<_\pi}(\beta(h_i)) = w_i'$ for $i = 1, \ldots, t$.

Then, we get

$$U' = \{\beta \in GL_n(F) : in_{<_\pi}(\beta(h_1) \wedge \cdots \wedge \beta(h_t)) = w_1' \wedge w_2' \wedge \cdots \wedge w_t'\}$$

is again a nonempty Zariski open subset of $GL_n(F)$ by [4, 4.1.5].

Let $A = (A_{ij}) \in GL_n(F)$ be the matrix acting on S via π, i.e. $A_{ij} = 1$ if $i = \pi(j)$ while $A_{ij} = 0$ for $i \neq \pi(j)$.

Note that A acts on $\wedge^t S_d$ as

$$A(f_1 \wedge \cdots \wedge f_t) = A(f_1) \wedge \cdots \wedge A(f_t).$$

Since ranking of the monomials in $<$ is preserved in $<_\pi$ after applying π, we obtain the following result.

Lemma 2.2 *For* $f, g \in \wedge^t S_d$, *we have*

1. $in_<(f) < in_<(g)$ *if and only if* $in_{<_\pi}(A(f)) <_\pi in_{<_\pi}(A(g))$.
2. $A(in_<(f)) = in_{<_\pi}(A(f))$.

Since A is the matrix of π, it obviously follows that.

Remark 2.3 *For* $f, g \in \wedge^t S_d$, *we get*

1. $in_<(f) < in_<(g)$ *if and only if* $in_{<_\pi}(\pi(f)) <_\pi in_{<_\pi}(\pi(g))$.
2. $\pi(in_<(f)) = in_{<_\pi}(\pi(f))$.

Theorem 2.4 $w_1' \wedge w_2' \wedge \cdots \wedge w_t' = A(w_1) \wedge \cdots \wedge A(w_t)$.

Proof Observe that

$$\begin{aligned} in_<(\alpha(f_1) \wedge \cdots \wedge \alpha(f_t)) =& in_<(\alpha(g_1) \wedge \cdots \wedge \alpha(g_t)) \\ =& in_<(\alpha(h_1) \wedge \cdots \wedge \alpha(h_t)) \\ =& w_1 \wedge w_2 \wedge \cdots \wedge w_t \end{aligned}$$

and

$$\begin{aligned} in_{<_\pi}(\beta(f_1) \wedge \cdots \wedge \beta(f_t)) =& in_{<_\pi}(\beta(g_1) \wedge \cdots \wedge \beta(g_t)) \\ =& in_{<_\pi}(\beta(h_1) \wedge \cdots \wedge \beta(h_t)) \\ =& w_1^{'} \wedge w_2^{'} \wedge \cdots \wedge w_t^{'}. \end{aligned}$$

Let $\alpha \in U$. By the previous observation and Lemma 2.2, we get

$$\begin{aligned} A(w_1) \wedge \cdots \wedge A(w_t) =& A(in_<(\alpha(g_1) \wedge \cdots \wedge \alpha(g_t))) \\ =& A(in_<(\alpha(h_1) \wedge \cdots \wedge \alpha(h_t))) \\ =& in_{<_\pi}(A(\alpha(h_1)) \wedge \cdots \wedge A(\alpha(h_t))). \end{aligned}$$

This implies $A(w_1) \wedge \cdots \wedge A(w_t) \leq w_1^{'} \wedge w_2^{'} \wedge \cdots \wedge w_t^{'}$ by the maximality in the definition of $w_1^{'} \wedge w_2^{'} \wedge \cdots \wedge w_t^{'}$.

Let $\beta \in U^{'}$. In the similar fashion, we have

$$\begin{aligned} w_1^{'} \wedge w_2^{'} \wedge \cdots \wedge w_t^{'} =& in_{<_\pi}(\beta(h_1) \wedge \cdots \wedge \beta(h_t)) \\ =& in_{<_\pi}(\beta(g_1) \wedge \cdots \wedge \beta(g_t)) \\ =& A(in_<(A^{-1}(\beta(g_1)) \wedge \cdots \wedge A^{-1}(\beta(g_t)))). \end{aligned}$$

By these equalities, we show

$$A^{-1}(w_1^{'} \wedge w_2^{'} \wedge \cdots \wedge w_t^{'}) = in_<(A^{-1}(\beta(g_1)) \wedge \cdots \wedge A^{-1}(\beta(g_t))).$$

Relying on the maximality condition in the definition of $w_1 \wedge w_2 \wedge \cdots \wedge w_t$, we reach $w_1^{'} \wedge w_2^{'} \wedge \cdots \wedge w_t^{'} \leq A(w_1) \wedge \cdots \wedge A(w_t)$ and this finishes the proof.

□

Corollary 2.5 $U^{'} = A(U) = \{A\alpha = A \circ \alpha : \alpha \in U\}$.

Proof Let $\alpha \in U$. By Theorem 2.4 and its proof, we obtain

$$w_1^{'} \wedge w_2^{'} \wedge \cdots \wedge w_t^{'} = in_{<_\pi}(A(\alpha(h_1)) \wedge \cdots \wedge A(\alpha(h_t))).$$

This proves $A\alpha \in U^{'}$. Similarly, if we take $\beta \in U^{'}$, then $A^{-1}\beta$ belongs to U by

$$w_1 \wedge w_2 \wedge \cdots \wedge w_t = in_<(A^{-1}(\beta(g_1)) \wedge \cdots \wedge A^{-1}(\beta(g_t))).$$

□

Theorem 2.6 *For a homogeneous ideal I, we get*

$$gin_{<\pi}(I) = \pi(gin_{<}(I)).$$

Proof Let $V, V^{'}$ be the Zariski open sets determining the generic initial ideals $gin_{<}(I)$ and $gin_{<\pi}(I)$, respectively. By the proof of [4, Theorem 4.1.2], we have that V is the intersection of U-type Zariski open sets.

Indeed, U as in [4, Lemma 4.1.5] is defined for $V \subset S_d$ and if we set $V = I_d$, the corresponding Zariski open set is denoted by U_d. Then, V is the intersection of some of these U_d's. Similarly, $V^{'}$ is the intersection of $U^{'}$-type Zariski open sets. Thus, we can say $V^{'} = A(V)$ by the previous corollary.

For $\alpha \in V$, we need to show

$$in_{<_\pi}(A\alpha I) = A(in_{<}(\alpha I)).$$

By Lemma 2.2, we know $in_{<_\pi}(A\alpha f) = A(in_{<}(\alpha f))$. Remembering that $A(f) = \pi(f)$, we get

$$in_{<_\pi}(\pi(\alpha f)) = \pi(in_{<}(\alpha f))$$

for any f in I and the assertion of the theorem follows.

□

Acknowledgement

The author is supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) under grant no. 119F181.

References

[1] Danış B, Sezer M. Generic initial ideals of modular polynomial invariants. Journal of Pure and Applied Algebra 2020; 224 (6). doi: 10.1016/j.jpaa.2019.106255

[2] Eisenbud D. Commutative Algebra with a View Toward Algebraic Geometry. New York, NY, USA: Springer-Verlag, 1995.

[3] Green ML. Generic initial ideals. In: Elias J, Giral JM, Mió-Roig RM, Zarzuela S. Six Lectures on Commutative Algebra. Basel, Switzerland: Birkhäuser, 1998, pp. 119-186.

[4] Herzog J, Hibi T. Monomial Ideals. London, England: Springer-Verlag London Limited, 2011.