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Banach algebra structure on strongly simple extensions

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Abstract: We consider strongly simple extensions of unitary commutative Banach algebras. We study these Banach algebra structure without assuming the continuity of the canonical injection. The link of the integrality with these extensions is studied. Several algebraic results are also obtained.

Key words: Banach algebra structure, simple extension, strongly simple extension, canonical injection, integral extension

1. Preliminaries and introduction

Let A and B be unital and commutative algebras. Then B will be called an extension of A if there exists an isomorphism of A into B that carries the identity of A into the identity of B. When convenient, we simply view A as a subalgebra of B that contains the identity of B. An extension of A is said to be strongly simple if it is the quotient of the algebra of polynomials $A[X]$ by a principal ideal. Let B be an extension of A. An element x of B is said to be an integral on A if it is the root of a unitary polynomial of $A[X]$. We say that the extension B is integral over A if every element of B is integral over A. The Jacobson radical of A denoted by $Rad(A)$ is the intersection of the maximal ideals of A. An algebra A is said to be semisimple if $Rad(A) = \{0\}$. The spectrum of an element x of A, denoted by $Sp_A(x)$, is defined by:

$$Sp_A(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ is not invertible}\}. \tag{1.1}$$

Let A be a commutative normed algebra. An element x of A is called a topological divisor of zero or topological zero divisor if there exists a sequence $(x_n)_n$ in A which does not tend to 0 such that $x_n x \longrightarrow 0$ as $n \longrightarrow \infty$. Let A be a unital and commutative algebra, a monic polynomial $\alpha(X) \in A[X]$ and $B = A[X]/\alpha(X)A[X]$. In [1], Arens and Hoffman showed that if A is normed (resp. Banach), then B is likewise.
In [3], the author considers extensions B of A in the topological sense, i.e. a Banach algebra topology on B which makes continuous the canonical injection $A \longrightarrow B$. He exactly considered the general case $B = A[X]/\alpha(X)A[X]$, where $\alpha(X)$ is not necessarily unitary and more generally in the case $B = A[X]/J$, where J is an arbitrary ideal of $A[X]$ with $J \cap A = \{0\}$ i.e. any simple extension. All the results obtained suppose the continuity of the canonical injection $A \longrightarrow B$.
In this paper, we consider strongly simple extension $B = A[X]/\alpha(X)A[X]$ of A, where $\alpha(X) = \sum_{0 \leq i \leq r} \alpha_i X^i$ with $\alpha_r \neq 0$, is not necessarily unitary and this without assuming the continuity of the canonical injection $A \longrightarrow B$.

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To do this, we first deal with purely algebraic situations which will provide support for the topological case. Without topological condition on the algebra A, we obtain a result (Proposition 2.1) which gives us algebraic information on the extension. Namely for a unital commutative algebra A, if $Sp_{A[\theta]}(\theta) \neq \mathbb{C}$ where $\theta = \overline{X}$ (that is the case where $A[\theta]$ admits a Banach algebra structure), then $A = A\alpha_0 + ... + A\alpha_r$ and the polynomial $\alpha(X)$ is not an algebraic divisor of zero in $A[X]$. In addition, if we assume that the elements $\alpha_0, ..., \alpha_{r-1} \in Rad(A)$, then α_r is invertible (Corollary 2.2). If $\alpha(X) \in A[X]$, with $A\alpha_1 + ... + A\alpha_r = A$, and $B = A[X]/\alpha(X)A[X]$ such that $\alpha(X)A[X] \cap A = \{0\}$, then every character of A extends to B (Proposition 2.3).
In the topological part, we consider a unital commutative algebra A in which every maximal ideal is of codimension 1 such that $B = A[X]/\alpha(X)A[X]$, with $A\alpha_1 + ... + A\alpha_r = A$, is provided with a Banach algebra norm. If

$$\alpha(X) = \alpha_r(X - \theta_1)...(X - \theta_r) \text{ where } \alpha_r \neq 0 \text{ and } \theta_1, ..., \theta_r \in B, \tag{1.2}$$

we show (Theorem 3.4) that α_r is invertible in A and so B is integral over A. We then prove (Theorem 3.5) that if there exists a Banach algebra structure on the algebra B, with $A\alpha_1 + ... + A\alpha_r = A$, then the coefficient $\alpha_r \neq 0$ is invertible in A if and only if it is not a topological divisor of zero in A. A presence of integrality is established in Theorem 3.9. Using a result of automatic continuity, we show (Theorem 3.10) that if there exists a structure of Banach algebra on B, with $A\alpha_1 + ... + A\alpha_r = A$, then $B = A + A\theta + ... + A\theta^n + Rad(A[\theta]))$ for some integer $n \geq 1$. Finally, in the particular case where the algebra A is semisimple, we prove (Theorem 3.12) the equivalence between the continuity of the canonical injection $A \longrightarrow B$, the integrality B over A and the closure of A in B.
Throughout the sequel, the algebras considered are complex and associative.

2. Algebraic considerations

This algebraic part is of paramount importance for Section 3 which is devoted to the Banach case. Without topological condition on algebra A, the following result gives a very useful algebraic information on the extension.

Proposition 2.1 *Let A be a unital and commutative algebra (not necessarily topological), $\alpha(X) = \sum_{0 \leq i \leq r} \alpha_i X^i \in A[X]$ with $\alpha_r \neq 0$ and*

$$B = A[X]/\alpha(X)A[X] = A[\theta] \tag{2.1}$$

where $\theta = \overline{X}$ is the class of X modulo $\alpha(X)A[X]$. Suppose that $Sp_B(\theta) \neq \mathbb{C}$ (that is the case where B admits a Banach algebra structure). Then:

i) $A\alpha_0 + ... + A\alpha_r = A$,

ii) The polynomial $\alpha(X)$ is not an algebraic divisor of zero in $A[X]$.

iii) If moreover B is an extension of A, then

$$ann(\alpha_1) \cap ... \cap ann(\alpha_r) = \{0\}, \tag{2.2}$$

where

$$ann(x) = \{a \in A : ax = 0\} \tag{2.3}$$

Proof i) Let $\lambda \in \mathbb{C}\backslash Sp_B(\theta)$. Then $\theta - \lambda e$ is invertible in B. So

$$(X-\lambda e)A[X]+\alpha(X)A[X]=A[X]. \tag{2.4}$$

Consider $u(X)$ and $v(X)$ in $A[X]$ such that:

$$(X-\lambda e)u(X)+\alpha(X)v(X)=e. \tag{2.5}$$

It follows that $\alpha(\lambda)v(\lambda)=e$. So

$$\alpha(\lambda)=\sum_{0\leq i\leq r}\lambda^i\alpha_i \tag{2.6}$$

is invertible in A. Hence the result.
ii) If $\alpha(X)$ is an algebraic divisor of zero in $A[X]$, then there exists $c \in A\backslash\{0\}$ such that $c\alpha(X)=0$ which cannot be the case by the assertion i).
iii) Let $a \in ann(\alpha_1)\cap ... \cap ann(\alpha_r)$. Then

$$a\alpha(X)=a\alpha_0 \in \alpha(X)A[X]\cap A=\{0\}. \tag{2.7}$$

Whence $a=0$ via the assertion i). □

As a consequence, one has:

Corollary 2.2 *With the hypotheses of Proposition 2.1 and if we assume that $\alpha_0, ..., \alpha_{r-1} \in Rad(A)$, then α_r is invertible.*

The second algebraic result concerns the extension of characters.

Proposition 2.3 *Let $\alpha(X) \in A[X]$ and $B = A[X]/\alpha(X)A[X]$, with $A\alpha_1 + ... + A\alpha_r = A$, such that $\alpha(X)A[X]\cap A = \{0\}$ and let χ be a character of A. Then χ extends into a character χ' of B.*

Proof Observe first that a character of A extends to a map $A[X] \longrightarrow \mathbb{C}[X]$. As $A\alpha_1 + ... + A\alpha_r = A$, the polynomial

$$\sum_{0\leq i\leq r}\chi(\alpha_i)X^i \in \mathbb{C}[X] \tag{2.8}$$

is of degree ≥ 1 and consequently it admits a root λ. The map $A[X] \longrightarrow \mathbb{C}$ sending X to λ vanishes on α and so induces a map on B. The last map on B is indeed a character of B extending χ.

□

Note that in connection with proposition 2.3, if B is an integral extension over A, then the prime (resp. maximal) ideals of A are the traces on A of the prime (resp. maximal) ideals of B ([4], Théorème I.9, p.142). In particular any character of A is extended into a character of B. Indeed, let χ be a character of A. According to ([4], Theorem I.9, p.142), there exists a maximal ideal M of B such that $M\cap A = \ker\chi$. It follows that the algebra B/M is an extension of $A/\ker\chi$. So B/M is an algebraic field extension of $\mathbb{C}$. As $\mathbb{C}$ is algebraically closed field, one has B/M is isomorphic to $\mathbb{C}$. So M is the kernel of a character of B. Moreover, this character extends χ.

3. Banach structure on strongly simple extensions and integrality

We start first with the following result which is in fact an application of the algebraic part.

Proposition 3.1 *Let A be a unital and commutative algebra and*

$$B = A[X]/(aX^n - b)A[X], \ (n \geq 1) \tag{3.1}$$

where $a, b \in A$ with

$$ann(a) \subset ann(b) \tag{3.2}$$

(condition which ensures equality $(aX^n - b)A[X] \cap A = \{0\}$). Suppose that there exists a Banach algebra structure on B. Then a is invertible in B.

Proof Let χ' be character of B such that $\chi'(a) = 0$. Then $\chi'(b) = \chi'(a\theta) = 0$. Thus $\chi'(a) = \chi'(b) = 0$ which is impossible since $Aa + Ab = A$ by i) of proposition 2.1. It follows that a is invertible in B. □

Notice that the existence of a normed algebra structure on $B = A[X]/(aX^n - b)A[X]$, even with continuous injection, does not necessarily entail the invertibility of a as shown by the following example:

Example 3.2 *Let A be a unital and commutative Banach algebra and a noninvertible element $a \in A$ such that the ideal aA is closed. We define an algebra norm on $A[X]$ by:*

$$\left\| \sum_{0 \leq i \leq n} a_i X^i \right\| = \sum_{0 \leq i \leq n} \|a_i\| . \tag{3.3}$$

One easily verify that $aXA[X]$ is a closed ideal of $A[X]$. So

$$A[X]/aXA[X] \tag{3.4}$$

is a normed algebra and the canonical injection:

$$A \longrightarrow A[X]/aXA[X] \tag{3.5}$$

is continuous ($A[X]/aXA[X]$ is indeed an extension of A). According to proposition 3.1, this extension cannot be provided with any Banach algebra structure.

In what follows, we suppose that $A\alpha_1 + ... + A\alpha_r = A$.

Remark 3.3 *The polynomial considered in Proposition 3.1 decomposes into linear factors in $A[\theta]$. Indeed let $(\omega_k)_{1 \leq k \leq n}$ be the nth roots of unity. Then $(\omega_k \theta)_{1 \leq k \leq n}$ are roots of $aX^n - b$. Denote by $(\sigma_k)_{1 \leq k \leq n}$ the elementary symmetric polynomials in n roots $\omega_1, ..., \omega_n$. Then one has $\sigma_k = 0$, for every $k = 1, ..., n-1$. Let $(\sigma'_k)_{1 \leq k \leq n}$ be the elementary symmetric polynomials in nth roots $\omega_1 \theta, ..., \omega_n \theta$ of the polynomial $\prod_{k=1}^{n} (X - \omega_k \theta)$. Then $\sigma'_k = 0$, for every $k = 1, ..., n-1$. Now as $\sigma'_k = \theta^k \sigma_k$, one has $\sigma'_k = 0$, for every $k = 1, ..., n-1$. Whence the result.*

We establish that the property given in Remark 3.3 implies the integrality of the extension as shown by the following result:

Theorem 3.4 *Let A be a unital and commutative algebra in which every maximal ideal is of codimension* 1,

$$\alpha(X) = \sum_{0 \leq i \leq r} \alpha_i X^i \in A[X] \text{ with } \alpha_r \neq 0 \tag{3.6}$$

and

$$B = A[X]/\alpha(X)A[X]. \tag{3.7}$$

Suppose that B is an extension of A provided with a Banach algebra norm such that

$$\alpha(X) = \alpha_r(X - \theta_1)...(X - \theta_r) \tag{3.8}$$

where $\theta_1, ..., \theta_r \in B$. Then α_r is invertible in A (so B is integral over A).

Proof One has

$$\alpha_{r-k} = (-1)^k \alpha_r s_k, \ 1 \leq k \leq r \tag{3.9}$$

where $s_1, ..., s_r$ are the elementary symmetric polynomials in $\theta_1, ..., \theta_r$. Consider χ a character of A. By Proposition 2.3, there exists a character χ' of B extending χ. Suppose that $\chi(\alpha_r) = 0$, then $\chi'(\alpha_r) = 0$ and consequently $\chi'(\alpha_{r-k}) = 0$ for all $k = 1, ..., r$. Finally $\chi(\alpha_k) = 0$ for every $k = 0, ..., r$. This cannot be the case since $A\alpha_1 + ... + A\alpha_r = A$. It follows that α_r is invertible in A. □

The following result gives a necessary and sufficient condition for a strongly simple extension to be integral.

Theorem 3.5 *Let A be a unital and commutative algebra in which every maximal ideal is of codimension* 1*. We assume that there exists a Banach algebra structure on the algebra*

$$A[\theta] = A[X]/\alpha(X)A[X] \tag{3.10}$$

where $\alpha(X) = \sum_{0 \leq i \leq r} \alpha_i X^i$ with $\alpha_r \neq 0$ and $\alpha(X)A[X] \cap A = \{0\}$. Then the following assertions are equivalent:

i) The coefficient α_r is invertible in A (i.e. $A[\theta]$ is integral over A).

ii) The coefficient α_r is not a topological divisor of zero in A (A being provided with the norm induced by that of $A[\theta]$).

For the proof, we will need the following lemma whose proof is analogous to that of ([2]), Proposition 4, p.18).

Lemma 3.6 *With the assumptions of Theorem 3.5. If $(a_n)_n$ is a sequence of invertible elements of A which converges to α_r, then α_r is invertible or a topological divisor of zero in A.*

Proof By Proposition 2.3, $G(A) = G(A[\theta]) \cap A$ where $G(A)$ (resp. $G(A[\theta])$) is the group of invertible elements of A (resp. $A[\theta]$). It follows that $G(A)$ is an open set of A. Suppose now that α_r is not invertible. As $a_n - \alpha_r = a_n(e - a_n^{-1}\alpha_r)$ and α_r is not invertible, the sequence $(e - a_n^{-1}\alpha_r)_n$ cannot tend to 0, and the lemma follows. □

Proof of theorem 3.5 It is clear that i)$\Longrightarrow$ii). Let us show the implication ii) $\Longrightarrow$i) by contraposition. Suppose that α_r is not invertible in A. Let $(\lambda_n)_n$ be a sequence of elements of $\mathbb{C}\backslash Sp_{A[\theta]}(\theta)$ which tends to infinity. Then, for every n, $\theta - \lambda_n e$ is invertible in $A[\theta]$. One has

$$(X - \lambda_n)A[X] + \alpha(X)A[X] = A[X]. \tag{3.11}$$

Consider $u(X)$ and $v(X)$ in $A[X]$ such that:

$$(X - \lambda_n)u(X) + \alpha(X)v(X) = e. \tag{3.12}$$

We see that $\sum\limits_{0\leq i\leq r} \lambda_n^i \alpha_i$ is invertible in A. Furthermore

$$(1/\lambda_n^r)\left(\sum_{0\leq i\leq r} \lambda_n^i \alpha_i\right) \underset{n\longrightarrow+\infty}{\longrightarrow} \alpha_r \qquad (*) \tag{3.13}$$

and $(1/\lambda_n^r)\left(\sum\limits_{0\leq i\leq r} \lambda_n^i \alpha_i\right)$ is invertible in A. And we conclude by Lemma 3.6. □

Remark 3.7 *The nature of the sequence considered in* $(*)$ *allows to affirm that if the coefficient* α_r *is not invertible in* A*, then it is a topological divisor of zero in* $(A, \|.\|)$ *for any Banach algebra norm* $\|.\|$ *on* A*.*

Example 3.8 *Let* $A = A(D)$ *be the Banach algebra of the disk,* $f \in A$ *where* $f(z) = z$ *and* $\alpha(X) \in A[X]$ *with dominant coefficient* f*. Since the ideal* Af *is closed and* A *is an integral domain, then* f *is not a topological divisor of zero. It follows from the above remark that the extension* $A[X]/\alpha(X)A[X]$ *of* A *cannot be endowed with any Banach algebra norm.*

With the continuity of the canonical injection, a simple extension provided with a Banach algebra structure is integral ([3]). Without this assumption of continuity, one has the presence of the following integrality.

Theorem 3.9 *Let* A *be a unital and commutative Banach algebra and*

$$A[\theta] = A[X]/\alpha(X)A[X] \tag{3.14}$$

be a strongly simple extension of A*. Suppose that there exists a Banach algebra norm on the algebra* $A[\theta]$*. Then for every* $\lambda \in \mathbb{C}\backslash Sp_B(\theta)$*, the element* $(\theta - \lambda e)^{-1}$ *is integral over* A*.*

Proof Let $a_0, ..., a_n \in A$ such that $(\theta - \lambda e)^{-1} = \sum\limits_{0\leq i\leq n} a_i \theta^i$. Then we have:

$$(\theta - \lambda e)^{-1} = \sum_{0\leq i\leq n} a_i((\theta - \lambda e) + \lambda e)^i. \tag{3.15}$$

Using Newton's binomial formula, we see that

$$(\theta - \lambda e)^{-1} = \sum_{0\leq j\leq n}\left(\sum_{j\leq i\leq n} \lambda^{i-j} C_i^j a_i (\theta - \lambda e)^j\right) \tag{3.16}$$

Finally the multiplication of the two members of this equality by $(\theta-\lambda e)^{-n+1}$ entails the integrality of $(\theta-\lambda e)^{-1}$ on A. □

Exploiting a result of automatic continuity, we obtain the following result involving the Jacobson radical of the algebra $A[\theta]$.

Theorem 3.10 *Let A be a unital and commutative Banach algebra and*

$$A[\theta] = A[X]/\alpha(X)A[X] \tag{3.17}$$

be a strongly simple extension of A. Suppose that there exists a Banach algebra norm on the algebra $A[\theta]$. Then there exists $n \geq 1$ such that:

$$A[\theta] = A + A\theta + ... + A\theta^{n-1} + Rad(A[\theta]) \tag{3.18}$$

Proof According to Proposition 2.3, one has $Rad(A) = Rad(A[\theta]) \cap A$. It follows that the algebra $A[\theta]/Rad(A[\theta])$ is an extension of the algebra $A/Rad(A)$. Furthermore, it is simple since

$$A[\theta]/Rad(A[\theta]) = (A/Rad(A))[\widetilde{\theta}], \tag{3.19}$$

where $\widetilde{\theta}$ is the class of θ modulo $Rad(A[\theta])$. Moreover, since $A[\theta]/Rad(A[\theta])$ is semisimple, the canonical injection $A/Rad(A) \longrightarrow A[\theta]/Rad(A[\theta])$ is continuous (cf. [2], Proposition 6, p.27). By ([3]), $A[\theta]/Rad(A[\theta])$ is integral over $A/Rad(A)$. There exists then $n \geq 1$ such that:

$$A[\theta]/Rad(A[\theta]) = (A/Rad(A)) + (A/Rad(A))\widetilde{\theta} + ... + (A/Rad(A))\widetilde{\theta^n} \tag{3.20}$$

Whence the result. □

As a consequence, we have:

Corollary 3.11 *With the hypothesis of Theorem 3.10, suppose, in addition, that the A-module $Rad(A[\theta])$ is of finite type. Then $A[\theta]$ is integral over A (i.e. the polynomial $\alpha(X)$ is unitary).*

Proof By theorem 3.10, $A[\theta]$ in an A-module of finite type. And we conclude by ([4], theorem I.3, p.135 and corollary 4, p.137).

□

We end this work with this result in the important particular case where the algebra A is semisimple.

Theorem 3.12 *Let A be a unital and commutative semisimple Banach algebra, $\alpha(X) \in A[X]$ and*

$$B = A[X]/\alpha(X)A[X]. \tag{3.21}$$

Suppose that there exists a Banach algebra norm on B. The following conditions are equivalent:

i) The canonical injection $A \longrightarrow B$ is continuous,

ii) B is integral over A,

iii) A is closed in B.

Proof i) $\Longrightarrow$ ii) follows from ([3]),

ii) $\Longrightarrow$ iii) follows from ([5], theorem 1, p.517),

iii) $\Longrightarrow$ i) As the algebra A is semisimple, the norm of A is equivalent with that induced by the norm of algebra B ([**2**], Corollary, p.27). Whence the result. □

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