Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 1300 – 1309
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doi:10.3906/mat-2102-11

A class of operators related to m-symmetric operators

Fei ZUO[1,*], **Salah MECHERI**[2]

[1]College of Mathematics and Information Science, Henan Normal University, Xinxiang, China

[2]Department of Mathematics, Tebessa University, Tebessa, Algeria

Received: 04.02.2021 • **Accepted/Published Online:** 01.04.2021 • **Final Version:** 20.05.2021

Abstract: m-symmetric operator plays a crucial role in the development of operator theory and has been widely studied due to unexpected intimate connections with differential equations, particularly conjugate point theory and disconjugacy. For positive integers m and k, an operator T is said to be a k-quasi-m-symmetric operator if $T^{*k}(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^j)T^k = 0$, which is a generalization of m-symmetric operator. In this paper, some basic structural properties of k-quasi-m-symmetric operators are established with the help of operator matrix representation. In particular, we also show that every k-quasi-3-symmetric operator has a scalar extension. Finally, we prove that generalized Weyl's theorem holds for k-quasi-3-symmetric operators.

Key words: K-quasi-m-symmetric operator, subscalarity, hyperinvariant subspace, Weyl's theorem

1. Introduction

Let $B(H)$ denote the algebra of all bounded linear operators on the complex separable Hilbert space H. An operator $T \in B(H)$ is called 3-symmetric if

$$T^{*3} - 3T^{*2}T + 3T^*T^2 - T^3 = 0,$$

where T^* is the adjoint operator of T. Helton in [13] introduced 3-symmetric operators as a generalization of selfadjoint operators. In a series of papers [12–14], he modelled these operators as multiplication t on a Sobolev space, established their connections to Sturm–Liouville operators, and showed, under some additional hypotheses, that they are the restriction to an invariant subspace of a Jordan operator of order two. Later in [1] Agler illustrated the connection between the above result and the classical disconjugacy theory by example. In [12] Helton initiated the study of m-symmetric operator, for a positive integer m, an operator $T \in B(H)$ is said to be m-symmetric if

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^j = 0.$$

Hence T is 1-symmetric if and only if T is selfadjoint. It is well known that if T is m-symmetric, then T is n-symmetric for all $n \geq m$. The notion of m-symmetric operator can be generalized in a natural way to k-quasi-m-symmetric operator as follows.

*Correspondence: zuofei2008@sina.com

2010 *AMS Mathematics Subject Classification:* 47B20; 47A10; 47A53

Definition 1.1 *For positive integers m and k, an operator $T \in B(H)$ is called k-quasi-m-symmetric if*

$$T^{*k}(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^j)T^k = 0.$$

In particular, for $m = 3$, an operator T is said to be k-quasi-3-symmetric if

$$T^{*k}(T^{*3} - 3T^{*2}T + 3T^*T^2 - T^3)T^k = 0.$$

Example 1.2 *Let* $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \in B(\mathbb{C}^3)$. *A simple calculation shows that T is a k-quasi-3-symmetric operator, but T is not a 3-symmetric operator.*

A bounded linear operator T on H is called scalar of order m if it possesses a spectral distribution of order m, i.e. if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbb{C}) \to B(H)$$

such that $\Phi(z) = T$, where z stands for the identity function on $\mathbb{C}$, and $C_0^m(\mathbb{C})$ stands for the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order m, $0 \le m \le \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to an invariant subspace. In 1984, Putinar [21] proved that every hyponormal operator has a scalar extension, which has been extended from hyponormal operators to analytic extensions of M-hyponormal operators [17]. In this paper, we study various properties of a k-quasi-3-symmetric operator. We show that every k-quasi-3-symmetric operator is subscalar. As an application, we prove that if T is a k-quasi-3-symmetric operator, then Weyl's theorem holds for $f(T)$ where f is an analytic function on an open neighborhood of $\sigma(T)$ and f is not constant on each connected component of the open set U containing $\sigma(T)$.

2. Preliminaries

Throughout this paper, the closure of a set M will be denoted by $\overline{M}$. If $T \in B(H)$, we shall write $N(T)$, $R(T)$, $\sigma(T)$ and iso$\sigma(T)$ for the null space, the range space, the spectrum and the isolated spectrum point of T, respectively.

An operator $T \in B(H)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$, abbreviated SVEP at λ_0, if for every open neighborhood G of λ_0, the only analytic function $f : G \to H$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in G$ is the function $f \equiv 0$. An operator T is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. For a Banach space $\mathcal{X}$, let $\xi(U, \mathcal{X})$ (resp., $\mathcal{O}(U, \mathcal{X})$) denote the Fréchet space of all infinite differentiable $\mathcal{X}$-value functions on U (resp., of all analytic $\mathcal{X}$-value functions on U). An operator $T \in B(\mathcal{X})$ is said to have property $(\beta)_\varepsilon$ at $\lambda \in \mathbb{C}$ if there exists a neighbourhood D of λ such that for every open subset U of D and $\mathcal{X}$-value functions sequence $\{f_n\}$ in $\xi(U, \mathcal{X})$, $(T - zI)f_n(z) \to 0$ in $\xi(U, \mathcal{X}) \Rightarrow f_n(z) \to 0$ in $\xi(U, \mathcal{X})$, and $T \in B(\mathcal{X})$ is said to have property (β) at $\lambda \in \mathbb{C}$ if there exists an $r > 0$ such that for every subset U of the open disc $D(\lambda; r)$ of radius r centered at λ and sequence $\{f_n\}$ of $\mathcal{X}$-value functions in $\mathcal{O}(U, \mathcal{X})$, $(T - zI)f_n(z) \to 0$ in $\mathcal{O}(U, \mathcal{X}) \Rightarrow f_n(z) \to 0$ in $\mathcal{O}(U, \mathcal{X})$. An operator $T \in B(H)$ is said to have property $(\beta)_\varepsilon$

(resp., (β)) if T has property $(\beta)_\varepsilon$ (resp., (β)) at every point $\lambda \in \mathbb{C}$. For $T \in B(H)$ and $x \in H$, the set $\rho_T(x)$ is defined to consist of elements $z_0 \in \mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in H, which verifies $(T - zI)f(z) = x$, and it is called the local resolvent set of T at x. We denote the complement of $\rho_T(x)$ by $\sigma_T(x)$, called the local spectrum of T at x, and define the local spectral subspace of T, $H_T(F) = \{x \in H : \sigma_T(x) \subset F\}$ for each subset F of $\mathbb{C}$. An operator $T \in B(H)$ is said to have Dunford's property (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. It is well known that

$$\text{property } (\beta)_\varepsilon \Rightarrow \text{property } (\beta) \Rightarrow \text{ Dunford's property } (C) \Rightarrow \text{ SVEP.}$$

An operator $T \in B(H)$ is called Fredholm if it has closed range with finite dimension null space and its range of finite codimension. Let $\alpha(T) := \dim N(T)$, $\beta(T) := \dim N(T^*)$. The index of a Fredholm operator $T \in B(H)$ is given by $i(T) = \alpha(T) - \beta(T)$. An operator $T \in B(H)$ is called Weyl if it is Fredholm of index zero. The Weyl spectrum $w(T)$ of T is defined by $w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$. Following [11], we say that Weyl's theorem holds for T if $\sigma(T) \setminus w(T) = \pi_{00}(T)$, where $\pi_{00}(T) := \{\lambda \in \text{iso}\sigma(T) : 0 < \dim N(T - \lambda) < \infty\}$. More generally, Berkani investigated B-Fredholm theory (see [6–8]). An operator T is called B-Fredholm if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and the induced operator $T_{[n]} : x \in R(T^n) \to Tx \in R(T^n)$ is Fredholm, i.e. $R(T_{[n]}) = R(T^{n+1})$ is closed, $\alpha(T_{[n]}) < \infty$ and $\beta(T_{[n]}) = \dim R(T^n)/R(T_{[n]}) < \infty$. Similarly, a B-Fredholm operator T is called B-Weyl if $i(T_{[n]}) = 0$. The B-Weyl spectrum $\sigma_{BW}(T)$ is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl}\}$. We say that generalized Weyl's theorem holds for T if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$, where $E(T)$ denotes the set of all isolated points of the spectrum which are eigenvalues (no restriction on multiplicity). Note that, if generalized Weyl's theorem holds for T, then so does Weyl's theorem [7].

3. Main results

We begin with the following theorem which is a structural theorem for k-quasi-m-symmetric operators.

Theorem 3.1 *Suppose that $R(T^k)$ is not dense. Then the following statements are equivalent:*
(1) *T is a k-quasi-m-symmetric operator;*
(2) *$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{R(T^k)} \oplus N(T^{*k})$, where T_1 is an m-symmetric operator and $T_3^k = 0$.*
Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof (1) $\Rightarrow$ (2) Consider the matrix representation of T with respect to the decomposition $H = \overline{R(T^k)} \oplus N(T^{*k})$:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$

Let P be the projection onto $\overline{R(T^k)}$. Since T is a k-quasi-m-symmetric operator, we have

$$P(\sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*m-j}T^j)P = 0.$$

Therefore

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{*m-j}T_1^j=0.$$

On the other hand, for any $x=(x_1,x_2)\in H,$ we have

$$(T_3^k x_2,x_2)=(T^k(I-P)x,(I-P)x)=((I-P)x,T^{*k}(I-P)x)=0,$$

which implies $T_3^k=0$. Since $\sigma(T_1)\cap\{0\}$ has no interior point, by [10, Corollary 7] $\sigma(T)=\sigma(T_1)\cup\{0\}$.

(2) $\Rightarrow$ (1) Suppose that $T=\begin{pmatrix} T_1 & T_2\\ 0 & T_3\end{pmatrix}$ on $H=\overline{R(T^k)}\oplus N(T^{*k})$, where T_1 is an m-symmetric operator and $T_3^k=0$. We have

$$T^k=\begin{pmatrix} T_1^k & \sum\limits_{j=0}^{k-1}T_1^jT_2T_3^{k-1-j}\\ 0 & 0\end{pmatrix}.$$

Since

$$\begin{aligned}
&T^{*k}\Big(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^j\Big)T^k\\
=&\begin{pmatrix} T_1^{*k} & 0\\ \big(\sum\limits_{j=0}^{k-1}T_1^jT_2T_3^{k-1-j}\big)^* & 0\end{pmatrix}\begin{pmatrix} \sum\limits_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{*m-j}T_1^j & *\\ * & *\end{pmatrix}\begin{pmatrix} T_1^k & \sum\limits_{j=0}^{k-1}T_1^jT_2T_3^{k-1-j}\\ 0 & 0\end{pmatrix}\\
=&\begin{pmatrix} T_1^{*k}\big(\sum\limits_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{*m-j}T_1^j\big)T_1^k & T_1^{*k}\big(\sum\limits_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{*m-j}T_1^j\big)\sum\limits_{j=0}^{k-1}T_1^jT_2T_3^{k-1-j}\\ \big(\sum\limits_{j=0}^{k-1}T_1^jT_2T_3^{k-1-j}\big)^*\big(\sum\limits_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{*m-j}T_1^j\big)T_1^k & \big(\sum\limits_{j=0}^{k-1}T_1^jT_2T_3^{k-1-j}\big)^*\big(\sum\limits_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{*m-j}T_1^j\big)\sum\limits_{j=0}^{k-1}T_1^jT_2T_3^{k-1-j}\end{pmatrix}\\
=&0,
\end{aligned}$$

for some nonspecified entries $*$. Hence T is a k-quasi-m-symmetric operator. $\square$

Corollary 3.2 *Suppose that T is a k-quasi-m-symmetric operator and $R(T^k)$ is dense. Then T is an m-symmetric operator.*

Proof This is a result of Theorem 3.1. $\square$

Corollary 3.3 *Suppose that T is a k-quasi-m-symmetric operator. Then T^n is also a k-quasi-m-symmetric operator for any $n\in\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.*

Proof If T^k has a dense range, then T is an m-symmetric operator, and so is T^n for any $n\in\mathbb{N}$ by [9, Theorem 2.4]. If T^k does not have a dense range, we decompose T as $T=\begin{pmatrix} T_1 & T_2\\ 0 & T_3\end{pmatrix}$ on $H=\overline{R(T^k)}\oplus N(T^{*k})$, where T_1 is an m-symmetric operator, and so is T_1^n. Since

$$T^n=\begin{pmatrix} T_1^n & \sum\limits_{j=0}^{n-1}T_1^jT_2T_3^{n-1-j}\\ 0 & T_3^n\end{pmatrix}\ \text{on}\ H=\overline{R(T^k)}\oplus N(T^{*k}),$$

it follows from Theorem 3.1 that T^n is a k-quasi-m-symmetric operator for any $n \in \mathbb{N}$. □

Remark The converse of Corollary 3.3 is not true in general as shown in the following example.

Example 3.4 *Let* $T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in B(\mathbb{C}^4)$. *A simple calculation shows that* $T^{*2}(T^{*6} - 3T^{*4}T^2 + 3T^{*2}T^4 - T^6)T^2 = 0$ *and* $T^*(T^{*3} - 3T^{*2}T + 3T^*T^2 - T^3)T \neq 0$. *So, we obtain that* T^2 *is a quasi-*3*-symmetric operator, but* T *is not a quasi-*3*-symmetric operator.*

Corollary 3.5 *Suppose that* T *is an invertible* k*-quasi-*m*-symmetric operator. Then* T^{-1} *is a* k*-quasi-*m*-symmetric operator.*

Proof Suppose that T is an invertible k-quasi-m-symmetric operator. Then T is an m-symmetric operator, and so is T^{-1}. Hence T^{-1} is a k-quasi-m-symmetric operator. □

Theorem 3.6 *Suppose that* T *is a* k*-quasi-*m*-symmetric operator and* M *is an invariant subspace for* T. *Then the restriction* $T|_M$ *is also a* k*-quasi-*m*-symmetric operator.*

Proof Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = M \oplus M^{\perp}$. Since T is a k-quasi-m-symmetric operator, i.e.

$$T^{*k}(\sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*m-j}T^j)T^k = 0,$$

we have

$$\begin{aligned}
& T^{*k}(\sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*m-j}T^j)T^k \\
= & \begin{pmatrix} T_1^{*k} & 0 \\ (\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j})^* & T_3^{*k} \end{pmatrix} \begin{pmatrix} \sum_{j=0}^{m}(-1)^j \binom{m}{j} T_1^{*m-j}T_1^j & \sharp \\ \sharp & \sharp \end{pmatrix} \begin{pmatrix} T_1^k & \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & T_3^k \end{pmatrix} \\
= & \begin{pmatrix} T_1^{*k}(\sum_{j=0}^{m}(-1)^j \binom{m}{j} T_1^{*m-j}T_1^j)T_1^k & \sharp \\ \sharp & \sharp \end{pmatrix} \\
= & 0,
\end{aligned}$$

for some nonspecified entries $\sharp$, which implies $T_1^{*k}(\sum_{j=0}^{m}(-1)^j \binom{m}{j} T_1^{*m-j}T_1^j)T_1^k = 0$. Hence $T|_M$ is a k-quasi-m-symmetric operator. □

Proposition 3.7 *Suppose that* $\{T_n\}$ *is a sequence of* k*-quasi-*m*-symmetric operators such that* $\lim_{n\to\infty} ||T_n - T|| = 0$. *Then* T *is a* k*-quasi-*m*-symmetric operator.*

Proof Suppose that $\{T_n\}$ is a sequence of k-quasi-m-symmetric operators such that $\lim_{n\to\infty}||T_n - T|| = 0$. Then

$$\begin{aligned}
&||T_n^{*k}(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T_n^{*m-j}T_n^j)T_n^k - T^{*k}(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^j)T^k||\\
\leq&||T_n^{*k}(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T_n^{*m-j}T_n^j)T_n^k - T_n^{*k}(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^j)T^k||\\
+&||T_n^{*k}(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^j)T^k - T^{*k}(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^j)T^k||\\
\leq&||T_n^{*k}||||\sum_{j=0}^{m}(-1)^j\binom{m}{j}T_n^{*m-j}T_n^{j+k} - \sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^{j+k}||\\
+&||T_n^{*k} - T^{*k}||||\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^{j+k}|| \to 0.
\end{aligned}$$

Since $\{T_n\}$ is a k-quasi-m-symmetric operator,

$$T_n^{*k}(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T_n^{*m-j}T_n^j)T_n^k = 0,$$

we have

$$T^{*k}(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^j)T^k = 0,$$

i.e. T is a k-quasi-m-symmetric operator. □

Now we are ready to prove that every k-quasi-3-symmetric operator has a scalar extension, we need the following lemmas.

Lemma 3.8 *Suppose that $T \in B(H)$ is a 3-symmetric operator. Then T is subscalar of order 4.*

Proof By [22] or [13], T is a 3-symmetric operator if and only if T is unitarily equivalent to

$$J|_M = \begin{pmatrix} V & E \\ 0 & V \end{pmatrix}|_M$$

for some selfadjoint operator V acting on some Hilbert space H and some operator E on H which commutes with V and some subspace M of $H \oplus H$ which is invariant for the block operator. Clearly, every selfadjoint operator is hyponormal and so J is a subscalar of order 4 by [16, Theorem 4.5]. Since T is unitarily equivalent to the restriction of J to an invariant subspace, it is subscalar of order 4. □

Lemma 3.9 ([19, Lemma 1]) *For $T \in B(\mathcal{X})$, the following statements are equivalent:*

(i) T is subscalar;

(ii) T has property $(\beta)_\varepsilon$.

Theorem 3.10 *Suppose that T is a k-quasi-3-symmetric operator. Then T is subscalar.*

Proof Assume that $R(T^k)$ is dense. Then T is a 3-symmetric operator, it is subscalar of order 4 by Lemma 3.8. So we may assume that T^k does not have dense range. Then by Theorem 3.1 the operator T can be decomposed as follows: $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{R(T^k)} \oplus N(T^{*k})$, where T_1 is a 3-symmetric operator and $T_3^k = 0$. Set $\sigma_{(\beta)_\varepsilon}(S) = \{\mu \in \sigma(S) : S \text{ does not satisfy property } (\beta)_\varepsilon \text{ at } \mu\}$. Since T_3 is nilpotent, $\sigma_{(\beta)_\varepsilon}(T_3) = \varphi$. Recall from [5, Theorem 2.1] that given operators S and R, $\lambda \in \sigma_{(\beta)_\varepsilon}(RS) \Leftrightarrow \lambda \in \sigma_{(\beta)_\varepsilon}(SR)$. Considering $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} I_1 & 0 \\ 0 & T_3 \end{pmatrix}\begin{pmatrix} I_1 & T_2 \\ 0 & I_2 \end{pmatrix}\begin{pmatrix} T_1 & 0 \\ 0 & I_2 \end{pmatrix}$, let $B = \begin{pmatrix} I_1 & 0 \\ 0 & T_3 \end{pmatrix}, E = \begin{pmatrix} I_1 & T_2 \\ 0 & I_2 \end{pmatrix}, A = \begin{pmatrix} T_1 & 0 \\ 0 & I_2 \end{pmatrix}$. Then $T = BEA$. Suppose $\lambda \in \sigma_{(\beta)_\varepsilon}(T) \Leftrightarrow \lambda \in \sigma_{(\beta)_\varepsilon}(BEA) = \sigma_{(\beta)_\varepsilon}(EAB)$. Hence, since E is invertible, $\lambda \in \sigma_{(\beta)_\varepsilon}(AB) = \sigma_{(\beta)_\varepsilon}(T_1 \oplus T_3) \Rightarrow \lambda \in \sigma_{(\beta)_\varepsilon}(T_1)$, contradiction. Thus T has property $(\beta)_\varepsilon$, i.e. T is subscalar. □

Corollary 3.11 *Suppose that T is a k-quasi-3-symmetric operator. Then T has property (β), Dunford's property (C) and SVEP.*

Proof It suffices to prove that T has property (β). Since property (β) is transmitted from an operator to its restrictions to closed invariant subspace, we are reduced by Theorem 3.10 to the case of a scalar operator. Since every scalar operator has property (β) [21], T has property (β). □

Corollary 3.12 *Suppose that T is a quasi-nilpotent k-quasi-3-symmetric operator. Then T is nilpotent.*

Proof Since a quasi-nilpotent subscalar operator is nilpotent, it follows by Theorem 3.10 that T is nilpotent. □

Recall that a closed subspace of infinite dimensional Hilbert space $\mathcal{H}$ is said to be hyperinvariant for T if it is invariant under every operator in the commutant $\{T\}^{'}$ of T.

Theorem 3.13 *Suppose that T is a k-quasi-3-symmetric operator with $T \neq \lambda I$ for any $\lambda \in \mathbb{C}$. If there exists a nonzero $x \in \mathcal{H}$ such that $\sigma_T(x) \subsetneqq \sigma(T)$, then T has a nontrivial hyperinvariant subspace.*

Proof Suppose that T is a k-quasi-3-symmetric operator. Then T has property (β) and Dunford's property (C). If there exists a nonzero $x \in \mathcal{H}$ such that $\sigma_T(x) \subsetneqq \sigma(T)$, set

$$\mathcal{M} = \{y \in \mathcal{H} : \sigma_T(y) \subseteq \sigma_T(x)\},$$

then $\mathcal{M}$ is a T-hyperinvariant subspace from [18]. Since $x \in \mathcal{M}$, we have that $\mathcal{M} \neq \{0\}$. Suppose that $\mathcal{M} = \mathcal{H}$. Since T has SVEP, it follows from [18] that

$$\sigma(T) = \bigcup \{\sigma_T(y) : y \in H\} \subseteq \sigma_T(x) \subsetneqq \sigma(T).$$

So we have a contradiction. Hence $\mathcal{M}$ is a nontrivial hyperinvariant subspace. □

It is known that an invariant subspace for an operator T may not be hyperinvariant. However, a sufficient condition that an invariant subspace be hyperinvariant can be derived from the following corollary.

Corollary 3.14 *Suppose that T is a k-quasi-3-symmetric operator with $T \neq \lambda I$ for any $\lambda \in \mathbb{C}$. If T has a nonzero invariant subspace $\mathcal{M}$ such that $\sigma(T|_{\mathcal{M}}) \subsetneqq \sigma(T)$, then T has a nontrivial hyperinvariant subspace.*

Proof For any nonzero $x \in \mathcal{M}$, we have

$$\sigma_T(x) \subseteq \sigma_{T|_{\mathcal{M}}}(x) \subseteq \sigma(T|_{\mathcal{M}}) \subsetneqq \sigma(T).$$

Hence T has a nontrivial hyperinvariant subspace by Theorem 3.13. □

Definition 3.15 *An operator $T \in B(H)$ is said to belong to the class $H(p)$ if there exists a natural number $p := p(\lambda)$ such that*

$$H_0(\lambda I - T) = N(\lambda I - T)^p \text{ for all } \lambda \in \mathbb{C},$$

where $H_0(\lambda I - T) := \{x \in H : \lim_{n\to\infty} ||(\lambda I - T)^n x||^{\frac{1}{n}} = 0\}$.

Theorem 3.16 ([20]) *Every subscalar operator $T \in B(H)$ is $H(p)$.*

Classical examples of subscalar operators are hyponormal operators. In this paper we will show that other important classes of operators are $H(p)$.

Definition 3.17 *An operator $T \in B(H)$ is said to be polaroid if every $\lambda \in iso\sigma(T)$ is a pole of the resolvent of T.*

The condition of being polaroid may be characterized by means of the quasi-nilpotent part:

Theorem 3.18 ([4]) *An operator $T \in B(H)$ is polaroid if and only if there exists $p := p(\lambda I - T) \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = N(\lambda I - T)^p \text{ for all } \lambda \in iso\sigma(T).$$

Note that every $H(p)$ operator is polaroid. By using Theorem 3.10 and Theorem 3.16, we deduce the following corollaries.

Corollary 3.19 *Every k-quasi-3-symmetric operator is $H(p)$.*

Corollary 3.20 *Every k-quasi-3-symmetric operator is polaroid.*

A bounded linear operator T is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. Note that if T is polaroid, then it is isoloid. However, the converse is not true. In the following, f is an analytic function on an open neighborhood of $\sigma(T)$ and f is not constant on each connected component of the open set U containing $\sigma(T)$.

Theorem 3.21 *Suppose that T is a k-quasi-3-symmetric operator. Then Weyl's theorem holds for $f(T)$.*

Proof We use the fact that if T is polaroid and T has SVEP, then T satisfies Weyl's theorem in [3, Theorem 3.3]. Suppose that T is a k-quasi-3-symmetric operator. By Corollary 3.11 and Corollary 3.20 we have that T satisfies Weyl's theorem. We show next that Weyl's theorem holds for $f(T)$. Since T is polaroid and has SVEP, then $f(T)$ is polaroid by [3, Lemma 3.11] and has SVEP by [2, Theorem 2.40]. Consequently, Weyl's theorem holds for $f(T)$. □

Lemma 3.22 *Suppose that T is a k-quasi-3-symmetric operator. Then Weyl's theorem holds for $T+F$ for any finite rank operator F commuting with T.*

Proof Since T is isoloid and Weyl's theorem holds for T, the result follows by [15, Theorem 3.3]. □

Theorem 3.23 *Suppose that T is a k-quasi-3-symmetric operator. Then Weyl's theorem holds for $f(T)+F$ for any finite rank operator F commuting with T.*

Proof Since T is isoloid, $f(T)$ is isoloid [15]. Since $f(T)$ obeys Weyl's theorem by Theorem 3.21 and $f(T)$ is isoloid, the result holds by Lemma 3.22. □

Since the SVEP for T entails that generalized Browder's theorem holds for T, i.e. $\sigma_{BW}(T) = \sigma_D(T)$, where $\sigma_D(T)$ denotes the Drazin spectrum, a sufficient condition for an operator T satisfying generalized Browder's theorem to satisfy generalized Weyl's theorem is that T is polaroid. We have the following result.

Theorem 3.24 *Suppose that T is a k-quasi-3-symmetric operator. Then generalized Weyl's theorem holds for T.*

Proof It is obvious from Corollary 3.11, Corollary 3.20 and the statements of the above. □

Acknowledgments The authors would like to express their thanks to anonymous referees for several delicate comments and suggestion to revise the original manuscript.

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