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On a fifth-order nonselfadjoint boundary value problem

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Abstract: In this paper we aim to share a way to impose some nonselfadjoint boundary conditions for the solutions of a formally symmetric fifth-order differential equation. Constructing a dissipative operator related with the problem we obtain some informations on spectral properties of the problem. In particular, using coordinate-free approach we construct characteristic matrix-function related with the contraction which is obtained with the aid of the dissipative operator.

Key words: Fifth-order boundary value problem, characteristic matrix, completeness theorem

1. Introduction

Model operator and characteristic function theories are some basic tools to get some information about the spectral properties of contractions in Hilbert spaces. First model operators have been introduced by Livshitz [10] and de Branges [6]. However, a proper description of model operators has been given by Sz.-Nagy and Foias with the aid of dilation theory for a single operator [18].

Definition 1.1 *Let A be an operator acting on a Hilbert space H. A dilation of A is an operator $\mathcal{A}$ such that $H \subset \mathcal{H}$ if for each $k \in \mathbb{N}$*

$$A^k = P_H \mathcal{A}^k | H,$$

where P_H is the orthogonal projection of $\mathcal{H}$ onto H. Here $\mathcal{H}$ is called the dilation space.

One of the most important results on dilations is the following theorem of Sz.-Nagy [19].

Theorem 1.2 *Let A be a contraction acting on a Hilbert space H. Then there exists a Hilbert space $\mathcal{H}$ containing H and a unitary operator U acting on $\mathcal{H}$ such that*

$$A^m = P_H U^m | H$$

The dilation $\mathcal{A}$ is said to be minimal if $span\{A^m H : m \in \mathbb{Z}\}. = \mathcal{H}$. In the case of the dilation $\mathcal{A}$ of A is unitary, A should be a contraction.

In 1965, Sarason [17] introduced the geometric structure of the dilation space. He showed that an operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is a dilation if and only if $\mathcal{H}$ decomposed as

$$\mathcal{H} = D_- \oplus H \oplus D_+,$$

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where $\quad \mathcal{A}D_- \subset D_-,\ \mathcal{A}^* D_+ \subset D_+$ with $P_H \mathcal{A} \mid H = A.$

Matrix representation of the minimal unitary dilation of a contraction acting on $\mathcal{H} = D_- \oplus H \oplus D_+$ is given by Nikolski and Vasyunin [13] (also see [14]). Their following theorem is very valuable in this context. Related important results on dilation theory, spectral analysis and Sturm–Liouville operators are given in [15, 16] and [1–3].

Theorem 1.3 *An operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is a minimal unitary dilation of $A : H \to H$ if and only if there exist subspaces D_- and D_+ of $\mathcal{H}$ such that*

$$\mathcal{H} = D_- \oplus H \oplus D_+$$

and with respect to this decomposition, $\mathcal{A}$ has matrix

$$\mathcal{A} = \begin{bmatrix} B_* & 0 & 0 \\ D_{A^*} U_*^* & A & 0 \\ -UA^* U_*^* & UD_A & B \end{bmatrix},$$

where $D_A = (I - A^*A)^{1/2}$ and $D_{A^*} = (I - AA^*)^{1/2}$ are the defect operators of A and $B = \mathcal{A} \mid D_+$, $B_*^* = \mathcal{A}^* \mid D_-$ are pure isometries, $U : \overline{D_A H} \to KerB^*$ and $U_* : \overline{D_{A^*} H} \to KerB_*$ are partial isometries.

Another description of the dilation operator $\mathcal{A}$ of A can be given with the following unitary mappings

$$\eta : E \to D_+ \ominus \mathcal{A}D_+,\ \eta_* : E_* \to D_- \ominus \mathcal{A}^* D_-,$$

where E and E_* are some auxiliary Hilbert spaces such that

$$\dim E = \dim \overline{D_A H} = \dim KerB, \qquad \dim E_* = \dim \overline{D_{A^*} H} = \dim KerB_*,$$

and hence the following functional embeddings

$$\begin{array}{l} \pi \left(\sum_{m\in\mathbb{Z}} z^m e_m\right) = \sum_{m\in\mathbb{Z}} \mathcal{A}^m \eta e_m,\ e_m \in E, \\ \pi_* \left(\sum_{m\in\mathbb{Z}} z^m e_{*m}\right) = \sum_{m\in\mathbb{Z}} \mathcal{A}^{m+1} \eta_* e_{*m},\ e_{*m} \in E_*. \end{array}$$

The function

$$\Theta_A = \pi_*^* \pi : L^2(E) \to L^2(E_*)$$

is called characteristic function of the contraction A. This can be written explicitly as

$$\Theta_A(\mu)\varphi = U_*(-A + \mu D_{A^*}(I - \mu A^*)^{-1} D_A) U^* \varphi, \tag{1.1}$$

where $\varphi \in E$ and $|\mu| \leq 1$.

This definition of the characteristic function coincides with the definition of the characteristic function given by Sz.-Nagy and Foias [18, 20]. However, this way of the construction of the characteristic function is different from the way introduced by Sz.-Nagy and Foias. Indeed, it does not contain the coordinates. For this reason this method is called coordinate-free method for the construction of the characteristic function. The spectral analysis of formally symmetric differential operators together with nonselfadjoint boundary conditions is one of the considerable field in the literature. Up to now, almost all differential operators have been constructed with the help of even-order differential expressions and some suitable boundary conditions [4, 5, 21, 28, 29].

Recently, odd-order differential expressions together with suitable boundary conditions have been studied in [22–27]. In these works, some problems have been considered as selfadjoint and the others have been considered as nonselfadjoint problems. As is known that the famous second-order differential expression is the following

$$\ell(f) = \frac{1}{w}[-(pf')' + qf], \tag{1.2}$$

where p, q and w are real-valued functions. The general form of the expression (1.2) has been introduced by Eckhardt et al. [7] as

$$\ell(f) = \frac{1}{w}[-(p[f' + sf])' + sp[f' + sf] + qf], \tag{1.3}$$

where s is also real-valued function. Clearly, when $s = 0$, Eq. (1.3) turns out to be equivalent to Eq. (1.2). But there is no need to consider s as zero. Therefore, Eq. (1.3) particularly shows that there are formally symmetric differential expressions that have not been discovered yet.

In [26], following fifth-order formally symmetric differential equation has been considered

$$i\left(q_2\left(q_2 f''\right)'\right)'' + \left(p_2 f''\right)'' - \left(p_1 f'\right)' + p_0 f + i\left[\left(q_0 f\right)' + q_0 f' - \left(q_1 f'\right)'' - \left(q_1 f''\right)'\right] = \mu w f \tag{1.4}$$

on a regular interval and imposing some separated and coupled selfadjoint boundary conditions, some spectral properties of the eigenvalues have been studied. In this paper, we will also consider a fifth-order formally symmetric differential equation but the equation that we consider will contain Eq. (1.4). Indeed, we will consider the following equation

$$\begin{aligned} &i\left(q_2\left(q_2 f''\right)'\right)'' + \left\{\left[p_2\left(f'' - s_1 f' + s_2 f\right)\right]' + p_2 s_1\left(f'' - s_1 f'\right) - p_1\left(f' + s_4 f\right) + s_3 f\right\}' \\ &+ p_2 s_2 f'' - s_3 f' + p_1 s_4\left(f' + s_4 f\right) + p_0 f + i\left[\left(q_0 f\right)' + q_0 f' - \left(q_1 f'\right)'' - \left(q_1 f''\right)'\right] = \mu w f, \end{aligned} \tag{1.5}$$

on an interval $[a, b]$. At a first glance, Eq. (1.5) may seem to be very complicated. However, for the cases $s_1 = s_2 = s_3 = s_4 \equiv 0$ on $[a, b]$, the Eq. (1.5) is the same with Eq. (1.4). We assume that all the coefficients in (1.5) are real-valued functions and regular, $q_2 \neq 0$, $w > 0$ on $[a, b]$ and μ is a complex number. Moreover, we assume that $q_2^{-1},\ s_1 q_2^{-1},\ q_1 q_2^{-1},\ p_2 q_2^{-1},\ p_2 s_1 q_2^{-1},\ p_1 s_4,\ s_3,\ q_0,\ p_1 s_4^2$ are integrable on $[a, b]$.

In this work we aim to study on the eigenvalues and eigen and associated functions of a nonselfadjoint boundary value problem generated by Eq. (1.5) and some nonselfadjoint separated boundary conditions. For this aim, we will construct a suitable operator such that the boundary value problem can be handled as the eigenvalue problem of that operator. In particular, this operator will be a dissipative operator in the Hilbert space $L^2([a, b]; w)$ and therefore it will admit to consider the connection between dissipative and contractive operator. Using the way of Nikolski and Vasyunin [12–14] on coordinate-free approach we will construct the characteristic function of the contraction relating with the dissipative operator and then we will prove some completeness theorems. It should be noted that the form (1.5) has not been introduced earlier and coordinate free-approach will be used for the first time for a fifth-order dissipative operator.

2. Basic results

In this paper we consider the Hilbert space $L^2([a, b]; w)$ as the standart Lebesgue space that consists of all functions f such that

$$\|f\|^2 = \int_a^b |f|^2 \, w dx < \infty,$$

where $\|f\|^2 = (f, f)$ and

$$(f, g) = \int_a^b f\overline{g} w dx.$$

Now we shall introduce the rth quasi-derivative $f^{[r]}$ of a function f as follows

$$\begin{aligned}
&f^{[0]} = f; \\
&f^{[1]} = f', \\
&f^{[2]} = -\tfrac{1+i}{\sqrt{2}} q_2 f'', \\
&f^{[3]} = iq_2 \left(q_2 f''\right)' + p_2 \left(f'' - s_1 f' + s_2 f\right) - iq_1 f', \\
&f^{[4]} = -i \left(q_2 \left(q_2 f''\right)'\right)' - \left[p_2 \left(f'' - s_1 f' + s_2 f\right)\right]' - p_2 s_1 \left(f'' - s_1 f'\right) \\
&+p_1 \left(f' + s_4 f\right) - s_3 f + iq_1 f'' - iq_0 f + i \left(q_1 f'\right)', \\
&f^{[5]} = \mu f.
\end{aligned}$$

Using these quasi-derivatives we can handle the Eq. (1.5) as the following Hamiltonian system

$$JF' = \left(\mu \mathcal{W} + \mathcal{P}\right) F, \tag{2.1}$$

where $x \in [a, b]$, $J, \mathcal{W}$ and $\mathcal{P}$ are 5×5 matrices, F is a 5×1 vector such that

$$\mathcal{W} = \begin{bmatrix} w & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}, \quad J = \begin{bmatrix} & & & & -1 \\ & & & -1 & \\ & & i & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}, \quad F = \begin{bmatrix} f \\ f^{[1]} \\ f^{[2]} \\ f^{[3]} \\ f^{[4]} \end{bmatrix}, \tag{2.2}$$

and

$$\mathcal{P} = \begin{bmatrix} -p_1 s_4^2 - p_0 & -p_1 s_4 + s_3 - iq_0 & \frac{1-i}{\sqrt{2}} \frac{p_2 s_2}{q_2} & & \\ -p_1 s_4 + s_3 + iq_0 & -p_2 s_1^2 - p_1 & \frac{1-i}{\sqrt{2}} \left(\frac{-p_2 s_1 + iq_1}{q_2}\right) & & 1 \\ \frac{1+i}{\sqrt{2}} \frac{p_2 s_2}{q_2} & \frac{1+i}{\sqrt{2}} \left(\frac{-p_2 s_1 - iq_1}{q_2}\right) & -\frac{p_2}{q_2^2} & -\frac{1+i}{\sqrt{2}} \frac{1}{q_2} & \\ & & -\frac{1-i}{\sqrt{2}} \frac{1}{q_2} & & \\ & 1 & & & \end{bmatrix}.$$

Here all the other elements of these matrices are zero. Eq. (2.1) particularly implies the following.

Lemma 2.1 *There exists a unique solution $f(x, \mu)$ of (1.5) satisfying the conditions*

$$f^{[s]}(l, \mu) = l_s,$$

where $l \in [a, b]$, $l_s \in \mathbb{C}$ and $s = 0, 1, ..., 4$. The functions $f^{[s]}(., \mu)$ of μ are entire functions.

Let $\mathcal{B}$ consist of the functions $f \in L^2([a,b];w)$ such that $f^{[s]}$ is absolutely continuous on $[a,b]$, where $s = 0,1,...,4,$ and $f^{[5]} \in L^2([a,b];w)$. Then we can introduce the following Lagrange's identity for the functions f and g belonging to $\mathcal{B}$

$$\int_a^b \left(f^{[5]}\overline{g} - f\overline{g}^{[5]}\right) wdx = [f,\overline{g}](b) - [f,\overline{g}](a), \tag{2.3}$$

where $[.,.]: \mathcal{B} \times \mathcal{B} \times [a,b] \to \mathbb{C}$ with the rule

$$[f,g] = fg^{[4]} - f^{[4]}g + f^{[1]}g^{[3]} - f^{[3]}g^{[1]} + if^{[2]}g^{[2]},\ x \in [a,b].$$

Following the same procedure of [11, pp. 63-64], we may share the following.

Lemma 2.2 *There exists* $f \in \mathcal{B}$ *subject to the complex numbers* $c_s, d_s,\quad s = 0,1,...,4,$ *satisfying*

$$f^{[s]}(a) = c_s,\ f^{[s]}(b) = d_s.$$

We define the maximal operator M on $\mathcal{B}$ using the following rule

$$Mf = f^{[5]}$$

and the minimal operator M_0 as the restriction of M to $\mathcal{B}_0$ that contains the functions $f \in \mathcal{B}$ such that

$$f^{[s]}(a) = 0,\ f^{[s]}(b) = 0,$$

where $s = 0,1,...,4.$ M_0 is the symmetric, closed operator such that $M_0^* = M$ [9, 11].

For the closed symmetric operators a basic theory is the deficiency indices theory for describing maximal selfadjoint and nonselfadjoint extensions of these operators. The deficiency indices m and n of a closed symmetric operator A with domain $Dom(A)$ are defined as the dimensions of the following subspaces

$$m = \dim\left(H \ominus (A - iI)\, Dom(A)\right),\ n = \dim\left(H \ominus (A + iI)\, Dom(A)\right),$$

where H is the Hilbert space.

Note that the deficiency indices of M_0 are $(5,5)$ [9, 11].

Gorbachuks' introduced a way to share the dissipative (maximal) extensions of a symmetric, closed operator A in the Hilbert space H with domain $Dom(A)$ using boundary value space [8]. Indeed, if for any $y, z \in Dom(A^*)$, where $Dom(A^*)$ is the domain of the adjoint operator A^* of A, the equation holds

$$(A^*y, z) - (y, A^*z) = (\tau_1 y, \tau_2 z)_{H_1} - (\tau_2 y, \tau_1 z)_{H_1},$$

where $\tau_{1,2}: Dom(A^*) \to H_1,$ H_1 is a Hilbert space, such that $\tau_1 y = y_1,$ $\tau_2 y = y_2,$ $y_1, y_2 \in H_1$ then the triple (H_1, τ_1, τ_2) is a boundary value space of A.

For $f \in \mathcal{B}$ we define the following mappings

$$\tau_1 f = \left\langle f^{[4]}(a), f^{[3]}(a), \frac{1}{2}f^{[2]}(a) + \frac{i}{2}f^{[2]}(b), f(b), f^{[1]}(b)\right\rangle$$

and

$$\tau_2 f = \left\langle f(a), f^{[1]}(a), if^{[2]}(a) + f^{[2]}(b), f^{[4]}(b), f^{[3]}(b) \right\rangle.$$

Then we have the following.

Theorem 2.3 *A boundary value space of M_0 is $(\mathbb{C}^5, \tau_1, \tau_2)$.*

Proof For $f, g \in B$ one has

$$(\tau_1 f, \tau_2 g) - (\tau_2 f, \tau_1 g) = [f, \overline{g}](b) - [f, \overline{g}](a). \tag{2.4}$$

On the other side we have

$$(Mf, g) - (f, Mg) = [f, \overline{g}](b) - [f, \overline{g}](a). \tag{2.5}$$

Therefore (2.4) and (2.5) together with Lemma 2.2 complete the proof. □

Using the boundary value space $(\mathbb{C}^5, \tau_1, \tau_2)$ and Gorbachuks' result [8] we can introduce the following theorem.

Theorem 2.4 *For a contraction K on $\mathbb{C}^5$ the boundary condition*

$$(K - I)\,\tau_1 f + i\,(K + I)\,\tau_2 f = 0, \;\; f \in B,$$

describes the maximal dissipative extension of M_0.

Corollary 2.5 *For $f \in B$ following separated boundary conditions*

$$\begin{aligned}
&f(a) + \gamma_1 f^{[4]}(a) = 0, \\
&f^{[1]}(a) + \gamma_2 f^{[3]}(a) = 0, \\
&(i + \gamma_3)\, f^{[2]}(a) + (1 + i\gamma_3)\, f^{[2]}(b) = 0, \\
&f^{[4]}(b) + \gamma_4 f(b) = 0, \\
&f^{[3]}(b) + \gamma_5 f^{[1]}(b) = 0,
\end{aligned} \tag{2.6}$$

where γ_1, γ_2 are real numbers, $\gamma_3, \gamma_4, \gamma_5$ are complex numbers with $Im\gamma_t > 0$, $t = 3, 4, 5$, $\gamma_3 \neq i$, describe the maximal dissipative extension of M_0.

We will handle the problem (1.5), (2.6) as the eigenvalue problem of the operator N generated by the rule

$$Nf = f^{[5]}$$

with domain $D(N)$ that contains the functions $f \in B$ satisfying the conditions (2.6). Note that N is maximal dissipative in $L^2([a, b]; w)$.

Theorem 2.6 *$D(N)$ does not contain a nontrivial subspace on which N has a selfadjoint part there.*

Proof For $f \in D(N)$ we obtain from (2.3) that

$$Im(Nf, f) = \frac{2Im\gamma_3}{|1 + i\gamma_3|^2} \left| f^{[2]}(a) \right|^2 + Im\gamma_4 \left| f(b) \right|^2 + Im\gamma_5 \left| f^{[1]}(b) \right|^2. \tag{2.7}$$

Let us assume that N is selfadjoint on $D_s(N) \subset D(N)$. Then for $f \in D_s(N)$ we obtain from (2.7) that $f(b) = f^{[1]}(b) = f^{[2]}(a) = 0$. From the conditions (2.6) we get that $f^{[4]}(b) = f^{[3]}(b) = f^{[2]}(b) = 0$. Therefore $f \equiv 0$. The proof is completed. □

Boundary conditions (2.6) can also be considered as the following

$$F(a) = \begin{bmatrix} \gamma_1 & & & & \\ & \gamma_2 & & & \\ & & 1+i\gamma_3 & & \\ & & & -1 & \\ & & & & -1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_2 \\ t_1 \end{bmatrix} = W.t \tag{2.8}$$

and

$$F(b) = \begin{bmatrix} -1 & & & & \\ & -1 & & & \\ & & -(i+\gamma_3) & & \\ & & & \gamma_5 & \\ & & & & \gamma_4 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_2 \\ t_1 \end{bmatrix} = P.t \tag{2.9}$$

where F is the vector generated by f with the rule (2.2), all the other elements of W and P are zero and $t_1, ..., t_5$ are some real numbers. Using (2.8) and (2.9) we obtain the following.

Theorem 2.7 *Eigenvalues of N belong to the half-plane $Im\mu > 0$ without any finite accumulation point. The only possible accumulation point may be infinity.*

Proof Corollary 2.5 and Theorem 2.6 imply that all eigenvalues lie in the open upper half-plane.

Let $f(x,\mu)$ be a solution of (1.5) and $F(x,\mu)$ be the corresponding vector according to the rule given in (2.2). Consider that $\mathcal{F}(x,\mu)$ is a 5×5 matrix whose columns are the solution of (2.1) satisfying $\mathcal{F}(a,\mu) = I$, where I is the 5×5 unit matrix. Then we have

$$F(x,\mu) = \mathcal{F}(x,\mu)F(a,\mu),$$

where $x \in [a,b]$. If $F(x,\mu)$ satisfies the conditions (2.8) and (2.9) we obtain that

$$(P - \mathcal{F}(b,\mu)W).t = 0. \tag{2.10}$$

For $t \neq 0$ we have from (2.10) that

$$\det (P - \mathcal{F}(b,\mu)W) = 0.$$

Therefore roots of $\det (P - \mathcal{F}(b,\mu)W)$ are the eigenvalues of N. Using Lemma 2.1 we complete the proof. □

3. Characteristic matrix-function

Using maximal dissipative operator N we shall consider the following operator

$$R = (N - iI)(N + iI)^{-1}.$$

In fact, R is the Cayley transform of N and the whole $L^2([a,b];w)$ is the domain of R. This implies that R is a contraction in $L^2([a,b];w)$, i.e. $\|R\| \leq 1$ and 1 cannot be an eigenvalue of R [18].

We shall remind that K on a Hilbert space H is called completely nonunitary (c.n.u.) if there is no nontrivial subspace of H on which K is unitary there.

Theorem 3.1 *On the Hilbert space $L^2([a,b];w)$ the operator R is c.n.u.*

Proof Let $f \in D(N)$ and let us set $(N+iI)f = h,$ where $h \in L^2([a,b];w)$. Now consider the inequality

$$\|Rh\| < \|h\| \tag{3.1}$$

or equivalently

$$\|(N-iI)f\| < \|(N+iI)f\|. \tag{3.2}$$

Inequality (3.2) satisfies if and only if N is completely dissipative and this is satisfied by Theorem 2.6 and hence from (3.1) we get that

$$\|R\| < 1. \tag{3.3}$$

The proof is completed. □

A contraction K belongs to the classes $C_{0.}$ and $C_{.0}$ if $\|K^n y\| \to 0$ and $\|K^{*n} y\| \to 0$ for all y, respectively, as $n \to \infty$. C_{00} is defined as the intersection of $C_{0.}$ and $C_{.0}$ [18].

Using (3.3) we can introduce the following.

Theorem 3.2 *R belongs to the class C_{00}.*

Defect operators of the contraction R are defined as follows [18]:

$$D_R := (I - R^*R)^{1/2},\ D_{R^*} := (I - RR^*)^{1/2}.$$

The closure of $L^2([a,b];w)$ under D_R and D_{R^*} are called, respectively, defect spaces and we will denote them by $\mathfrak{D}_R$ and $\mathfrak{D}_{R^*}$, respectively. Finally, the dimensions of $\mathfrak{D}_R$ and $\mathfrak{D}_{R^*}$ are called defect indices of R and we will denote them by $\mathfrak{d}_R$ and $\mathfrak{d}_{R^*}$, respectively.

Let $\vartheta(x,\mu)$, $\varrho(x,\mu)$ and $\psi(x,\mu)$ be the solutions of (1.5) satisfying

$$\vartheta(a,\mu) = \gamma_1,\ \vartheta^{[4]}(a,\mu) = -1,\ \vartheta^{[1]}(a,\mu) = \vartheta^{[2]}(a,\mu) = \vartheta^{[3]}(a,\mu) = 0,$$

$$\varrho(a,\mu) = \varrho^{[1]}(a,\mu) = \varrho^{[3]}(a,\mu) = \varrho^{[4]}(a,\mu) = 0,\ \varrho^{[2]}(a,\mu) = 1 + i\gamma_3,$$

and

$$\psi^{[1]}(a,\mu) = \gamma_2,\ \psi^{[3]}(a,\mu) = -1,\ \psi(a,\mu) = \psi^{[2]}(a,\mu) = \psi^{[4]}(a,\mu) = 0.$$

Clearly, $\vartheta, \varrho, \psi \in B$. From Lemma 2.2 we may infer that ϑ, ϱ and ψ also satisfy the following

$$\vartheta^{[2]}(b,\mu) = 0,\ \varrho^{[2]}(b,\mu) = -(i + \gamma_3),\ \psi^{[2]}(b,\mu) = 0$$

and take arbitrary values for the other quasi-derivatives at b. Therefore ϑ, ϱ and ψ satisfy the first three boundary conditions in (2.6).

Lemma 3.3 *$\mathfrak{D}_R$ and $\mathfrak{D}_{R^*}$ are spanned by $\{\vartheta(x,i),\psi(x,i)+\varrho(x,i)\}$ and $\{\vartheta(x,-i),\psi(x,-i)+\varrho(x,-i)\}$, respectively. In other words,* $\mathfrak{d}_R=\mathfrak{d}_{R^*}=2$.

Proof Firstly, we note that $R^*=(N^*+iI)(N^*-iI)^{-1}$. Then setting $(N+iI)f=g$, where $f\in D(N)$ and $g\in L^2([a,b];w)$ we obtain that

$$D_R^2 g=(I-R^*R)g=(N+iI)f-(N^*+iI)(N^*-iI)^{-1}(N-iI)f. \tag{3.4}$$

Let us set the following

$$(N^*-iI)^{-1}(N-iI)f=h. \tag{3.5}$$

From (3.5) we get

$$(N-iI)f=(N^*-iI)h. \tag{3.6}$$

Therefore from (3.6) we may infer that $f-h$ is a solution of

$$(f-h)^{[5]}=i\,(f-h) \tag{3.7}$$

such that $f-h\in L^2([a,b];w)$ and $f-h$ satisfies the first three boundary conditions in (2.6). Eq. (3.7) can also be handled as

$$f-h=c_1\vartheta(x,i)+c_2\left[\psi(x,i)+\varrho(x,i)\right], \tag{3.8}$$

where c_1,c_2 are constants. Therefore from (3.4) and (3.8) we see that $\vartheta(x,i)$ and $\psi(x,i)+\varrho(x,i)$ span D_R and hence $\mathfrak{d}_R=2$.

The other assertion can be proved similarly and this completes the proof. □

Recall that a contraction K belongs to the class C_0 if for a nonzero function $z\in H^\infty$, where H^p denotes the Hardy class, satisfies $z(K)=0$.

Since R has finite defect indices and belongs to C_{00} we have the following [18, p. 273].

Theorem 3.4 *R belongs to the class C_0.*

To obtain the characteristic function of R we will use the formula (1.1). Note that (1.1) can also be considered as

$$D_{R^*}U_*^*\Theta_R(\delta)=(\delta I-R)\left(I-\delta R^*\right)^{-1}D_RU^*,\ \ |\delta|\le 1 \tag{3.9}$$

Then we have the following.

Theorem 3.5 *The characteristic matrix-function $\Theta_R(\delta)$ of R can be introduced as*

$$\Theta_K(\delta)=\begin{bmatrix} -\dfrac{\Gamma(\vartheta;\overline{\gamma}_4;i)}{\Gamma(\vartheta;\gamma_4;-i)}\dfrac{\Gamma(\vartheta;\gamma_4;\mu)}{\Gamma(\vartheta;\overline{\gamma}_4;\mu)} & 0\\ 0 & -\dfrac{\Gamma(\vartheta;\overline{\gamma}_5;i)}{\Gamma(\vartheta;\gamma_5;-i)}\dfrac{\Gamma(\vartheta;\gamma_5;\mu)}{\Gamma(\vartheta;\overline{\gamma}_5;\mu)}\end{bmatrix}$$

or

$$\Theta_K(\delta)=\left[\begin{array}{ll} -\dfrac{\Gamma(\psi;\overline{\gamma}_4;i)}{\Gamma(\psi;\gamma_4;-i)}\dfrac{\Gamma(\psi;\gamma_4;\mu)}{\Gamma(\psi;\overline{\gamma}_4;\mu)} & 0 \\ 0 & -\dfrac{\Gamma(\psi;\overline{\gamma}_5;i)}{\Gamma(\psi;\gamma_5;-i)}\dfrac{\Gamma(\psi;\gamma_5;\mu)}{\Gamma(\psi;\overline{\gamma}_5;\mu)} \end{array}\right],$$

where $\Gamma(\chi;\gamma_4;\lambda):=\chi^{[4]}(b,\lambda)+\gamma_4\chi(b,\lambda)$ *and* $\Gamma(\chi;\gamma_5;\lambda):=\chi^{[3]}(b,\lambda)+\gamma_5\chi^{[1]}(b,\lambda)$ *and* $\delta=(\mu-i)/(\mu+i)$, $Im\mu>0$.

Proof To construct the characteristic function $\Theta_R(\delta)$ of R we will use Eq. (3.9) and the function $\vartheta(x,i)$. We should note that one may also use the function $\psi(x,i)+\varrho(x,i)$.

Since $\mathfrak{d}_R=\mathfrak{d}_{R^*}=2$ we let $E=E_*=\mathbb{C}^2$. Now for $c=\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right]\in\mathbb{C}^2$ we define the following

$$U\vartheta(x,i)=\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right]. \tag{3.10}$$

Using (3.9) we obtain that

$$\begin{aligned} &D_{R^*}U_*^*\Theta_R(\delta)\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right]=(\delta I-R)\left(I-\delta R^*\right)^{-1}\vartheta(x,i) \\ &=-\left(N-\mu I\right)\left(N+iI\right)^{-1}\left(N^*-iI\right)\left(N^*-\mu I\right)^{-1}D_R\vartheta(x,i). \end{aligned} \tag{3.11}$$

At this stage we should note that $D_R^2\vartheta(x,i)=\vartheta(x,i)$ and $R\vartheta(x,i)=0$. Furthermore $D_R\vartheta(x,i)=\vartheta(x,i)$.

Now we shall consider the following sum

$$\vartheta(x,\mu)+d_1\vartheta(x,\zeta).$$

This sum satisfies the fourth boundary condition in (2.6) if

$$d_1=d_1(\mu,\zeta,\gamma_4)=-\frac{\Gamma(\vartheta;\gamma_4;\mu)}{\Gamma(\vartheta;\gamma_4;\zeta)}.$$

Similarly the following sum

$$\vartheta(x,\mu)+d_2\vartheta(x,\zeta)$$

satisfies the last boundary condition in (2.6) if

$$d_2=d_2(\mu,\zeta,\gamma_5)=-\frac{\Gamma(\vartheta;\gamma_5;\mu)}{\Gamma(\vartheta;\gamma_5;\zeta)}.$$

Using the equation

$$\left(N-\mu I\right)^{-1}\vartheta(x,\zeta)=\frac{\vartheta(x,\zeta)+d_k(\mu,\zeta,\gamma_{k+3})\vartheta(x,\mu)}{\zeta-\mu},$$

where $k=1,2,$ and (3.11) one obtains

$$\begin{aligned} D_{R^*}U_*^*\Theta_R(\delta)\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right] &=-\frac{\Gamma(\vartheta;\overline{\gamma}_4;i)}{\Gamma(\vartheta;\gamma_4;-i)}\frac{\Gamma(\vartheta;\gamma_4;\mu)}{\Gamma(\vartheta;\overline{\gamma}_4;\mu)}\vartheta(x,-i) \\ &=-\frac{\Gamma(\psi;\overline{\gamma}_5;i)}{\Gamma(\psi;\gamma_5;-i)}\frac{\Gamma(\psi;\gamma_5;\mu)}{\Gamma(\psi;\overline{\gamma}_5;\mu)}\vartheta(x,-i). \end{aligned} \tag{3.12}$$

From (3.10), (3.12) and the fact that $\|\vartheta(x,i)\| = \left\|\overline{\vartheta(x,-i)}\right\| = \|\vartheta(x,-i)\|$ we have

$$\Theta_R(\delta) = \begin{bmatrix} -\dfrac{\Gamma(\vartheta;\overline{\gamma}_4;i)}{\Gamma(\vartheta;\gamma_4;-i)}\dfrac{\Gamma(\vartheta;\gamma_4;\mu)}{\Gamma(\vartheta;\overline{\gamma}_4;\mu)} & 0 \\ 0 & -\dfrac{\Gamma(\vartheta;\overline{\gamma}_5;i)}{\Gamma(\vartheta;\gamma_5;-i)}\dfrac{\Gamma(\vartheta;\gamma_5;\mu)}{\Gamma(\vartheta;\overline{\gamma}_5;\mu)} \end{bmatrix}$$

and this completes the proof. □

We shall remind that a function $\Theta(\delta)$ that has a power series expansion

$$\Theta(\delta) = \sum_{n=0}^{\infty} \delta^n \Theta_n$$

and the values of $\Theta(\delta)$ are bounded operators from a separable Hilbert space H_1 to a separable Hilbert space H_2 such that $\|\Theta(\delta)\| \leq 1$ is called a contractive analytic function. $\Theta(\delta)$ is called inner provided that $\Theta(e^{ir})$ is an isometry from H_1 into H_2 for almost all r.

Theorem 3.6 $\Theta_R(\delta)$ *is inner.*

Proof The result comes from the fact that $R \in C_{.0}$. □

Since $R \in C_0$ we have the following [18, p. 273].

Corollary 3.7 $\det \Theta_R(\delta)$ *is inner.*

Recall that a weak contraction K is a contraction K on a Hilbert space H with the property that the unit disk is not fulfilled by the spectrum of K and $I - K^*K$ is a trace class operator. It is known that a contraction belonging to the class C_0 and having finite defects is a weak contraction. Therefore we can introduce the following result

Theorem 3.8 R *is a weak contraction.*

Now Corollary 3.7 implies the following.

Theorem 3.9 $\det \Theta_R(\delta)$ *is a Blaschke product.*

Proof For $\delta = (\mu - i)\,/\,(\mu + i)\,,\ Im\mu > 0,$ we have

$$\det \Theta_R(\delta) = \mathbb{B}(\mu) e^{i\mu s}, \tag{3.13}$$

where $\mathbb{B}(\mu)$ is a Blaschke product and $s \geq 0$. Using (3.13) and setting $\mu_k = ik$ we get that

$$\left| \frac{\Gamma(\vartheta;\gamma_4;\mu_k)}{\Gamma(\vartheta;\overline{\gamma}_4;\mu_k)} \frac{\Gamma(\vartheta;\gamma_5;\mu_k)}{\Gamma(\vartheta;\overline{\gamma}_5;\mu_k)} \right| \leq e^{-ks} \tag{3.14}$$

as

$$\left| \frac{\Gamma(\vartheta;\overline{\gamma}_4;i)}{\Gamma(\vartheta;\gamma_4;-i)} \right| = \left| \frac{\Gamma(\vartheta;\overline{\gamma}_5;i)}{\Gamma(\vartheta;\gamma_5;-i)} \right| = 1.$$

As $k \to \infty$ we obtain from (3.14) that one of the followings holds:

(i) $\Gamma(\vartheta;\gamma_4;\mu_k) \to 0$ as $k \to \infty$,

(ii) $\Gamma(\vartheta;\gamma_5;\mu_k) \to 0$ as $k \to \infty$,

(iii) $\Gamma(\vartheta;\gamma_4;\mu_k) \to 0$ and $\Gamma(\vartheta;\gamma_5;\mu_k) \to 0$ as $k \to \infty$.

The case (iii) is possible and in this case $\vartheta(x,\mu_\infty)$ becomes an eigenfunction and μ_∞ becomes an eigenvalue of N and equivalently 1 becomes an eigenvalue of R. However, this is not possible and this completes the proof. □

Therefore we have the following.

Theorem 3.10 *Root functions of R are complete in $L^2([a,b];w)$.*

Since the completeness of R and N are equivalent we can introduce the following.

Theorem 3.11 *Root functions of N are complete in $L^2([a,b];w)$.*

Collecting all the result for the boundary value problem (1.5), (2.6) we can introduce the following.

Corollary 3.12 *For an eigenvalue μ of the boundary value problem (1.5), (2.6) one has $Im\mu > 0$. The only possible limiting point of these eigenvalues is infinity. However, infinity cannot be an eigenvalue of the problem (of the operator N). All eigenfunctions and associated functions of the boundary value problem (of the operator N) generates $L^2([a,b];w)$.*

4. Conclusion

In this paper a fifth-order formally symmetric differential equation of the form (1.5) has been introduced for the first time and imposing some well-defined nonselfadjoint boundary conditions for the solutions of (1.5) we have constructed a maximal, simple dissipative operator. Passing to the corresponding Cayley transform we have constructed the characteristic matrix-function of the Cayley transform. Finally we have introduced some completeness theorems.

The contraction R, as was introduced in Theorem 3.8, is a weak contraction. Since R also belongs to C_0 we may introduce some additional results. However, for this aim we need to share some definitions. These definitions can be found in [18] or [12].

A Hilbert space H_1 that is a subspace of the Hilbert space H is called a wandering space for an isometry A if for $p,q \geq 0$, $p \neq q$, A^pH_1 is orthogonal to A^qH_1.

Let $H_1 \subset H$ be a wandering subspace of the isometry A. It is called that A is a unilateral shift if

$$\bigoplus_{0}^{\infty} A^n H_1 = H.$$

The dimension of $H \ominus AH$ is called the multiplicity of A.

Now let H^p denote the Hardy space and A be a unilateral shift on H^2 with multiplicity one. Jordan block $A(\chi)$, where χ is an inner function such that $\chi \in H^\infty$, is defined on

$$\mathcal{H}(\chi) = H^2 \ominus \chi H^2$$

as

$$A(\chi) = P_{\mathcal{H}(\chi)} A \mid \mathcal{H}(\chi).$$

Note that $A(\chi) \in C_0$.

Consider that $\Xi = \{\chi_j\}_{j \geq 0} \subset H^\infty$ is a sequence of inner functions χ_j such that $\chi_j \mid \chi_{j+1}$ for all $j \geq 0$. The Jordan operator $J(\chi_j)$ is defined by

$$J(\chi_j) = \bigoplus_0^\infty A(\chi_j).$$

Since $R \in C_0$ we can introduce the following.

Theorem 4.1 *The Jordan model of R is*

$$J(\chi_j) = \bigoplus_0^\infty A(\chi_j),$$

where $\chi_j = \theta_j / \theta_{j+1}$, θ_j is the greatest common divisor of all minors of corank j of the characteristic matrix-function Θ_R.

The cyclic multiplicity m_K of a contraction K acting on the Hilbert space H is defined as the smallest cardinality of a subset $H_1 \subset H$ satisfying the equation

$$span\,\{K^m \omega : \omega \in H_1, m \geq 0\} = H.$$

It is known that $m_K \leq d_{K^*}$ for a contraction $K \in C_{.0}$ [18]. Therefore we can introduce the following.

Theorem 4.2 $m_R \leq 2$.

Theorem 3.8 implies the following.

Theorem 4.3 *$I - \Theta_R(\delta)$ is of trace class.*

Using Theorem 3.9 we can also introduce the following.

Theorem 4.4 *Let $k(\mu)$ be the rank of the Riesz projection at an eigenvalue μ of R. Then*

$$\det |\Theta_R(0)|^2 = \prod_{\mu \neq 0} |\mu|^{2k(\mu)}.$$

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