Gauss–Bonnet theorems and the Lorentzian Heisenberg group

Tong WU©, Sining WEI©, Yong WANG∗©
School of Mathematics and Statistics, Northeast Normal University, Changchun, China

Received: 09.11.2020 • Accepted/Published Online: 17.01.2021 • Final Version: 26.03.2021

Abstract: In this paper, we compute sub-Riemannian limits of Gaussian curvature for a $C^2$-smooth surface in the Lorentzian Heisenberg group for the second Lorentzian metric and the third Lorentzian metric and signed geodesic curvature for $C^2$-smooth curves on surfaces. We get Gauss–Bonnet theorems in the Lorentzian Heisenberg group for the second Lorentzian metric and the third Lorentzian metric.

Key words: Lorentzian Heisenberg group, the second Lorentzian metric, the third Lorentzian metric, Gauss–Bonnet theorem, sub-Riemannian limit

1. Introduction

In [6], Diniz and Veloso prove a version of the Gauss–Bonnet theorem in sub-Riemannian Heisenberg space $\mathbb{H}^1$ and define a Gaussian curvature for nonhorizontal surfaces in sub-Riemannian Heisenberg space $\mathbb{H}^1$. The definition was analogous to Gauss curvature of surfaces in $\mathbb{R}^3$ with particular normal to surface and Hausdorff measure of area. The image of Gauss map was in the cylinder of radius one. In [2], Balogh et al. propose a suitable candidate for the notion of intrinsic Gaussian curvature for Euclidean $C^2$-smooth surface in the first Heisenberg group $\mathbb{H}^1$. These results were then used to prove a Heisenberg version of the Gauss–Bonnet theorem. In [10], Veloso verified that Gaussian curvature of surfaces and normal curvature of curves in surfaces introduced by [6] and by [2] to prove Gauss–Bonnet theorems in Heisenberg space $\mathbb{H}^1$ were unequal and he applied the same formalism of [6] to get the curvatures of [2]. With the obtained formulas, it is possible to prove the Gauss–Bonnet theorem as a straightforward application of Stokes theorem. In [8] Gilkey and Park use analytic continuation to derive the Chern–Gauss–Bonnet theorem for pseudo-Riemannian manifolds with boundary directly from the corresponding result in the Riemannian setting. In [9], Rahmani and Rahmani proved that there are three different metrics of the sense of isometry in the Lorentzian groups. In [11], Wang and Wei gave sub-Riemannian limits of Gaussian curvature for a $C^2$-smooth surface in the Lorentzian Heisenberg group for the first Lorentzian metric and the Lorentzian group of rigid motions of the Minkowski plane away from characteristic points and signed geodesic curvature for $C^2$-smooth curves on surfaces. And they got Gauss–Bonnet theorems in the Lorentzian Heisenberg group for the first Lorentzian metric and the Lorentzian group of rigid motions of the Minkowski plane. In this paper, we will use similar methods to show that Gauss–Bonnet theorem holding in the Lorentzian Heisenberg group for the second Lorentzian metric and for the third Lorentzian metric.

*Correspondence: wangy581@nenu.edu.cn
2010 AMS Mathematics Subject Classification: 53C40; 53C42.
* Correspondence: wangy581@nenu.edu.cn

This work is licensed under a Creative Commons Attribution 4.0 International License.
The Riemannian approximation scheme used in [2], can in general depend upon the choice of the complement to the horizontal distribution. In the context of \( \mathbb{H}^1 \) the choice which they have adopted is rather natural. The existence of the limit defining the intrinsic curvature of a surface depends crucially on the cancellation of certain divergent quantities in the limit. Such cancellation stems from the specific choice of the adapted frame bundle on the surface, and on symmetries of the underlying left-invariant group structure on the Heisenberg group. In [2], they proposed an interesting question which is solved in this paper to understand to what extent similar phenomena hold in other sub-Riemannian geometric structures. The main results of this paper are Gauss–Bonnet type theorems for Lorentzian surfaces in Lorentzian Heisenberg group for the second Lorentzian metric and the Lorentzian Heisenberg group for the third Lorentzian metric.

In Section 2, we compute the sub-Riemannian limit of curvature of curves in the Lorentzian Heisenberg group for the second Lorentzian metric. In Section 3, we compute sub-Riemannian limits of geodesic curvature of curves on Lorentzian surfaces and the Riemannian Gaussian curvature of surfaces in the Lorentzian Heisenberg group for the second Lorentzian metric. We prove the Gauss–Bonnet theorem in the Lorentzian Heisenberg group for the second Lorentzian metric. In Section 4, we compute the sub-Riemannian limit of curvature of curves in the Lorentzian Heisenberg group for the third Lorentzian metric. In Section 5, we compute sub-Riemannian limits of geodesic curvature of curves on Lorentzian surfaces and the Riemannian Gaussian curvature of surfaces in the Lorentzian Heisenberg group for the third Lorentzian metric. We prove the Gauss–Bonnet theorem in the Lorentzian Heisenberg group for the third Lorentzian metric.

2. The sub-Riemannian limit of curvature of curves in the Lorentzian Heisenberg group for the second Lorentzian metric

In this section we prepare some basic notions in the Lorentzian Heisenberg group. Let \( \mathbb{H} \) be the Heisenberg group where the noncommutative group law is given by

\[
(x, y, z) \ast (x', y', z') = (x + x', y + y', z + z - xy + x'y).
\]

Let

\[
X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2} - x_1 \partial_{x_3}, \quad X_3 = \partial_{x_3},
\]

then

\[
\partial_{x_1} = X_1, \quad \partial_{x_2} = X_2 + x_1 X_3, \quad \partial_{x_3} = X_3,
\]

and \( \text{span}\{X_1, X_2, X_3\} = T\mathbb{H} \). Let \( H = \text{span}\{X_1, X_2\} \) be the horizontal distribution on \( \mathbb{H} \). Let \( \omega_1 = dx_1, \quad \omega_2 = dx_2, \quad \omega = x_1 dx_2 + dx_3 \). Then \( H = \text{Ker}\omega \). For the constant \( L > 0 \), let \( g_L = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 - L \omega_1 \otimes \omega \), \( g = g_1 \) be a Lorentzian metric on \( \mathbb{H} \) (see [9]). We call \((\mathbb{H}, g_L)\) a Lorentzian Heisenberg group and write \( \mathbb{H}^2_L \) instead of \((\mathbb{H}, g_L)\). Then \( X_1, X_2, \tilde{X}_3 := L^{-\frac{1}{2}} X_3 \) are orthonormal basis on \( T\mathbb{H}^2_L \) with respect to \( g_L \). We have

\[
[X_1, X_2] = -X_3, \quad [X_2, X_3] = 0, \quad [X_1, X_3] = 0.
\]

We say that a nonzero vector \( x \in \mathbb{H}^2_L \) is spacelike, null or timelike if \( \langle x, x \rangle > 0 \), \( \langle x, x \rangle = 0 \) or \( \langle x, x \rangle < 0 \) respectively. The norm of the vector \( x \in \mathbb{H}^2_L \) is defined by \( ||x|| = \sqrt{\langle x, x \rangle} \). Let \( \gamma : I \to \mathbb{H}^2_L \) be a regular curve, where \( I \) is an open interval in \( \mathbb{R} \). The regular curve \( \gamma \) is called a spacelike curve, timelike curve or null curve if \( \gamma'(t) \) is a spacelike vector, timelike vector or null vector at any \( t \in I \) respectively.

Let \( \nabla^L \) be the Levi–Civita connection on \( \mathbb{H}^2_L \) with respect to \( g_L \). By the Koszul formula, we have

\[
2\langle \nabla^L_{X_i} X_j, X_k \rangle_L = \langle [X_i, X_j], X_k \rangle_L - \langle [X_j, X_k], X_i \rangle_L + \langle [X_k, X_i], X_j \rangle_L.
\]

719
where \(i, j, k = 1, 2, 3\). By (2.3) and (2.4), we have

**Lemma 2.1**

\[
\nabla^L_{X_i} X_j = 0, \quad 1 \leq j \leq 3, \quad \nabla^L_{X_i} X_2 = -\frac{1}{2} X_3, \quad \nabla^L_{X_2} X_1 = \frac{1}{2} X_3, \quad (2.5)
\]

\[
\nabla^L_{X_i} X_3 = \nabla^L_{X_i} X_1 = -\frac{L}{2} X_2, \quad \nabla^L_{X_2} X_3 = \nabla^L_{X_3} X_2 = \frac{L}{2} X_1.
\]

**Definition 2.2** Let \(\gamma : I \to \mathbb{H}^2_L\) be a \(C^1\)-smooth curve. We say that \(\gamma\) is regular if \(\dot{\gamma} \neq 0\) for every \(t \in I\). Moreover we say that \(\gamma(t)\) is a horizontal point of \(\gamma\) if

\[
\omega(\dot{\gamma}(t)) = \gamma_1(t)\dot{\gamma}_2(t) + \gamma_3(t) = 0.
\]

We recall that in Riemannian geometry the standard definition of curvature for a curve \(\gamma\) parametrized by arc length is \(k^L_\gamma := \|\nabla^L_{\dot{\gamma}} \gamma\|\). For curves with an arbitrary parametrization, we give the definitions as follows:

**Definition 2.3** Let \(\gamma : I \to \mathbb{H}^2_L\) be a \(C^2\)-smooth regular curve.

(1) If \(\nabla^L_{\dot{\gamma}} \gamma\) is a spacelike vector, the curvature \(k^L_\gamma\) of \(\gamma(t)\) is defined as

\[
k^L_\gamma := \sqrt{\frac{||\nabla^L_{\dot{\gamma}} \gamma||_L^2}{||\dot{\gamma}||^2_L} - \frac{\langle \nabla^L_{\dot{\gamma}} \gamma, \dot{\gamma} \rangle^2_L}{\langle \dot{\gamma}, \dot{\gamma} \rangle^2_L}}. \quad (2.6)
\]

(2) If \(\nabla^L_{\dot{\gamma}} \gamma\) is a timelike vector, the curvature \(k^L_\gamma\) of \(\gamma\) at \(\gamma(t)\) is defined as

\[
k^L_\gamma := \sqrt{\frac{||\nabla^L_{\dot{\gamma}} \gamma||^2_L}{||\dot{\gamma}||^2_L} + \frac{\langle \nabla^L_{\dot{\gamma}} \gamma, \dot{\gamma} \rangle^2_L}{\langle \dot{\gamma}, \dot{\gamma} \rangle^2_L}}. \quad (2.7)
\]

**Lemma 2.4** Let \(\gamma : I \to \mathbb{H}^2_L\) be a \(C^2\)-smooth regular curve.

(1) If \(\nabla^L_{\dot{\gamma}} \gamma\) is a spacelike vector, then

\[
k^L_\gamma = \left\{ \left\{ [\gamma_1 + L\gamma_2 \omega(\dot{\gamma}(t))]^2 + [\gamma_2 - L\gamma_1 \omega(\dot{\gamma}(t))]^2 - L \left[ \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right]^2 \right\} \right\}^{\frac{1}{2}}.
\]

In particular, if \(\gamma(t)\) is a horizontal point of \(\gamma\), then

\[
k^L_\gamma = \left\{ \left\{ \gamma_1^2 + \gamma_2^2 - L \left[ \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right]^2 \right\} \cdot [\gamma_1^2 + \gamma_2^2]^{-2} \right\}^{\frac{1}{2}}. \quad (2.9)
\]
(2) If $\nabla^L_\gamma \dot{\gamma}$ is a timelike vector, then

\[
 k^L_\gamma = \left\{ -\left[ \gamma_1 + L\gamma_2 \omega(\dot{\gamma}(t)) \right]^2 + \left[ \gamma_2 - L\gamma_1 \omega(\dot{\gamma}(t)) \right]^2 - L \left[ \frac{d}{dt} (\omega(\dot{\gamma}(t))) \right]^2 \right\} \cdot \left[ \frac{\dot{\gamma}_1^2 + \dot{\gamma}_2^2}{\left(\dot{\gamma}_1^2 + \dot{\gamma}_2^2 - L\omega^2(\dot{\gamma}(t))\right)} \right]^{\frac{1}{2}}
\]  \tag{2.10}

In particular, if $\gamma(t)$ is a horizontal point of $\gamma$, then

\[
 k^L_\gamma = \left\{ -\left[ \gamma_1 + L\gamma_2 \omega(\dot{\gamma}(t)) \right]^2 + \left[ \gamma_2 - L\gamma_1 \omega(\dot{\gamma}(t)) \right]^2 - L \left[ \frac{d}{dt} (\omega(\dot{\gamma}(t))) \right]^2 \right\} \cdot \left[ \frac{\dot{\gamma}_1^2 + \dot{\gamma}_2^2}{\left(\dot{\gamma}_1^2 + \dot{\gamma}_2^2 - L\omega^2(\dot{\gamma}(t))\right)} \right]^{\frac{1}{2}} + \left[ \gamma_1 \gamma_1 + \gamma_2 \gamma_2 \right] \cdot \left[ \frac{\dot{\gamma}_1^2 + \dot{\gamma}_2^2}{\left(\dot{\gamma}_1^2 + \dot{\gamma}_2^2 - L\omega^2(\dot{\gamma}(t))\right)} \right]^{\frac{3}{2}}
\]  \tag{2.11}

**Proof** By (2.2), we have

\[
 \dot{\gamma}(t) = \dot{\gamma}_1 X_1 + \dot{\gamma}_2 X_2 + \omega(\dot{\gamma}(t))X_3.
\]  \tag{2.12}

By Lemma 2.1 and (2.12), we have

\[
 \nabla^L_\gamma X_1 = -\frac{L}{2} [\gamma_1(t)\gamma_2(t) + \gamma_3(t)] X_2 + \frac{\gamma_2(t)}{2} X_3,
\]  \tag{2.13}

\[
 \nabla^L_\gamma X_2 = \frac{L}{2} [\gamma_1(t)\gamma_2(t) + \gamma_3(t)] X_1 - \frac{\gamma_1(t)}{2} X_3,
\]

\[
 \nabla^L_\gamma X_3 = \frac{L}{2} \dot{\gamma}_2(t) X_1 - \frac{L}{2} \dot{\gamma}_1(t) X_2.
\]

By (2.12) and (2.13), we have

\[
 \nabla^L_\gamma \dot{\gamma} = [\dot{\gamma}_1 + L\gamma_2 \omega(\dot{\gamma}(t))] X_1 + [\dot{\gamma}_2 - L\gamma_1 \omega(\dot{\gamma}(t))] X_2 + \left[ \frac{d}{dt} (\omega(\dot{\gamma}(t))) \right] X_3.
\]  \tag{2.14}

By Definition 2.3, (2.12) and (2.14), we get Lemma 2.4. 

**Definition 2.5** Let $\gamma : I \to \mathbb{H}^2_L$ be a $C^2$-smooth regular curve. We define the intrinsic curvature $k^\infty_\gamma$ of $\gamma$ at $\gamma(t)$ to be

\[
 k^\infty_\gamma := \lim_{L \to +\infty} k^L_\gamma,
\]

if the limit exists.

We introduce the following notation: for continuous functions $f_1, f_2 : (0, +\infty) \to \mathbb{R}$,

\[
 f_1(L) \sim f_2(L), \quad \text{as} \quad L \to +\infty \iff \lim_{L \to +\infty} \frac{f_1(L)}{f_2(L)} = 1.
\]  \tag{2.15}
Lemma 2.6 Let $\gamma : I \to \mathbb{H}_L^2$ be a $C^2$-smooth regular curve. Then

(1) If $\nabla^L_\gamma \dot{\gamma}$ is a spacelike vector, then

$$k_\gamma^\infty = \sqrt{\dot{\gamma}_1^2 + \dot{\gamma}_2^2} / |\omega(\dot{\gamma}(t))|, \quad \text{if} \quad \omega(\dot{\gamma}(t)) \neq 0;$$

(2.16)

$$k_\gamma^\infty = \left\{ (\dot{\gamma}_1^2 + \dot{\gamma}_2^2) \cdot (\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^{-2} - (\dot{\gamma}_1 \dot{\gamma}_1 + \dot{\gamma}_2 \dot{\gamma}_2)^2 \cdot (\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^{-3} \right\}^{\frac{1}{2}},$$

(2.17)

if $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$;

$$\lim_{L \to +\infty} k^L_\gamma / \sqrt{L} = \sqrt{-\frac{d}{dt}(\omega(\dot{\gamma}(t)))^2} / |\dot{\gamma}_1^2 + \dot{\gamma}_2^2|, \quad \text{if} \quad \omega(\dot{\gamma}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0.$$  

(2.18)

Therefore, this situation does not exist.

(2) If $\nabla^L_\gamma \dot{\gamma}$ is a timelike vector, then

$$k_\gamma^\infty = \sqrt{-\dot{\gamma}_1^2 + \dot{\gamma}_2^2} / |\omega(\dot{\gamma}(t))|, \quad \text{if} \quad \omega(\dot{\gamma}(t)) \neq 0;$$

(2.19)

$$k_\gamma^\infty = \left\{ - (\dot{\gamma}_1^2 + \dot{\gamma}_2^2) \cdot (\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^{-2} + (\dot{\gamma}_1 \dot{\gamma}_1 + \dot{\gamma}_2 \dot{\gamma}_2)^2 \cdot (\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^{-3} \right\}^{\frac{1}{2}},$$

(2.20)

if $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$.

$$\lim_{L \to +\infty} k^L_\gamma / \sqrt{L} = |\frac{d}{dt}(\omega(\dot{\gamma}(t)))| / |\dot{\gamma}_1^2 + \dot{\gamma}_2^2|, \quad \text{if} \quad \omega(\dot{\gamma}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0.$$  

(2.21)

Therefore, this situation does not exist.

**Proof** Using the notation introduced in (2.15), when $\omega(\dot{\gamma}(t)) \neq 0$, we have

$$\langle \nabla^L_\gamma \dot{\gamma}, \nabla^L_\gamma \dot{\gamma} \rangle_L \sim |\omega(\dot{\gamma}(t))|^2 (\dot{\gamma}_1^2 + \dot{\gamma}_2^2)L^2, \quad \text{as} \quad L \to +\infty,$$

$$\langle \dot{\gamma}, \dot{\gamma} \rangle_L \sim -L \omega(\dot{\gamma}(t))^2, \quad \text{as} \quad L \to +\infty,$$

$$\langle \nabla^L_\gamma \dot{\gamma}, \dot{\gamma} \rangle_L^2 \sim O(L^2) \quad \text{as} \quad L \to +\infty.$$

Therefore

$$\frac{\langle \nabla^L_\gamma \dot{\gamma}, \nabla^L_\gamma \dot{\gamma} \rangle_L}{||\dot{\gamma}||^2_L} \to \frac{\dot{\gamma}_1^2 + \dot{\gamma}_2^2}{|\omega(\dot{\gamma}(t))|^2}, \quad \text{as} \quad L \to +\infty,$$

$$\frac{\langle \nabla^L_\gamma \dot{\gamma}, \dot{\gamma} \rangle_L^2}{\langle \dot{\gamma}, \dot{\gamma} \rangle_L^2} \to 0, \quad \text{as} \quad L \to +\infty.$$
We will say that a surface $\Sigma$ is regular if $\Sigma$ is a $C^2$-smooth compact and oriented surface. In particular, we will assume that there exists a $C^2$-smooth function $u : \mathbb{H}^2_L \to \mathbb{R}$ such that

$$\Sigma = \{(x_1, x_2, x_3) \in \mathbb{H}^2_L : u(x_1, x_2, x_3) = 0\}$$

and $\nabla_{\mathbb{H}^2_L} u = u_x \partial_{x_1} + u_x \partial_{x_2} + u_x \partial_{x_3} \neq 0$. A point $x \in \Sigma$ is called characteristic if $\nabla_{\mathbb{H}_L} u(x) = (0, 0)$. Our computations will be local and away from characteristic points of $\Sigma$.

Let us define first

$$p := X_1 u, \quad q := X_2 u, \quad r := \bar{X}_3 u.$$ 

Because $p^2 + q^2 > 0$, we say that $\Sigma \subset \mathbb{H}^2_L$ is Horizontal spacelike surface. When $L \to +\infty$, then $p^2 + q^2 - r^2 > 0$. And we define

$$l := \sqrt{p^2 + q^2}, \quad l_L := \sqrt{p^2 + q^2 - r^2}, \quad p := \frac{p}{l},$$

$$q := \frac{q}{l}, \quad p_L := \frac{p}{l_L}, \quad q_L := \frac{q}{l_L}, \quad r := \frac{r}{l_L}.$$ 

(3.1)

In particular, $p^2 + q^2 = 1$. These functions are well defined at every non-characteristic point. Let

$$v_L = p_L X_1 + q_L X_2 - r_L \bar{X}_3, \quad e_1 = q X_1 - p X_2, \quad e_2 = r X_1 + q X_2 - \frac{l}{l_L} \bar{X}_3,$$

(3.2)

then $v_L$ is the unit spacelike normal vector to $\Sigma$ and $e_1$ is the unit spacelike vector, $e_2$ is the unit timelike vector. $\{e_1, e_2\}$ are the orthonormal basis of $\Sigma$. We call $\Sigma$ a Lorentzian surface in the Lorentzian Heisenberg group.

Let $\dot{\gamma} = \lambda_1 e_1 + \lambda_2 e_2$. If $\gamma : I \to \mathbb{H}^2_L$ be a $C^2$-smooth timelike curve, then we define $J_L(\dot{\gamma}) := -\lambda_1 e_2 - \lambda_2 e_1$. If $\gamma : I \to \mathbb{H}^2_L$ be a $C^2$-smooth spacelike curve, then $J_L(\dot{\gamma}) := \lambda_1 e_2 + \lambda_2 e_1$. Then $\langle \dot{\gamma}, J_L(\dot{\gamma}) \rangle = 0$ and $\langle \dot{\gamma}, J_L(\dot{\gamma}) \rangle$ has the same orientation with $\{e_1, e_2\}$.

For every $U, V \in T\Sigma$, we define $\nabla_{\Sigma,L} V = \pi \nabla_{\Sigma,L} V$ where $\pi : T\mathbb{H}^2_L \to T\Sigma$ is the projection. Then $\nabla_{\Sigma,L}$ is the Levi–Civita connection on $\Sigma$ with respect to the metric $g_L$. By (2.14), (3.2) and

$$\nabla_{\dot{\gamma}} V = \langle \nabla_{\dot{\gamma}} V, e_1 \rangle e_1 - \langle \nabla_{\dot{\gamma}} V, e_2 \rangle e_2,$$

(3.3)
we have
\[
\nabla_{\dot{\gamma}_L}^\Sigma \dot{\gamma} = \{ \overline{\gamma}_1 + L \dot{\gamma}_2 \omega(\dot{\gamma}(t)) \} - \overline{\gamma}_2 - L \dot{\gamma}_1 \omega(\dot{\gamma}(t)) \} e_1
\]
\[
- \left( \overline{\tau} \p [\overline{\gamma}_1 + L \dot{\gamma}_2 \omega(\dot{\gamma}(t))] + \overline{\tau} \bar{\omega}[\overline{\gamma}_2 - L \dot{\gamma}_1 \omega(\dot{\gamma}(t))] \right)
\]
\[
+ \frac{1}{L^2} \left( \frac{d}{dt} \omega(\dot{\gamma}(t)) \right) e_2.
\]
Moreover if \( \omega(\dot{\gamma}(t)) = 0 \), then
\[
\nabla_{\dot{\gamma}_L}^\Sigma \dot{\gamma} = [\overline{\gamma}_1 - \overline{\gamma}_2] e_1 - \left( \overline{\tau} \p \overline{\gamma}_1 + \overline{\tau} \bar{\omega}[\overline{\gamma}_2 + \frac{L^2}{L} \left( \frac{d}{dt} \omega(\dot{\gamma}(t)) \right) \right) e_2
\]

**Definition 3.1** Let \( \Sigma \subset \mathbb{H}^2_L \) be a regular Lorentzian surface. Let \( \gamma : I \to \Sigma \) be a \( C^2 \)-smooth regular curve.

(1) If \( \nabla_{\dot{\gamma}_L}^\Sigma \dot{\gamma} \) is spacelike vectors, the geodesic curvature \( k_{\dot{\gamma}}^{L, \Sigma} \) of \( \gamma \) at \( \dot{\gamma}(t) \) is defined as
\[
k_{\dot{\gamma}}^{L, \Sigma} := \sqrt{\frac{||\nabla_{\dot{\gamma}_L}^\Sigma \dot{\gamma}||^2_{\Sigma,L}}{||\dot{\gamma}||^2_{\Sigma,L} - \frac{\langle \nabla_{\dot{\gamma}_L}^\Sigma \dot{\gamma}, \dot{\gamma} \rangle^2_{\Sigma,L}}{\langle \dot{\gamma}, \dot{\gamma} \rangle^2_{\Sigma,L}}}.
\]

(2) If \( \nabla_{\dot{\gamma}_L}^\Sigma \dot{\gamma} \) is timelike vectors, the curvature \( k_{\dot{\gamma}}^{L} \) of \( \gamma \) at \( \dot{\gamma}(t) \) is defined as
\[
k_{\dot{\gamma}}^{L, \Sigma} := \sqrt{\frac{||\nabla_{\dot{\gamma}_L}^\Sigma \dot{\gamma}||^2_{\Sigma,L}}{||\dot{\gamma}||^2_{\Sigma,L} + \frac{\langle \nabla_{\dot{\gamma}_L}^\Sigma \dot{\gamma}, \dot{\gamma} \rangle^2_{\Sigma,L}}{\langle \dot{\gamma}, \dot{\gamma} \rangle^2_{\Sigma,L}}}.
\]

**Definition 3.2** Let \( \Sigma \subset \mathbb{H}^2_L \) be a regular Lorentzian surface. Let \( \gamma : I \to \Sigma \) be a \( C^2 \)-smooth regular curve.

We define the intrinsic geodesic curvature \( k_{\dot{\gamma}}^{\infty, \Sigma} \) of \( \gamma \) at \( \dot{\gamma}(t) \) to be
\[
k_{\dot{\gamma}}^{\infty, \Sigma} := \lim_{L \to +\infty} k_{\dot{\gamma}}^{L, \Sigma},
\]
if the limit exists.

**Lemma 3.3** Let \( \Sigma \subset \mathbb{H}^2_L \) be a regular Lorentzian surface. Let \( \gamma : I \to \Sigma \) be a \( C^2 \)-smooth regular curve.

(1) If \( \nabla_{\dot{\gamma}_L}^\Sigma \dot{\gamma} \) is a spacelike vector, then
\[
k_{\dot{\gamma}}^{\infty, \Sigma} = \frac{||\overline{\gamma}_1 - \overline{\gamma}_2||}{||\omega(\dot{\gamma}(t))||}, \text{ if } \omega(\dot{\gamma}(t)) \neq 0;
\]
\[
k_{\dot{\gamma}}^{\infty, \Sigma} = 0, \text{ if } \omega(\dot{\gamma}(t)) = 0 \text{ and } \frac{d}{dt} \omega(\dot{\gamma}(t)) = 0;
\]
\[
\lim_{L \to +\infty} k_{\dot{\gamma}}^{L, \Sigma} = \sqrt{\frac{\frac{d}{dt} \omega(\dot{\gamma}(t))^2}{(\overline{\gamma}_1 - \overline{\gamma}_2)^2}}, \text{ if } \omega(\dot{\gamma}(t)) = 0 \text{ and } \frac{d}{dt} \omega(\dot{\gamma}(t)) \neq 0.
\]
Therefore, this situation does not exist.

(2) If $\nabla_{^\gamma}^\Sigma L\gamma$ is a timelike vector, then

$$k_{^\gamma,\Sigma}^\infty = \frac{\sqrt{- (p\gamma_1 + q\gamma_2)^2}}{|\omega(\dot{\gamma}(t))|}, \text{ if } \omega(\dot{\gamma}(t)) \neq 0;$$

(3.11)

Therefore, this situation does not exist.

$$k_{^\gamma,\Sigma}^\infty = 0, \text{ if } \omega(\dot{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0;$$

(3.12)

$$\lim_{L \to +\infty} \frac{k_{^\gamma,\Sigma}^L}{\sqrt{L}} = \left| \frac{\frac{d}{dt}(\omega(\dot{\gamma}(t)))}{(p\gamma_1 - q\gamma_2)^2} \right|, \text{ if } \omega(\dot{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \quad (3.13)$$

Proof  By (2.12) and $\dot{\gamma} \in T\Sigma$, we have

$$\dot{\gamma} = (q\gamma_1 - p\gamma_2)e_1 - \frac{l_L}{L} L^2 \omega(\dot{\gamma}(t)) e_2. \quad (3.14)$$

By (3.4), we have

$$\langle \nabla_{^\gamma}^\Sigma L\gamma, \nabla_{^\gamma}^\Sigma L\gamma \rangle_{L,\Sigma} = \{ \bar{q} [\gamma_1 + L\gamma_2 \omega(\dot{\gamma}(t))] - \bar{p} [\gamma_2 - L\gamma_1 \omega(\dot{\gamma}(t))] \}^2$$

$$\{ \frac{r_L}{r_L} p [\gamma_1 - L\gamma_2 \omega(\dot{\gamma}(t))] + \bar{r}_L \bar{q} [\gamma_2 - L\gamma_1 \omega(\dot{\gamma}(t))]$$

$$+ \frac{l_L}{l_L} L^2 \left[ \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right] \}^2. \quad (3.15)$$

Similarly, we have that when $\omega(\dot{\gamma}(t)) \neq 0$,

$$\langle \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma, L} = (\bar{q}\gamma_1 - \bar{p}\gamma_2)^2 - \left[ \frac{l_L}{L} L^2 \omega(\dot{\gamma}(t)) \right]^2 \sim - L |\omega(\dot{\gamma}(t))|^2, \text{ as } L \to +\infty. \quad (3.16)$$

By (3.4) and (3.14), we have

$$\langle \nabla_{^\gamma}^\Sigma L\gamma, \dot{\gamma} \rangle_{\Sigma, L} = (\bar{q}\gamma_1 - \bar{p}\gamma_2) \cdot (\bar{q} [\gamma_1 + L\gamma_2 \omega(\dot{\gamma}(t))]$$

$$- \bar{p} [\gamma_2 - L\gamma_1 \omega(\dot{\gamma}(t))] - \left[ \frac{l_L}{L} L^2 \omega(\dot{\gamma}(t)) \right] \cdot \{ \frac{r_L}{r_L} p [\gamma_1 + L\gamma_2 \omega(\dot{\gamma}(t))]$$

$$+ \frac{l_L}{l_L} L^2 \left[ \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right] \} \sim M_0 L, \quad (3.17)$$

where $M_0$ does not depend on $L$. By Definition 3.1, (3.15)–(3.17), we get (3.8).

When $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$, we have

$$\langle \nabla_{^\gamma}^\Sigma L\gamma, \nabla_{^\gamma}^\Sigma L\gamma \rangle_{L,\Sigma} = (\bar{q}\gamma_1 - \bar{p}\gamma_2)^2 + \left( \frac{r_L}{r_L} p \gamma_1 + \frac{r_L}{r_L} q \gamma_2 \right)^2$$

$$\sim (\bar{q}\gamma_1 - \bar{p}\gamma_2)^2, \text{ as } L \to +\infty. \quad (3.18)$$
\begin{align*}
\langle \gamma, \dot{\gamma} \rangle_{\Sigma, L} &= (p \dot{\gamma}_1 - q \dot{\gamma}_2)^2,
\langle \nabla^L \gamma, \dot{\gamma} \rangle_{\Sigma, L} &= (p \dot{\gamma}_1 - q \dot{\gamma}_2) \cdot (p \ddot{\gamma}_1 - q \ddot{\gamma}_2)
\end{align*}
(3.19)

By (3.18)–(3.20) and Definition 3.1, we get \( k_{\gamma, L}^\infty = 0 \).

When \( \omega(\dot{\gamma}(t)) = 0 \) and \( \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0 \), we have

\begin{align*}
\langle \nabla^L \gamma, \nabla^L \dot{\gamma} \rangle_{L, \Sigma} &\sim - \frac{d}{dt}(\omega(\dot{\gamma}(t)))^2,
\langle \nabla^L \gamma, \dot{\gamma} \rangle_{\Sigma, L} &= O(1),
\end{align*}
so we get (3.10).

Definition 3.4 Let \( \Sigma \subset \mathbb{H}^2_L \) be a regular surface. Let \( \gamma : I \rightarrow \Sigma \) be a \( C^2 \)-smooth regular curve. The signed geodesic curvature \( k_{\gamma, \Sigma}^{L,s} \) of \( \gamma \) at \( \gamma(t) \) is defined as

\[ k_{\gamma, \Sigma}^{L,s} := \frac{\langle \nabla^L \gamma, J_L(\dot{\gamma}(t)) \rangle_{\Sigma, L}}{||\dot{\gamma}||_{\Sigma, L}^3}. \]
(3.21)

Definition 3.5 Let \( \Sigma \subset \mathbb{H}^2_L \) be a regular surface. Let \( \gamma : [a, b] \rightarrow \Sigma \) be a \( C^2 \)-smooth regular curve. We define the intrinsic geodesic curvature \( k_{\gamma, \Sigma}^\infty \) of \( \gamma \) at the noncharacteristic point \( \gamma(t) \) to be

\[ k_{\gamma, \Sigma}^\infty := \lim_{L \rightarrow +\infty} k_{\gamma, \Sigma}^{L,s}, \]

if the limit exists.

Lemma 3.6 Let \( \Sigma \subset \mathbb{H}^2_L \) be a regular Lorentzian surface.

(1) If \( \gamma : I \rightarrow \Sigma \) be a spacelike \( C^2 \)-smooth regular curve, then

\[ k_{\gamma, \Sigma}^\infty = - \frac{p \dot{\gamma}_1 + q \dot{\gamma}_2}{\sqrt{-\omega(\dot{\gamma}(t))^2}}, \text{ if } \omega(\dot{\gamma}(t)) \neq 0. \]
(3.22)

Therefore, the situation does not exist.

\[ k_{\gamma, \Sigma}^\infty = 0, \text{ if } \omega(\dot{\gamma}(t)) = 0, \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0, \]
(3.23)

\[ \lim_{L \rightarrow +\infty} \frac{k_{\gamma, \Sigma}^{L,s}}{\sqrt{L}} = \frac{(p \dot{\gamma}_1 - q \dot{\gamma}_2)(\frac{d}{dt}(\omega(\dot{\gamma}(t))))}{(p \dot{\gamma}_1 + q \dot{\gamma}_2)^3}, \text{ if } \omega(\dot{\gamma}(t)) = 0, \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \]
(3.24)

(2) If \( \gamma : I \rightarrow \Sigma \) be a timelike \( C^2 \)-smooth regular curve, then

\[ k_{\gamma, \Sigma}^\infty = - \frac{p \dot{\gamma}_1 + q \dot{\gamma}_2}{|\omega(\dot{\gamma}(t))|}, \text{ if } \omega(\dot{\gamma}(t)) \neq 0. \]
(3.25)
Theorem 3.7

We define the second fundamental form \( II \) of the embedding of \( \Sigma \) into \( \mathbb{H}^2_L \):

\[
II^L = \begin{pmatrix}
\langle \nabla^L_{v_L} v_L, e_1 \rangle_L & \langle \nabla^L_{v_L} v_L, e_2 \rangle_L \\
\langle \nabla^L_{e_2 v_L} v_L, e_1 \rangle_L & \langle \nabla^L_{e_2 v_L} v_L, e_2 \rangle_L
\end{pmatrix}.
\]

Similarly to Theorem 4.3 in [5], we have

**Theorem 3.7** The second fundamental form \( II^L \) of the embedding of \( \Sigma \) into \( \mathbb{H}^2_L \) is given by

\[
II^L = \begin{pmatrix}
\frac{1}{p_L} [X_1(\overline{p})] + X_2(\overline{q}), & -\frac{l_L}{p_L} (e_1, \nabla_H(\overline{p_L})) - \frac{\sqrt{l}}{p_L} \\
-\frac{l_L}{p_L} (e_1, \nabla_H(\overline{q_L})) - \frac{\sqrt{l}}{p_L} & -\frac{l_L}{p_L} (e_2, \nabla_H(\overline{q_L})) L + \overline{X_3}(\overline{p_L})
\end{pmatrix}.
\]
The mean curvature $H_L$ of $\Sigma$ is defined by

$$H_L := \text{tr}(II_L).$$

Define the curvature of a connection $\nabla$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$ 

(3.34)

Let

$$K^\Sigma_L(e_1,e_2) = \langle R^\Sigma_L(e_1,e_2)e_1,e_2 \rangle_{\Sigma,L}, \quad K^L(e_1,e_2) = \langle R^L(e_1,e_2)e_1,e_2 \rangle_L.$$ 

(3.35)

By the Gauss equation, we have

$$K^\Sigma_L(e_1,e_2) = K^L(e_1,e_2) + \det(II_L).$$ 

(3.36)

Proposition 3.8 Away from characteristic points, the horizontal mean curvature $H_\infty$ of $\Sigma \subset \mathbb{H}$ is given by

$$H_\infty = \lim_{L \to +\infty} H_L = X_1(p) + X_2(\bar{q}).$$ 

(3.37)

Proof By

$$\frac{l^2}{l_L^2} \langle e_2, \nabla_H(\frac{r}{l}) \rangle_L = \frac{\bar{p}r}{l} X_1(\bar{r}l) + \frac{\bar{q}r}{l} X_2(\bar{r}l) \to O(L^{-1})$$

we get (3.37).

By Lemma 2.1 and (3.35), we have

Lemma 3.9 Let $\mathbb{H}_L^2$ be the Lorentzian Heisenberg space, then

$$R^L(X_1,X_2)X_1 = -\frac{3}{4}LX_2, \quad R^L(X_1,X_2)X_2 = \frac{3}{4}LX_1, \quad R^L(X_1,X_2)X_3 = 0,$$

(3.38)

$$R^L(X_1,X_3)X_1 = \frac{1}{4}LX_3, \quad R^L(X_1,X_3)X_2 = 0, \quad R^L(X_1,X_3)X_3 = \frac{L^2}{4}X_1,$$

$$R^L(X_2,X_3)X_1 = 0, \quad R^L(X_2,X_3)X_2 = \frac{L}{4}X_3, \quad R^L(X_2,X_3)X_3 = \frac{L^2}{4}X_2.$$

Proposition 3.10 Away from characteristic points, we have

$$K^\Sigma_L(e_1,e_2) \to A_0 + O(L^{-1}), \quad \text{as} \quad L \to +\infty,$$

(3.39)

where

$$A_0 := -\langle e_1, \nabla_H(\frac{X_3u}{|\nabla_H u|}) + \frac{(X_3u)^2}{l^2} \rangle.$$ 

(3.40)
Proof By (3.2), we have
\begin{align*}
\langle R^L(e_1,e_2)e_1,e_2 \rangle_L &= \langle \bar{r}^L(R^L(X_1,X_2)X_1,X_2)_{\bar{L}} \rangle + 2\frac{l}{L^2} \langle \bar{q}^L(R^L(X_1,X_2)X_2,X_3)_{\bar{L}} \rangle + \frac{l^2}{L^4} \langle \bar{q}^L(R^L(X_1,X_3)X_1,X_3)_{\bar{L}} \rangle + 2\frac{l}{L^2} \langle \bar{r}^L(R^L(X_2,X_3)X_2,X_3)_{\bar{L}} \rangle.
\end{align*}

By Lemma 3.9, we have
\begin{align*}
\mathcal{K}^L(e_1,e_2) &= \frac{L}{4}(\frac{l}{L^2})^2 + 3\frac{L}{4}\bar{r}^L^2.
\end{align*}

By (3.33) and \(\nabla_H(\bar{r}_L) = L^{-\frac{1}{2}}\nabla_H(\frac{X_3u}{|\nabla_H u|}) + O(L^{-1})\) as \(L \to +\infty\), we get
\begin{align*}
det(II^L) &= \frac{L}{4} - \langle e_1, \nabla_H(\frac{X_3u}{|\nabla_H u|}) \rangle + O(L^{-1}) \quad \text{as } L \to +\infty.
\end{align*}

By (3.33),(3.42),(3.43) we get (3.39). \(\square\)

Let us first consider the case of a timelike curve \(\gamma : I \to \mathbb{H}^2_L\). We define the Riemannian length measure \(ds = ||\dot{\gamma}||_Ldt\).

Lemma 3.11 Let \(\gamma : I \to \mathbb{H}^2_L\) be a \(C^2\)-smooth timelike curve. Let
\begin{align*}
ds := |\omega(\dot{\gamma}(t))|dt, \quad \bar{d}s := \frac{1}{2|\omega(\dot{\gamma}(t))|} \left( -\dot{\gamma}_1^2 - \dot{\gamma}_2^2 \right) dt.
\end{align*}

Then
\begin{align*}
\lim_{L \to +\infty} \frac{1}{\sqrt{L}} \int_\gamma \bar{d}s_L = \int_a^b ds.
\end{align*}

When \(\omega(\dot{\gamma}(t)) \neq 0\), we have
\begin{align*}
\frac{1}{\sqrt{L}} ds_L = ds + dsL^{-1} + O(L^{-2}) \quad \text{as } L \to +\infty.
\end{align*}

When \(\omega(\dot{\gamma}(t)) = 0\), we have
\begin{align*}
\frac{1}{\sqrt{L}} ds_L = \frac{1}{\sqrt{L}} \sqrt{-\dot{\gamma}_1^2 - \dot{\gamma}_2^2}.
\end{align*}

Therefore, the situation does not exist.
Proof We know that \( ||\dot{\gamma}(t)||_L = \sqrt{-\dot{\gamma}_1^2 - \dot{\gamma}_2^2 + \omega(\dot{\gamma}(t))^2} \), similar to the proof of Lemma 6.1 in [11], we can prove (3.45). When \( \omega(\dot{\gamma}(t)) \neq 0 \), we have

\[
\frac{1}{\sqrt{L}} ds_L = \sqrt{L^{-1} (-\dot{\gamma}_1^2 - \dot{\gamma}_2^2) + \omega(\dot{\gamma}(t))^2} dt.
\]

Using the Taylor expansion, we can prove (3.45). From the definition of \( ds_L \), and \( \omega(\dot{\gamma}(t)) = 0 \), we get (3.47).

\[ \square \]

Proposition 3.12 Let \( \Sigma \subset \mathbb{H}_L^2 \) be a regular Lorentzian \( C^2 \)-smooth surface. Let \( d\sigma_L \) denote the surface measure on \( \Sigma \) with respect to the Riemannian metric \( g_L \). Let

\[
d\sigma_{\Sigma} := (\overline{p}_\omega - \overline{q}_\omega) \wedge \omega, \quad d\sigma_{\Sigma} := \frac{X_3 u}{l} \omega_1 \wedge \omega_2 + \frac{(X_3 u)^2}{2l^2} (\overline{p}_\omega - \overline{q}_\omega) \wedge \omega.
\]

Then

\[
\frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} = d\sigma_{\Sigma} + d\sigma_{\Sigma} L^{-1} + O(L^{-2}), \quad \text{as} \quad L \to +\infty.
\]

If \( \Sigma = f(D) \) with

\[
f = f(u_1, u_2) = (f_1, f_2, f_3) : D \subset \mathbb{R}^2 \to \mathbb{H}_L^2,
\]

then

\[
\lim_{L \to +\infty} \frac{1}{\sqrt{L}} \int_{\Sigma} d\sigma_{\Sigma,L} = \int_{D} \left\{ \left[ (f_2 u_1 (f_3 u_2 - (f_3) u_1 (f_2) u_2) \right]^2 
+ \left[ (f_1 u_2 (f_2) u_1 - (f_2) u_2 (f_1) u_1) f_1 + (f_3) u_1 (f_1) u_2 - (f_1) u_1 (f_3) u_2 \right]^2 \right]^\frac{1}{2} \right\}^\frac{1}{2} du_1 du_2.
\]

Proof We know that

\[
g_L(X_1, \cdot) = \omega_1, \quad g_L(X_2, \cdot) = \omega_2, \quad g_L(X_3, \cdot) = -L\omega,
\]

so

\[
e_1^* = g_L(e_1, \cdot) = \overline{q}_\omega - \overline{p}_\omega, \quad e_2^* = g_L(e_2, \cdot) = \overline{r}_L \overline{p}_\omega + \overline{r}_L \overline{q}_\omega - \frac{l}{l_L} L^\frac{3}{2} \omega.
\]

Then

\[
\frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} = \frac{1}{\sqrt{L}} e_1^* \wedge e_2^* = \frac{l}{l_L} (\overline{p}_\omega - \overline{q}_\omega) \wedge \omega + \frac{1}{\sqrt{L}} \overline{r}_L \omega_1 \wedge \omega_2.
\]

By

\[
\overline{r}_L = \frac{(X_3 u) L^{\frac{3}{2}}}{\sqrt{p^2 + q^2 - L^{-1} (X_3 u)^2}}
\]

and the Taylor expansion

\[
\frac{1}{l_L} = \frac{1}{l} - \frac{1}{2l^3} (X_3 u)^2 L^{-1} + O(L^{-2}) \quad \text{as} \quad L \to +\infty
\]

we get (3.55). By (2.2), we have

\[
f_{u_1} = (f_1)_{u_1} \partial_{x_1} + (f_2)_{u_1} \partial_{x_2} + (f_3)_{u_1} \partial_{x_3}
= (f_1)_{u_1} X_1 + (f_2)_{u_1} X_2 + \sqrt{L} [f_1 (f_2)_{u_1} + (f_3)_{u_1}] \tilde{X}_3,
\]

730
and
\[ f_{u_2} = (f_1)_{u_2}^X + (f_2)_{u_2}X_2 + \sqrt{L} [f_1(f_2)_{u_2} + (f_3)_{u_2}]X_3. \] (3.54)
Let
\[ \nabla_L = \begin{pmatrix} X_1, & X_2, & -\hat{X}_3 \\ (f_1)_{u_1}, & (f_2)_{u_1}, & \sqrt{L} [f_1(f_2)_{u_1} + (f_3)_{u_1}] \\ (f_1)_{u_2}, & (f_2)_{u_2}, & \sqrt{L} [f_1(f_2)_{u_2} + (f_3)_{u_2}] \end{pmatrix}. \] (3.55)
We know that
\[ d\sigma_{\Sigma,L} = -\det(g_{ij}) du_1 du_2, \quad g_{ij} = g_L(f_u, f_w), \quad \det(g_{ij}) = ||\nabla_L||_L^2 = -(\nabla, \nabla), \]
so by the dominated convergence theorem, we get (3.51).

**Theorem 3.13** Let \( \Sigma \subset \mathbb{H}_L^2 \) be a regular Lorentzian surface with finitely many boundary components \((\partial \Sigma)_i, i \in \{1, \ldots, n\}\), given by \( C^2 \)-smooth regular and closed timelike curves \( \gamma_i : [0, 2\pi] \to (\partial \Sigma)_i \). Suppose that the characteristic set \( C(\Sigma) \) satisfies \( \mathcal{H}^1(C(\Sigma)) = 0 \) and that \( ||\nabla_H u||_H^{-1} \) is locally summable with respect to the 2-dimensional Hausdorff measure near the characteristic set \( C(\Sigma) \). Let \( A_0 \) be defined by (3.40) and \( d\sigma_\Sigma \) be defined by (3.48) and \( k_{\gamma,\Sigma}^{\infty,s} \) be the sub-Riemannian signed geodesic curvature of \( \gamma \) relative to \( \Sigma \). Then we have
\[
\int_{\Sigma} A_0 d\sigma_{\Sigma} + \sum_{i=1}^{n} \int_{\gamma_i} k_{\gamma_i,\Sigma}^{\infty,s} ds = 0. \] (3.56)

**Proof** Using the discussions in [2], we may assume that all points satisfy \( \omega(\gamma_i(t)) \neq 0 \) on \( \gamma_i \). Then by Lemma 3.6, we have
\[
k_{\gamma,\Sigma}^{L,s} = k_{\gamma,\Sigma}^{\infty,s} + O(L^{-1}). \] (3.57)
By the Gauss-Bonnet theorem (see [8] page 90 Theorem 1.4), we have
\[
\int_{\Sigma} k_{\gamma,\Sigma}^{\infty,s} \frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} + \sum_{i=1}^{n} \int_{\gamma_i} k_{\gamma_i,\Sigma}^{L,s} \frac{1}{\sqrt{L}} ds_L = 0. \] (3.58)
So by (3.57),(3.58),(3.40),(3.48),(3.49), we get
\[
\left( \int_{\Sigma} A_0 d\sigma_{\Sigma} + \sum_{i=1}^{n} \int_{\gamma_i} k_{\gamma_i,\Sigma}^{\infty,s} ds \right) + O(L^{-\frac{1}{2}}) = 0. \] (3.59)
Let \( L \) go to the infinity and using the dominated convergence theorem, then we get (3.56).

**Example 3.14** In \( \mathbb{H}_L^2 \), let \( u = x_1^2 + x_2^2 + x_3^2 - 1 \) and \( \sum = S^2 \cdot \sum \) is a regular surface. By (2.1), we get
\[
X_1(u) = 2x_1; \quad X_2(u) = 2x_2 - 2x_1x_3. \] (3.60)
Solve the equations \( X_1(u) = X_2(u) = 0 \), then we get
\[
C(\Sigma) = \{(0, 0, 1), (0, 0, -1)\}
\]
and \( H^1(C(\Sigma)) = 0 \).

A parametrization of \( \Sigma \) is

\[
\begin{align*}
x_1 &= \cos(\varphi)\cos(\theta), \quad x_2 = \cos(\varphi)\sin(\theta), \\
x_3 &= \sin(\varphi), \quad \text{for} \quad \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}), \quad \theta \in [0,2\pi).
\end{align*}
\]

Then

\[
\| \nabla_H u \|^2_H = X_1(u)^2 + X_2(u)^2 = 4x_1^2 + 4(x_2 - x_1x_3)^2 = 4\cos(\varphi)^2 + 4\cos(\varphi)^2\sin(\varphi)^2 - 8\cos(\varphi)^2\cos(\theta)^2 - 8\cos(\varphi)^2\sin(\varphi)\sin(\theta).
\]

By the definitions of \( w_j \) for \( 1 \leq j \leq 3 \) and (3.48), we have

\[
d\sigma_{\Sigma} = \frac{1}{\| \nabla_H u \|^2_H} \left[ (X_1(u))dx_2 - (X_2(u))dx_1 \right] \wedge (x_1dx_2 + dx_3) = \frac{1}{\| \nabla_H u \|^2_H} 2\cos(\varphi)^3 \left[ 1 - 2\sin(\theta)\cos(\varphi) + \cos(\theta)^2\sin(\varphi)^2 \right] d\theta \wedge d\varphi.
\]

By (3.62) and (3.63), we have \( \| \nabla_H u \|^2_H^{-1} \) is locally summable around the isolated characteristic points with respect to the measure \( d\sigma_{\Sigma} \).

### 4. The sub-Riemannian limit of curvature of curves in the Lorentzian Heisenberg group for the third Lorentzian metric

We consider the Lorentzian Heisenberg group for the third lorentzian metric, let \( \mathbb{H} \) be the Heisenberg group where the noncommutative group law is given by

\[
(\overline{x}, \overline{y}, \overline{z}) \ast (x, y, z) = (\overline{x} + x, \overline{y} + y, \overline{z} + z - \overline{xy} + x\overline{y}).
\]

Let

\[
X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2} + (1 - x_1)\partial_{x_3}, \quad X_3 = \partial_{x_2} - x_1\partial_{x_3}.
\]

Then

\[
\partial_{x_1} = X_1, \quad \partial_{x_2} = x_1X_2 + (1 - x_1)X_3, \quad \partial_{x_3} = X_2 - X_3,
\]

and span\{\(X_1, X_2, X_3\)\} = \(T\mathbb{H}\). Let \( H = \text{span}\{X_1, X_2\} \) be the horizontal distribution on \( \mathbb{H} \). Let \( \omega_1 = dx_1, \quad \omega_2 = x_1dx_2 + dx_3, \quad \omega = (1 - x_1)dx_2 - dx_3 \). Then \( H = \text{Ker}\omega \). For the constant \( L > 0 \), let \( g_L = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 - L\omega \otimes \omega, \quad g = g_1 \) be the third Lorentzian metric on \( \mathbb{H} \) (see [8]). We call \((\mathbb{H}, g_L)\) a Lorentzian Heisenberg group for the Lorentzian third metric and write \( \mathbb{H}^L_3 \) instead of \((\mathbb{H}, g_L)\). Then \( X_1, X_2, X_3 := L^{-\frac{1}{2}}X_3 \) are orthonormal basis on \( T\mathbb{H}^L_3 \) with respect to \( g_L \). We have

\[
[X_1, X_2] = X_3 - X_2, \quad [X_2, X_3] = 0, \quad [X_1, X_3] = X_3 - X_2.
\]

We say that a nonzero vector \( x \in \mathbb{H}^L_3 \) is spacelike, null or timelike if \( \langle x, x \rangle > 0 \), \( \langle x, x \rangle = 0 \) or \( \langle x, x \rangle < 0 \) respectively. The norm of the vector \( x \in \mathbb{H}^L_3 \) is defined by \( \| x \| = \sqrt{|\langle x, x \rangle|} \).
Let $\gamma : I \rightarrow \mathbb{H}_L^3$ be a regular curve, where $I$ is an open interval in $\mathbb{R}$. The regular curve $\gamma$ is called a spacelike curve, timelike curve or null curve if $\gamma'(t)$ is a spacelike vector, timelike vector or null vector at any $t \in I$ respectively.

Let $\nabla^L$ be the Levi–Civita connection on $\mathbb{H}_L^3$ with respect to $g_L$. By the Koszul formula and (4.3), we have

**Lemma 4.1** Let $\mathbb{H}_L^3$ be the Lorentzian Heisenberg group for the third metric, then

\[
\begin{align*}
\nabla^L_{X_1} X_1 &= 0, & \nabla^L_{X_1} X_3 &= \frac{L-1}{2} X_2, \nabla^L_{X_2} X_2 = -X_1, \\
\nabla^L_{X_1} X_2 &= \frac{L-1}{2 L} X_3, & \nabla^L_{X_2} X_1 &= \frac{-L-1}{2 L} X_3 + X_2, & \nabla^L_{X_3} X_3 &= -LX_1, \\
\nabla^L_{X_3} X_1 &= \frac{1+L}{2} X_2 - X_3, & \nabla^L_{X_2} X_3 &= \nabla^L_{X_3} X_2 = -\frac{1+L}{2} X_1.
\end{align*}
\]

**Definition 4.2** Let $\gamma : I \rightarrow \mathbb{H}_L^3$ be a $C^1$-smooth curve. We say that $\gamma(t)$ is a horizontal point of $\gamma$ if

\[
\omega(\dot{\gamma}(t)) = (1 - \gamma_1(t))\gamma_2(t) - \gamma_3(t) = 0.
\]

Similar to the Definition 2.3 and Definition 2.5, we can define $k^L_\gamma$ and $k^\infty_\gamma$ for the Lorentzian Heisenberg group for the third Lorentzian metric, we have

**Lemma 4.3** Let $\gamma : I \rightarrow \mathbb{H}_L^3$ be a $C^2$-smooth regular curve.

(1) If $\nabla^L_{\dot{\gamma}} \dot{\gamma}$ is a spacelike vector, then

\[
k^\infty_\gamma = \frac{\sqrt{\gamma_1^2 + \gamma_2^2}}{|\omega(\dot{\gamma}(t))|}, \quad \text{if } \omega(\dot{\gamma}(t)) \neq 0,
\]

(4.5)

\[
k^\infty_\gamma = \left\{ \begin{array}{l}
\left[ \gamma_1 - \gamma_1^2 \gamma_2 - \gamma_2^2 - 2 \gamma_1 \gamma_2 \gamma_3 \right]^2 + [\gamma_1 \gamma_1 \gamma_2 + \gamma_1 \gamma_2 \gamma_3 + \gamma_1 \gamma_2 + \gamma_1 \gamma_2 + \gamma_3]^2 \\
\left\{ \gamma_1^2 + (\gamma_1 \gamma_2 + \gamma_3)^2 \right\}^2
\end{array} \right.
\]

(4.6)

\[
- \left\{ \gamma_1 \left[ \gamma_1 - \gamma_1^2 \gamma_2 - \gamma_2^2 - 2 \gamma_1 \gamma_2 \gamma_3 \right] + (\gamma_1 \gamma_2 + \gamma_3) \left[ \gamma_1 \gamma_1 \gamma_2 + \gamma_1 \gamma_2 \gamma_3 + \gamma_1 \gamma_2 + \gamma_1 \gamma_2 + \gamma_3 \right] \right\}^2
\]

\[
\left\{ \gamma_1^2 + (\gamma_1 \gamma_2 + \gamma_3)^2 \right\}^3
\]

\[
\text{if } \omega(\dot{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0
\]

\[
\lim_{L \to +\infty} \frac{k^L_\gamma}{\sqrt{L}} = \sqrt{\frac{-\frac{d}{dt}(\omega(\dot{\gamma}(t)))}{|\gamma_1^2 + (\gamma_1 \gamma_2 + \gamma_3)^2|}}, \quad \text{if } \omega(\dot{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0.
\]

(4.7)

Therefore, the situation does not exist.

(2) If $\nabla^L_{\dot{\gamma}} \dot{\gamma}$ is a timelike vector, then

\[
k^\infty_\gamma = \frac{\sqrt{-\gamma_1^2 - \gamma_2^2}}{|\omega(\dot{\gamma}(t))|}, \quad \text{if } \omega(\dot{\gamma}(t)) \neq 0,
\]

(4.8)
\[
\lim_{L \to +\infty} k_{\gamma_1}^L \frac{\rho}{|\bar{\gamma}|} = \frac{|d\bar{\gamma}/dt(\bar{\gamma}(t))|}{|2\gamma_1 - \bar{p}\gamma_2 + \gamma_3|^2}, \quad \text{if} \quad \omega(\bar{\gamma}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\bar{\gamma}(t))) \neq 0.
\]
(4.9)

Therefore, the situation does not exist.

\[
k_{\gamma_1}^\infty = \left\{ \begin{array}{l}
- \left[ \gamma_1 - \gamma_2 \gamma_2 - \gamma_3^2 - 2\gamma_1 \gamma_2 \gamma_3 \right]^2 + \left[ \gamma_1 \gamma_2 + \gamma_3 \gamma_2 + \gamma_1 \gamma_2 + \gamma_2 \gamma_2 + \gamma_3 \right]^2 \\
\left\{ \gamma_1^2 + (\gamma_1 \gamma_2 + \gamma_3)^2 \right\}^2
\end{array} \right.
\]
(4.10)

if \( \omega(\gamma(t)) = 0 \) and \( \frac{d}{dt}(\omega(\gamma(t))) = 0 \),

\[
\lim_{L \to +\infty} k_{\gamma_1}^L \frac{\rho}{|\bar{\gamma}|} = \frac{|d\bar{\gamma}/dt(\bar{\gamma}(t))|}{|\bar{\gamma}|^2 + (\bar{\gamma}_3 + \gamma_3)^2}, \quad \text{if} \quad \omega(\bar{\gamma}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\bar{\gamma}(t))) \neq 0.
\]
(4.11)

**Proof** By (4.2), we have

\[
\bar{\gamma}(t) = \gamma_1 X_1 + (\gamma_1 \gamma_2 + \gamma_3) X_2 + \omega(\gamma(t)) X_3.
\]
(4.12)

By Lemma 4.1 and (4.12), we have

\[
\nabla_\gamma^L X_1 = \left[ \gamma_1 \gamma_2 + \gamma_3 + \frac{L + 1}{2} \omega(\gamma(t)) \right] X_2 - \left[ \frac{L + 1}{2L} (\gamma_1 \gamma_2 + \gamma_3) + \omega(\gamma(t)) \right] X_3,
\]
(4.13)

\[
\nabla_\gamma^L X_2 = - \left[ \gamma_1 \gamma_2 + \gamma_3 + \frac{L + 1}{2} \omega(\gamma(t)) \right] X_1 + \frac{L - 1}{2L} \gamma_1 X_3,
\]

\[
\nabla_\gamma^L X_3 = \left[ \frac{L - 1}{2} \gamma_3 + \frac{L - 1}{2} \gamma_1 \gamma_2 - L \gamma_2 \right] X_1 + \frac{L - 1}{2} \gamma_1 X_2.
\]

By (4.12) and (4.13), we have

\[
\nabla^L \bar{\gamma} = \left[ \gamma_1 - \gamma_2 \gamma_2 - \gamma_3^2 - 2\gamma_1 \gamma_2 \gamma_3 + (-L \gamma_2 - \gamma_1 \gamma_2 - \gamma_3) \omega(\gamma(t)) \right] X_1
\]

\[
+ \left[ \gamma_1 \gamma_2 + \gamma_1 \gamma_3 + \gamma_1 \gamma_2 + \gamma_2 \gamma_1 + \gamma_3 + L \gamma_1 \omega(\gamma(t)) \right] X_2
\]

\[
+ \left[ - \frac{L}{2} \gamma_1 \gamma_2 \gamma_1 - \frac{L}{2} \gamma_3 \gamma_1 - \gamma_1 \omega(\gamma(t)) + \frac{d}{dt}(\omega(\gamma(t))) \right] X_3
\]
(4.14)

By (4.12) and (4.14), when \( \omega(\gamma(t)) \neq 0 \), we have

\[
\langle \nabla^L \bar{\gamma}, \nabla^L \bar{\gamma} \rangle_L \sim \left[ \gamma_1^2 + \gamma_2^2 \right] \omega(\gamma(t))^2 L^2, \quad \text{as} \quad L \to +\infty,
\]

\[
\langle \bar{\gamma}, \bar{\gamma} \rangle_L \sim -L \omega(\gamma(t))^2, \quad \text{as} \quad L \to +\infty,
\]

\[
\langle \nabla^L \bar{\gamma}, \bar{\gamma} \rangle_L^2 \sim O(L^2) \quad \text{as} \quad L \to +\infty.
\]

Therefore

\[
\frac{\langle \nabla^L \bar{\gamma}, \nabla^L \bar{\gamma} \rangle_L}{\|\bar{\gamma}\|_L^2} \to \frac{[\gamma_1^2 + \gamma_2^2]}{\omega(\gamma(t))^2}, \quad \text{as} \quad L \to +\infty,
\]

734
Lemma 5.1

\[
\frac{(\nabla^L_\gamma \dot{\gamma})^2}{||\dot{\gamma}||^2_L} \to 0, \quad \text{as } L \to +\infty.
\]

So by (2.6), we have (4.5) and (4.8). (4.6) and (4.9) come from (4.12), (4.14), (2.6) and \( \omega(\dot{\gamma}(t)) = 0 \) and \( \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0 \), we have

\[
(\nabla^L_\gamma \dot{\gamma}, \nabla^L_\gamma \dot{\gamma})_L \sim -L \frac{d}{dt}(\omega(\dot{\gamma}(t)))^2, \quad \text{as } L \to +\infty,
\]

\[
\langle \dot{\gamma}, \dot{\gamma} \rangle_L = \{\dot{\gamma}_1^2 + (\gamma_1 \dot{\gamma}_2 + \dot{\gamma}_3)^2\},
\]

\[
(\nabla^L_\gamma (\dot{\gamma}), \dot{\gamma})_L = O(1) \quad \text{as } L \to +\infty.
\]

By (2.6), we get (4.7).

\[\square\]

5. Lorentzian surfaces and a Gauss–Bonnet theorem in the Lorentzian Heisenberg group for the second Lorentzian metric

We will consider a regular surface \( \Sigma_1 \subset \mathbb{H}^3_L \) and regular curve \( \gamma \subset \Sigma_1 \). We will assume that there exists a \( C^2 \)-smooth function \( u : \mathbb{H}^3_L \to \mathbb{R} \) such that

\[
\Sigma_1 = \{(x_1, x_2, x_3) \in \mathbb{H}^3_L : u(x_1, x_2, x_3) = 0\}.
\]

Similar to Section 3, we define \( p, q, r, l, L, p, q, p_L, q_L, r_L, v_L, e_1, e_2, J_L, k^L_{\gamma, \Sigma_1}, k^L_{\gamma, \Sigma_1}, k^L_{\gamma, \Sigma_1}, k^\infty_{\gamma, \Sigma_1}, k^\infty_{\gamma, \Sigma_1} \). We call \( \Sigma \) a Lorentzian surface in the Lorentzian Heisenberg group for the third lorentzian metric. By (3.3) and (5.12), we have

\[
\nabla^L_{\gamma \Sigma_1, L} \dot{\gamma} = \left\{ p' \gamma_1 - \gamma_1^2 \gamma_2^2 - \gamma_3^2 - 2\gamma_1 \gamma_2 \gamma_3 + (-L \gamma_2 - \gamma_2 \gamma_1 + \gamma_3) \omega(\dot{\gamma}(t)) \right\} e_1
\]

\[
- \left\{ p_L' \gamma_1 - \gamma_1^2 \gamma_2^2 - \gamma_3^2 - 2\gamma_1 \gamma_2 \gamma_3 + (-L \gamma_2 - \gamma_2 \gamma_1 - \gamma_3) \omega(\dot{\gamma}(t)) \right\} e_2
\]

\[
+ \left\{ \frac{1}{L} \frac{d}{dt}(\omega(\dot{\gamma}(t))) - \frac{1}{L} \gamma_1 \gamma_2 - \frac{1}{L} \gamma_1 \gamma_3 - \gamma_1 \omega(\dot{\gamma}(t)) \right\} e_2
\]

\[
= N_1 e_1 + N_2 e_2.
\]

By (4.12) and \( \dot{\gamma}(t) \in T \Sigma_1 \), we have

\[
\dot{\gamma}(t) = [p' \gamma_1 - p(\gamma_1 \gamma_2 + \gamma_3)] e_1 - \frac{l_L}{L} \frac{1}{\omega(\dot{\gamma}(t))} \omega(\dot{\gamma}(t)) e_2.
\]

We have

**Lemma 5.1** Let \( \Sigma_1 \subset \mathbb{H}^3_L \) be a regular Lorentzian surface. Let \( \gamma : I \to \Sigma_1 \) be a \( C^2 \)-smooth regular curve.

(1) If \( \nabla^L_{\gamma \Sigma_1, L} \dot{\gamma} \) is a spacelike vector, then

\[
k^\infty_{\gamma, \Sigma_1} = \frac{|p' \gamma_1 + p_L' \gamma_2|}{|\omega(\dot{\gamma}(t))|}, \quad \text{if } \omega(\dot{\gamma}(t)) \neq 0,
\]

\[\square\]

735
Therefore, the situation does not exist.

(2) If \( \nabla_{\gamma}^{\Sigma_1, L} \) is a timelike vector, then

\[
k_{\gamma, \Sigma_1}^\infty = \frac{- (p\gamma_1 + p\gamma_2)^2}{|\omega(\dot{\gamma}(t))|}, \quad \text{if } \omega(\dot{\gamma}(t)) \neq 0,
\]

Therefore, the situation does not exist.

\[
l_{\gamma, \Sigma_1}^\infty = 0, \quad \text{if } \omega(\dot{\gamma}(t)) = 0, \quad \text{and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0,
\]

\[
\lim_{L \to +\infty} \frac{k_{\gamma, \Sigma_1}^L}{L} = \frac{|\frac{d}{dt}(\omega(\dot{\gamma}(t)))|}{|p\gamma_1 - p(\gamma_2 \gamma_1 + \gamma_3)|^2}, \quad \text{if } \omega(\dot{\gamma}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0.
\]

**Proof** By (5.1), we have

\[
\langle \nabla_{\gamma}^{\Sigma_1, L} \dot{\gamma}, \nabla_{\gamma}^{\Sigma_1, L} \dot{\gamma} \rangle_{L, \Sigma_1} \sim L^2 \omega(\dot{\gamma}(t)) \left[ p\gamma_2 + p\gamma_1 \right]^2,
\]

as \( L \to +\infty. \)

By (5.2), we have that when \( \omega(\dot{\gamma}(t)) \neq 0, \)

\[
\dot{\gamma}, \dot{\gamma} \rangle_{\Sigma_1, L} \sim -L[\omega(\dot{\gamma}(t))]^2, \quad \text{as } L \to +\infty.
\]

By (5.1) and (5.2), we have

\[
\langle \nabla_{\gamma}^{\Sigma_1, L} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma_1, L} \sim M_0 L,
\]

where \( M_0 \) does not depend on \( L. \) By Definition 3.1, (5.7) - (5.9), we get (5.3). When \( \omega(\dot{\gamma}(t)) = 0 \) and \( \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0, \) we have

\[
\langle \nabla_{\gamma}^{\Sigma_1, L} \dot{\gamma}, \nabla_{\gamma}^{\Sigma_1, L} \dot{\gamma} \rangle_{L, \Sigma_1} \sim N_1^2
\]

as \( L \to +\infty; \)

\[
\langle \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma_1, L} = \left[ p\gamma_1 - p(\gamma_1 \gamma_2 + \gamma_3) \right]^2,
\]

(5.11)

\[
\langle \nabla_{\gamma}^{\Sigma_1, L} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma_1, L} = \left[ p\gamma_1 - p(\gamma_1 \gamma_2 + \gamma_3) \right] N_1.
\]

By (5.10) - (5.12) and Definition 3.1, we get \( k_{\gamma, \Sigma_1}^{\infty} = 0. \) When \( \omega(\dot{\gamma}(t)) = 0 \) and \( \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0, \) we have

\[
\langle \nabla_{\gamma}^{\Sigma_1, L} \dot{\gamma}, \nabla_{\gamma}^{\Sigma_1, L} \dot{\gamma} \rangle_{L, \Sigma_1} \sim -L\left[ \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right]^2,
\]
\[
\langle \nabla^\Sigma_{\bar{\gamma}, \bar{\gamma}} J_L, \bar{J}_L \rangle_{\Sigma_1, L} = O(1),
\]
so we get (5.6).

**Lemma 5.2** Let \( \Sigma_1 \subset H_3^L \) be a regular Lorentzian surface.

(1) If \( \gamma : I \to \Sigma_1 \) be a spacelike \( C^2 \)-smooth curve, then
\[
k_{\gamma, \Sigma_1}^{\infty, s} = -\frac{\overline{\gamma}_2 + \overline{\gamma}_1}{\sqrt{-\omega(\bar{\gamma}(t))}}^2, \quad \text{if} \quad \omega(\bar{\gamma}(t)) \neq 0. \tag{5.13}
\]
Therefore, the situation does not exist.

\[
k_{\gamma, \Sigma_1}^{\infty, s} = 0 \quad \text{if} \quad \omega(\bar{\gamma}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\bar{\gamma}(t))) = 0; \tag{5.14}
\]
\[
\lim_{L \to +\infty} \frac{k_{\gamma, \Sigma_1}^{L, s}}{\sqrt{L}} = \frac{|\frac{d}{dt}(\omega(\bar{\gamma}(t)))|}{|\overline{\gamma}_1 - \overline{\gamma}(\gamma_1 \gamma_2 + \gamma_3)|^2}, \quad \text{if} \quad \omega(\bar{\gamma}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\bar{\gamma}(t))) \neq 0. \tag{5.15}
\]

(2) If \( \gamma : [a, b] \to \Sigma_1 \) be a timelike \( C^2 \)-smooth curve, then
\[
k_{\gamma, \Sigma_1}^{\infty, s} = -\frac{\overline{\gamma}_2 + \overline{\gamma}_1}{|\omega(\bar{\gamma}(t))|}, \quad \text{if} \quad \omega(\bar{\gamma}(t)) \neq 0. \tag{5.16}
\]
\[
||\dot{\gamma}||_{L, \Sigma} = \sqrt{-[\overline{\gamma}_1 - \overline{\gamma}(\gamma_2 \gamma_1 + \gamma_3)]^2, \quad \text{if} \quad \omega(\bar{\gamma}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\bar{\gamma}(t))) = 0; \tag{5.17}
\]
Therefore, the situation does not exist.

\[
||\dot{\gamma}||_{L, \Sigma} = \sqrt{-[\overline{\gamma}_1 - \overline{\gamma}(\gamma_2 \gamma_1 + \gamma_3)]^2, \quad \text{if} \quad \omega(\bar{\gamma}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\bar{\gamma}(t))) \neq 0. \tag{5.18}
\]
Therefore, the situation does not exist.

**Proof** For (1), by (3.3) and (5.2), we have
\[
J_L(\bar{\gamma}) = -\frac{l_L}{L} - \omega(\bar{\gamma}(t)) e_1 + [\overline{\gamma}_1 - \overline{\gamma}(\gamma_1 \gamma_2 + \gamma_3)] e_2. \tag{5.19}
\]
By (5.1) and (5.19), we have
\[
\langle \nabla^\Sigma_{\bar{\gamma}, \bar{\gamma}} J_L(\bar{\gamma}), J_L(\bar{\gamma}) \rangle_{\Sigma_1, L} \sim L^2 \omega(\bar{\gamma}(t))^2 [\overline{\gamma}_2 + \overline{\gamma}_1], \quad \text{as} \quad L \to +\infty. \tag{5.20}
\]
When \( \omega(\bar{\gamma}(t)) = 0 \) and \( \frac{d}{dt}(\omega(\bar{\gamma}(t))) = 0, \) we get
\[
\langle \nabla^\Sigma_{\bar{\gamma}, \bar{\gamma}} J_L(\bar{\gamma}), J_L(\bar{\gamma}) \rangle_{L, \Sigma_1} \sim M_0 L^{-\frac{1}{2}} \quad \text{as} \quad L \to +\infty. \tag{5.21}
\]
By Lemma 4.4, the Lorentzian Heisenberg group for the third Lorentzian metric. Similarly to Theorem 7.38, Lemma 5.5, Proposition 5.4, Theorem 5.3, so we get (5.15).

Next, we compute the sub-Riemannian limit of the Riemannian Gaussian curvature of surfaces in the Lorentzian Heisenberg group for the third Lorentzian metric. Similarly to Theorem 4.3 in [5], we have

**Theorem 5.3** The second fundamental form $II^L_1$ of the embedding of $\Sigma_1$ into $\mathbb{H}^3_L$ is given by

$$II^L_1 = \left( \begin{array}{cc} h_{11} & h_{12} \\ h_{21} & h_{22} \end{array} \right),$$

where

$$h_{11} = \frac{l}{L} [X_1(\overline{p}) + X_2(\overline{q})] + \overline{p} - q + \frac{r_0}{L} - \overline{p}r_0 L^{-\frac{1}{2}},$$

$$h_{12} = h_{21} = -\frac{l}{L} \langle e_1, \nabla_H(\overline{r}) \rangle_L + \overline{p} L^{-1} - \frac{l}{2} L + 1 - \frac{r_0}{L} + \frac{r_0}{L} L^{-\frac{1}{2}} - \overline{p} - \frac{r_0}{L} L^{-\frac{1}{2}},$$

$$h_{22} = -\frac{l}{L} \langle e_2, \nabla_H(\overline{r}) \rangle_L + \tilde{X}_3(\overline{r}) - \overline{p} L^{-1} \frac{l}{L} L^{-\frac{1}{2}} + \overline{p} \frac{L}{L} L^{-\frac{1}{2}} + \frac{r_0}{L} L^{-\frac{1}{2}} - \overline{p} L^{-\frac{1}{2}} \frac{l}{L}.$$

Similar to Proposition 3.8, we have

**Proposition 5.4** Away from characteristic points, the horizontal mean curvature $\mathcal{H}^L_\infty$ of $\Sigma_1 \subset \mathbb{H}^3_L$ is given by

$$\mathcal{H}^L_\infty = X_1(\overline{p}) + X_2(\overline{q}) + 2 \overline{p}.$$

By Lemma 4.1, we have

**Lemma 5.5** Let $\mathbb{H}^3_L$ be the Lorentzian Heisenberg group for the third metric, then

$$R^L(X_1, X_2)X_1 = \frac{-3L^2 + 2L + 1}{4L} X_2 + \frac{L - 1}{L} X_3, \quad R^L(X_1, X_2)X_2 = \frac{3L^2 - 2L - 1}{4L} X_1, \quad R^L(X_1, X_2)X_3 = \frac{L^2 + 2L - 3}{4L} X_3, \quad (5.25)$$

$$R^L(X_1, X_3)X_2 = (L - 1)X_1, \quad R^L(X_1, X_3)X_3 = (1 - L)X_2 + \frac{L^2 + 2L - 3}{4L} X_3,$n

$$R^L(X_2, X_3)X_1 = 0, \quad R^L(X_2, X_3)X_2 = \frac{L^2 - 2L + 1}{4L} X_3, \quad R^L(X_2, X_3)X_3 = \frac{L^2 - 2L + 1}{4L} X_3.$$
Proposition 5.6  Away from characteristic points, we have
\[ K^{\Sigma_1:\infty}(e_1, e_2) = \mathcal{A} + O(L^{-1}). \] (5.26)

where
\[ \mathcal{A} = \frac{(X_3 u)^2}{t^2} - 2\tau \frac{X_3 u}{|\nabla_H u|} + \frac{q^2 + p^2}{2} + (e_1, \nabla_H \left( \frac{X_3 u}{|\nabla_H u|} \right)) + p [X_1(p) + X_2(\bar{p})]. \] (5.27)

Proof  By (3.41) and Lemma 5.5, we have
\[ K^{\Sigma_1}(e_1, e_2) = r_L^2 \frac{3L^2 - 2L - 1}{4L} - 2\tau_L q_L L^{-\frac{1}{4}} (L - 1) + \frac{q_L^2 L^2 + 2L - 3}{4L} + P_L \frac{L^2 - 2L + 1}{4L}. \]

Similar to (3.43), we have
\[ \det(IH^1) = -\frac{L}{4} + (e_1, \nabla_H \left( \frac{X_3 u}{|\nabla_H u|} \right)) + p [X_1(p) + X_2(\bar{p})] + O(L^{-1}) \quad \text{as} \quad L \to +\infty. \] (5.28)

By (5.27) and (5.28), we have (5.26).

Similar to (3.46) and (3.49), for the Lorentzian Heisenberg group for the third Lorentzian metric. Let \( \gamma : I \to \mathbb{H}_L^3 \) be a timelike \( C^2 \)-smooth curve, we have
\[ \lim_{L \to +\infty} \frac{1}{\sqrt{L}} ds_L = ds, \quad \lim_{L \to +\infty} \frac{1}{\sqrt{L}} d\sigma_{\Sigma_1, L} = d\sigma_{\Sigma_1}. \] (5.29)

By (5.26), (5.29) and Lemma 5.2, similar to the proof of Theorem 3.13, we have

Theorem 5.7  Let \( \Sigma_1 \subset \mathbb{H}_L^3 \) be a regular Lorentzian surface with finitely many boundary components \( (\partial \Sigma_1)_i, i \in \{1, \ldots, n\} \), given by \( C^2 \)-smooth regular and closed timelike curves \( \gamma_i : [0, 2\pi] \to (\partial \Sigma_1)_i \). Suppose that the characteristic set \( C(\Sigma_1) \) satisfies \( \mathcal{H}^1(C(\Sigma_1)) = 0 \) and that \( ||\nabla_H u||_{H^{-1}} \) is locally summable with respect to the 2-dimensional Hausdorff measure near the characteristic set \( C(\Sigma_1) \), then
\[ \int_{\Sigma_1} K^{\Sigma_1:\infty} d\sigma_{\Sigma_1} + \sum_{i=1}^n \int_{\gamma_i} k^{\infty,e}_{\gamma_i, \Sigma_1} ds = 0. \] (5.30)

Example 5.8  In \( \mathbb{H}_L^3 \), let \( u = x_1^2 + x_2^2 + x_3^2 - 1 \) and \( \Sigma_1 = S^2 \). \( \Sigma_1 \) is a regular surface. By (4.1), we get
\[ X_1(u) = 2x_1; \quad X_2(u) = 2x_2 + 2(1 - x_1)x_3. \] (5.31)

Solve the equations \( X_1(u) = X_2(u) = 0 \), then we get
\[ C(\Sigma_1) = \left\{(0, \frac{\sqrt{2}}{2}, - \frac{\sqrt{2}}{2}), (0, - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\right\} \]
and \( \mathcal{H}^1(C(\Sigma_1)) = 0 \).

A parametrization of \( \Sigma_1 \) is
\[ x_1 = \cos(\varphi)\cos(\theta), \quad x_2 = \cos(\varphi)\sin(\theta), \quad x_3 = \sin(\varphi), \quad \text{for} \ \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}), \ \theta \in [0, 2\pi). \] (5.32)
Then
\[ \|\nabla_H u\|_H^2 = X_1(u)^2 + X_2(u)^2 \] (5.33)
\[ = 4x_1^2 + 4 [x_2 + (1 - x_1)x_3]^2 \]
\[ = 4 + 4\cos(\varphi)^2\cos(\theta)^2\sin(\varphi)^2 - 8\cos(\varphi)\cos(\theta)\sin(\varphi)^2 \]
\[ - 8\cos(\varphi)^2\sin(\varphi)\cos(\theta)\sin(\theta) + 8\cos(\varphi)\sin(\varphi)\sin(\theta). \]

By the definitions of \( w_j \) for \( 1 \leq j \leq 3 \) and (5.29), we have
\[ d\sigma_{\Sigma_1} = \frac{1}{\|\nabla_H u\|_H} [(X_1(u)) (x_1 dx_2 + dx_3) - (X_2(u)) dx_1] \wedge [(1 - x_1) dx_2 - dx_3] \] (5.34)
\[ = - \frac{1}{\|\nabla_H u\|_H} 2\cos(\varphi) \lambda_0 d\theta \wedge d\varphi. \]

where
\[ \lambda_0 = \cos(\varphi)^2 - 2\cos(\varphi)^2\cos(\theta)\sin(\varphi)\sin(\theta) - 2\cos(\varphi)\cos(\theta)\sin(\varphi)^2 \]
\[ + 2\cos(\varphi)\sin(\theta)\sin(\varphi) + \sin(\varphi)^2 + \cos(\varphi)^2\sin(\varphi)^2\cos(\theta)^2. \] (5.35)

And \( \lambda_0 \) is a bounded smooth function on \( \Sigma_1 \). By (5.33) and (5.34), we have \( \|\nabla_H u\|_H^{-1} \) is locally summable around the isolated characteristic points with respect to the measure \( d\sigma_{\Sigma_1} \).

Acknowledgment

The third author was supported in part by NSFC No.11771070.

References


