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Preservers of the local spectral radius zero of Jordan product of operators

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Abstract: Let $\mathscr{B}(X)$ be the algebra of all bounded linear operators on an infinite-dimensional complex Banach space X, and denote by $r_T(x)$ the local spectral radius of any operator $T \in \mathscr{B}(X)$ at any vector $x \in X$. In this paper, we characterize surjective maps φ on $\mathscr{B}(X)$ satisfying

$$r_{\varphi(T)\varphi(A)+\varphi(A)\varphi(T)}(x) = 0 \text{ if and only if } r_{TA+AT}(x) = 0$$

for all $x \in X$ and $A, T \in \mathscr{B}(X)$.

Key words: Nonlinear preservers, quasinilpotent part, local spectral radius, Jordan product

1. Introduction

Let X be an infinite-dimensional complex Banach space and let $\mathscr{B}(X)$ be the algebra of all bounded linear operators on X. The local resolvent set of an operator $T \in \mathscr{B}(X)$ at a vector $x \in X$, $\rho_T(x)$, is the set of all λ in $\mathbb{C}$ for which there exist an open neighborhood U_λ of λ and a function $f : U_\lambda \longrightarrow X$ such that $(T - \mu)f(\mu) = x$ for all $\mu \in \mathbb{C}$. Its complement in $\mathbb{C}$, denoted by $\sigma_T(x)$, is called the local spectrum of T at x, and is a closed subset (possibly empty) of $\sigma(T)$, the spectrum of T. The local spectral radius of an operator $T \in \mathscr{B}(X)$ at a point $x \in X$ is defined by:

$$r_T(x) = \limsup_{n\to+\infty} \| T^n x \|^{\frac{1}{n}} .$$

The quasinilpotent part of an operator $T \in \mathcal{B}(X)$ is defined by:

$$H_0(T) = \{x \in X : \limsup_{n\to+\infty} \| T^n x \|^{\frac{1}{n}} = 0\}.$$

For more information on general local spectral theory, the interested reader may consult the remarkable books in [2] and [20].

Over the last years, the study of additive and linear local spectra preserver problems has attracted the attention of many authors. Bourhim and Ransford [12] were the first ones to consider this type of preserver problems. They studied additive maps on $\mathscr{B}(X)$ preserving the local spectrum at each point $x \in X$ and showed that if $\varphi : \mathscr{B}(X) \longrightarrow \mathscr{B}(X)$ is an additive map satisfying $\sigma_{\varphi(T)}(x) = \sigma_T(x)$ for all $T \in \mathscr{B}(X)$ and for all

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$x \in X$, then φ is the identity on $\mathscr{B}(X)$. These results opened the way for some authors to consider a more general problem of characterizing additive or linear maps on matrices or operators preserving different local spectral sets and quantities such as the local spectrum, the local spectral radius and the local inner spectral radius; see for instance [1, 4–13, 15–19] and the references therein.

In [16], Costara described surjective linear maps on $\mathscr{B}(X)$ which preserve operators of local spectral radius zero at points of X. He showed, in particular, that if φ is a surjective linear map on $\mathscr{B}(X)$ such that for every $x \in X$ and $T \in \mathscr{B}(X)$, we have:

$$r_{\varphi(T)}(x) = 0 \text{ if and only if } r_T(x) = 0$$

then there exists a nonzero scalar $\alpha \in \mathbb{C}$ such that $\varphi(T) = \alpha T$ for all $T \in \mathscr{B}(X)$.

In [7], Bourhim and Mashreghi obtained a similar result to that in [16] without assuming that φ is linear. They proved that if φ is a surjective map on $\mathscr{B}(X)$ that satisfies:

$$r_{\varphi(T)-\varphi(S)}(x) = 0 \text{ if and only if } r_{T-S}(x) = 0,$$

for every $x \in X$ and $T, S \in \mathscr{B}(X)$, then there are a nonzero scalar $\alpha \in \mathbb{C}$ and an operator $A \in \mathscr{B}(X)$ such that $\varphi(T) = \alpha T + A$ for all $T \in \mathscr{B}(X)$.

This result was extended by Elhodaibi and Jaatit in [17] to maps φ assumed to be nonsurjective. For generalized product of operators, Abdelali et al. showed in [1] that a surjective map φ on $\mathscr{B}(X)$ satisfies:

$$r_{\varphi(T_1)*\cdots*\varphi(T_k)}(x) = 0 \Longleftrightarrow r_{T_1*\cdots*T_k}(x) = 0$$

for all $x \in X$ and all $T_1, T_2, \cdots, T_k \in \mathscr{B}(X)$ if and only if there exists a map $\gamma : \mathscr{B}(X) \longrightarrow \mathbb{C}\backslash\{0\}$ such that $\varphi(T) = \gamma(T)T$ for all $T \in \mathscr{B}(X)$.

In this paper, we show that a surjective map φ on $\mathscr{B}(X)$ satisfies:

$$r_{\varphi(T)\varphi(A)+\varphi(A)\varphi(T)}(x) = 0 \text{ if and only if } r_{TA+AT}(x) = 0 \qquad (x \in X \text{ and } A, T \in \mathscr{B}(X)$$

if and only if there exists a function $\gamma : \mathscr{B}(X) \longrightarrow \mathbb{C}\backslash\{0\}$ such that $\varphi(T) = \gamma(T)T$ for all $T \in \mathscr{B}(X)$.

2. Preliminaries

For any operator $T \in \mathscr{B}(X)$, let $N(T)$ be the kernel of T and $\operatorname{ran}(T)$ be its range. For a subspace Y of X, denote by $\dim(Y)$ and $\operatorname{codim}(Y)$ its dimension and codimension, respectively.

Let x be a nonzero vector in X and f be a nonzero functional in the topological dual X^* of X. We denote, as usual, by $x \otimes f$ the rank-one operator given by $(x \otimes f)z = f(z)x$ for $z \in X$. Note that $x \otimes f$ is an idempotent operator if and only if $f(x) = 1$, and it is nilpotent if and only if $f(x) = 0$. For a positive integer n, let $\mathscr{F}_n(X)$ be the set of all operators in $\mathscr{B}(X)$ of rank at most n.

The quasinilpotent part of a rank-one operator is given by:

$$H_0(x \otimes f) = \begin{cases} X & \text{if} \quad f(x) = 0 \\ N(f) & \text{if} \quad f(x) \neq 0 \end{cases}$$

For the proof of the mains result, we need some auxiliary lemmas. The first one summaries some elementary properties of the quasinilpotent following parts of operators.

Lemma 2.1 *(See [2]) Let $T \in \mathscr{B}(X)$. The following statements hold.*

1. $N(T) \subseteq H_0(T)$.
2. $H_0(\lambda T) = H_0(T)$ *for all nonzero scalar* $\lambda \in \mathbb{C}$.
3. $H_0(T^n) = H_0(T)$ *for all* $n \in \mathbb{N}^*$.
4. T *is quasinilpotent if and only if* $H_0(T) = X$.
5. *Let* M *be a finite dimensional subspace of* X *such that* $TM = M$ *then* $M \cap H_0(T) = \{0\}$ *and* $dim(M) \leq codim(H_0(T))$.

In this lemma, the last assertion is obvious.

The next lemma characterizes rank-one operators $A \in \mathscr{B}(X)$ in terms of the quasinilpotent following parts of the Jordan product of A with any operator $T \in \mathscr{B}(X)$.

Lemma 2.2 *For a nonzero operator* $A \in \mathscr{B}(X)$*, the following statements are equivalent:*

1. A *is a rank-one operator.*
2. $codim(H_0(AT + TA)) \leq 2$ *for all* $T \in \mathscr{B}(X)$.

Proof $(1) \Longrightarrow (2)$: Let A be a rank-one operator and write $A = x \otimes f$ with $f \in X^*, x \in X$ and let $T \in \mathscr{B}(X)$ be an arbitrary operator. Then $N(f) \cap N(T^* f) \subset H_0(AT + TA)$ and thus $\text{codim}(H_0(AT + TA)) \leq 2$.

$(2) \Longrightarrow (1)$: Assume that $\text{rank}(A) \geq 2$, and let us show that there exists $T \in \mathscr{B}(X)$ such that $\text{codim}(H_0(AT + TA)) \geq 3$.

Firstly, suppose that $\text{rank}(A) \geq 3$. Let $y_1 = Ax_1, y_2 = Ax_2$, and $y_3 = Ax_3$ such that $\{y_1, y_2, y_3\}$ are linearly independent. Take an operator $T \in \mathscr{B}(X)$ such that

$$Ty_1 = x_1, \quad Ty_2 = x_2 \quad \text{and} \quad Ty_3 = x_3, \tag{2.1}$$

and set $R = AT + TA$.

Case 1: Assume that $\dim(\text{span}\{x_1, x_2, x_3, y_1, y_2, y_3\}) = 3$.

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ be such that $x_1 = \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3$. We have $Tx_1 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, $ATx_1 = \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = x_1$ and $TAx_1 = x_1$. We do the same calculations for x_2, x_3, and $\{x_1, x_2, x_3\} \subseteq N(R - 2I)$.

Set $M = \text{span}\{x_1, x_2, x_3\}$ and $\dim M = 3$ then we have $\text{ran}\, M = M$ and $\text{codim}(H_0(R)) \geq 3$.

Case 2: Assume that $\dim(\text{span}\{x_1, x_2, x_3, y_1, y_2, y_3\}) = 4 = \dim(\text{ span}\{x_1, y_1, y_2, y_3\})$.

In this case, we can choose easily an operator $T \in \mathscr{B}(X)$ satisfying both (2.1) and $Tx_1 = 0$. Then there exist $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $Rx_2 = -\alpha_1 x_1 + 2x_2$ and $Rx_3 = -\alpha_2 x_1 + 2x_3$. Hence:

$$\begin{cases} Rx_1 &= x_1 \\ Rx_2 &= -\alpha_1 x_1 + 2x_2 \\ Rx_3 &= -\alpha_2 x_1 + 2x_3 \end{cases}$$

the matrix of R in basis $\{x_1,x_2,x_3\}$ is $\begin{pmatrix}1 & -\alpha_1 & -\alpha_2\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}$.

Set $M=$span$\{x_1,x_2,x_3\}$ and $\dim M=3$ then we have ran $M=M$ and codim$(H_0(R))\geq 3$.

Case 3: Assume that dim(span$\{x_1,x_2,x_3,y_1,y_2,y_3\})=5=$dim(span$\{x_1,x_2,y_1,y_2,y_3\}$).

In this case, we choose $T\in\mathscr{B}(X)$ to be an operator satisfying both (2.1) and $Tx_1=0,Tx_2=0$. Then there exist $\alpha_1,\alpha_2\in\mathbb{C}$ such that $Rx_3=-\alpha_1x_1-\alpha_2x_2+2x_3$. Hence:

$$\begin{cases} Rx_1 &= x_1\\ Rx_2 &= x_2\\ Rx_3 &= -\alpha_1x_1-\alpha_2x_2+2x_3\end{cases}$$

As before, codim$(H_0(R))\geq 3$.

Case 4: Assume that dim(span$\{x_1,x_2,x_3,y_1,y_2,y_3\})=6$.

In this case, we choose $T\in\mathscr{B}(X)$ to be an operator satisfying both (2.1) and $Tx_1=0,Tx_2=0,Tx_3=0$. We have:

$$\begin{cases} Rx_1 &= x_1\\ Rx_2 &= x_2\\ Rx_3 &= x_3\end{cases}$$

Then $\{x_1,x_2,x_3\}\subseteq N(R-I)$. Set $M=$span$\{x_1,x_2,x_3\}$ and $\dim M=3$ then we have ran $M=M$ and codim$(H_0(R))\geq 3$.

Secondly, suppose that rank$(A)=2$ then $N(A)\neq 0$.

Set A such that $y_1=Ax_1$, $y_2=Ax_2$, and $\{y_1,y_2\}$ are linearly independent. Take an operator $T\in\mathscr{B}(X)$ with the following properties

$$Ty_1=x_1 \text{ and } Ty_2=x_2, \tag{2.2}$$

and consider the following operator $R=AT+TA$.

Assume that dim$(span\{x_1,x_2,y_1,y_2\})=2$, we can replace x_1 by x_1+z with $z\in N(A)$ a nonzero vector such that $\{x_1+z,y_1,y_2\}$ are linearly independent. Then without loss of generality, we may assume that $\{x_1,y_1,y_2\}$ and $\{x_1,x_2,y_1\}$ are linearly independent. Thus, we distinguish two cases:

Case 1: Assume that dim(span$\{x_1,x_2,y_1,y_2\})=3$.

In this case, we choose $T\in\mathscr{B}(X)$ to be an operator satisfying both (2.2) and $Tx_1=0$. Take $\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3\in\mathbb{C}$ satisfying $x_2=\alpha_1x_1+\beta_1y_1+\beta_2y_2$ and $Ay_1=\alpha_2y_1+\beta_3y_2$, $ATy_1=y_1$ then $TAx_2=x_2$, $Tx_2=\beta_1x_1+\beta_2x_2$, and $ATx_2=\beta_1y_1+\beta_2y_2=x_2-\alpha x_1$. Hence:

$$\begin{cases} Rx_1 &= x_1\\ Rx_2 &= -\alpha_1x_1+2x_2\\ Ry_1 &= \alpha_2x_1+\beta_3x_2+y_1\end{cases}$$

As before, codim$(H_0(R))\geq 3$.

Case 2: Assume that $\dim(\operatorname{span}\{x_1, x_2, y_1, y_2\}) = 4$. In this case, we choose $T \in \mathscr{B}(X)$ to be an operator satisfying both (2.2) and $Tx_1 = 0, Tx_2 = 0$. Let $\alpha, \beta \in \mathbb{C}$ such that $Ay_1 = \alpha y_1 + \beta y_2$.

We have $Rx_1 = x_1$, $Rx_2 = x_2$, and $ATy_1 = y_1$. Then:

$$\begin{cases} Rx_1 &= x_1 \\ Rx_2 &= x_2 \\ Ry_1 &= \alpha x_1 + \beta x_2 + y_1 \end{cases}$$

As before, $\operatorname{codim}(H_0(R)) \geq 3$. □

In terms of the quasinilpotent following parts of the Jordan product of operators, the last lemma characterizes when two operators in $\mathscr{B}(X)$ are scalar multiple of each other.

Lemma 2.3 *For two operators A, B in $\mathscr{B}(X)\backslash\{0\}$, the following statements are equivalent.*

1. *$B = \lambda A$ for some nonzero scalar $\lambda \in \mathbb{C}$.*

2. *$H_0(AT + TA) = H_0(BT + TB)$ for all $T \in \mathscr{F}_2(X)$.*

In particular if A has a finite rank then we can replace $\mathscr{F}_2(X)$ by $\mathscr{F}_1(X)$.

Proof (1) $\Longrightarrow$ (2) For some nonzero scalar $\lambda \in \mathbb{C}$ we have $B = \lambda A$, then

$$H_0(BT + TB) = H_0(\lambda AT + \lambda TA) = H_0(\lambda(AT + TA)) = H_0(AT + TA)$$

(2) $\Longrightarrow$ (1) Let $A, B \in \mathscr{B}(X)$ be two operators such that

$$H_0(BT + TB) = H_0(AT + TA)$$

for all $T \in \mathscr{F}_2(X)$. If $A = 0$, then $H_0(BT + TB) = X$ for all $T \in \mathscr{F}_2(X)$, and $\sigma_{BT+TB}(x) = \{0\}$ for all nonzero $x \in X$ and all $T \in \mathscr{F}_2(X)$. By [6, Corollary 4.4], we conclude that $B = 0$. Similarly, if $B = 0$, then $A = 0$; thus, we may and shall assume that both A and B are nonzero operators.

Assume that there exists a nonzero vector $x \in X$ such that $\{x, Ax, Bx\}$ are linearly independent. Then there exists a linear functional $f \in X^*$ such that $f(x) = f(Ax) = 0$ and $f(Bx) = 1$. We choose $T = x \otimes f$ and set $R = Ax \otimes f + x \otimes A^* f, S = Bx \otimes f + x \otimes B^* f$. It follows that $\operatorname{ran}(R) = \operatorname{span}\{x, Ax\}$ and

$$\begin{cases} Rx &= 0 \\ RAx &= f(A^2x)x \end{cases}$$

Therefore, R is nilpotent and $H_0(R) = X$. We have $Sx = (BT + TB)x = (Bx \otimes f + x \otimes B^* f)x = x$, and then

$$x \notin H_0(BT + TB) = H_0(AT + TA) = X.$$

This contradiction shows that $Bx \in \operatorname{span}\{x, Ax\}$ for all $x \in X$. Now, we shall discuss three cases:

Case 1: If $A = \mu I$ for some nonzero scalar $\mu \in \mathbb{C}$ and there exists $x \in X$ such that x and Bx are linearly independent. Take $T = x \otimes f$ such that $f(x) = 0$ and $f(Bx) = 1$, R is nilpotent and $Sx = x$. This is a contradiction then $B = \nu I$, with $\nu \in \mathbb{C}\backslash\{0\}$.

For $A \notin \mathbb{C}I$, $B = \alpha I + \lambda A$ for some $(\alpha, \lambda) \in \mathbb{C} \times \mathbb{C}^*$, see [21, Lemma 2.4] . Without loss of generality, we can choose $\lambda = 1$.

Case 2: There exists a nonzero vector $x \in X$ such that $\{x, Ax, A^2x\}$ are linearly independent, and there exists a linear functional $f \in X^*$ such that $f(Ax) = f(A^2x) = 0$ and $f(x) \neq 0$, take $T = x \otimes f$ then we have $S = R + 2\alpha T$ and R is a rank two operator, S is less than rank two operator. We have:

$$\begin{aligned} R^2 &= (Ax \otimes f + x \otimes A^* f)(Ax \otimes f + x \otimes A^* f) \\ &= f(Ax)Ax \otimes f + f(x)Ax \otimes A^* f + f(A^2x)x \otimes f + f(Ax)x \otimes A^* f \\ &= f(x)Ax \otimes A^* f. \end{aligned}$$

Also:

$$\begin{aligned} S^2 &= (Ax \otimes f + x \otimes A^* f + 2\alpha x \otimes f)(Ax \otimes f + x \otimes A^* f + 2\alpha x \otimes f) \\ &= f(x)(Ax + 2\alpha x) \otimes (A^* f + 2\alpha f). \end{aligned}$$

Then R is nilpotent. $H_0(S) = H_0(R) = X$; therefore, for $z = Ax + 2\alpha x$, we have $S^2 z = 0$ then $4\alpha^2 f(x)^2 = 0$; hence, $\alpha = 0$.

Case 3: If not then for all $x \in X$, $A^2x \in \text{span}\{x, Ax\}$, we omit the proof for this, $A^2 = aA + bI$ for some $a, b \in \mathbb{C}$, see [21, Lemma 2.4].

If $b = 0$, then there exists $f \in X^*$ and $x \in X$ such that $f(x) \neq 0$ and $f(Ax) = 0$; hence, $f(A^2x) = 0$, see case 2. Without loss of generality we can choose $b = 1$. Hence, in this case, we infer that $A^2 = aA + I$ and there exists $x \in X$ such that x and Ax are linearly independent. Consequently, there exist two linear functionals $(f, g) \in (X^*)^2$ such that $f(Ax) = g(x) = 0$ and $f(x) \neq 0, g(Ax) \neq 0$, take $T = x \otimes f + Ax \otimes g$ we have:

$$\begin{aligned} R &= TA + AT \\ &= (x \otimes f + Ax \otimes g)A + A(x \otimes f + Ax \otimes g) \\ &= x \otimes A^* f + Ax \otimes A^* g + Ax \otimes f + A^2x \otimes g \\ &= x \otimes (A^* f + g) + Ax \otimes (f + A^* g + ag). \end{aligned}$$

Also:

$$\begin{aligned} S &= R + 2\alpha T \\ &= x \otimes (A^* f + g + 2\alpha f) + Ax \otimes (f + A^* g + ag + 2\alpha g). \end{aligned}$$

Set $t = f(x) + g(Ax)$, we have:

$$\begin{cases} Sx &= 2\alpha f(x)x + tAx \\ SAx &= tx + 2(a + \alpha)g(Ax)Ax \end{cases}$$

the matrix of S in basis (x, Ax) is $U = \begin{pmatrix} 2\alpha f(x) & t \\ t & 2(a + \alpha)g(Ax) \end{pmatrix}$, we have:

$$det(U) = -4\alpha(a + \alpha)f(x)^2 + 4\alpha(a + \alpha)f(x)t - t^2.$$

• If $\alpha \neq -a$ and $\alpha \neq 0$, we choose $t \neq 0$ such that S is a noninvertible operator.

Also we have:

$$\begin{cases} Rx = & tAx \\ RAx & = & tx + 2ag(Ax)Ax \end{cases}$$

the matrix of R in basis (x, Ax) is $\begin{pmatrix} 0 & t \\ t & 2ag(Ax) \end{pmatrix}$, then R is an invertible operator and $\operatorname{codim}(H_0(R)) \geq 2$, this is a contradiction.

• If $\alpha = -a \neq 0$ and $t = 0$ then:

$$\begin{cases} RAx & = & 2ag(Ax)Ax \\ SAx & = & 0. \end{cases}$$

Therefore, $Ax \notin H_0(R)$ and $Ax \in H_0(S)$. This contradiction completes the proof. □

3. The main result

In this section, we state and prove the main result of this paper that gives a complete description of all surjective maps on $\mathscr{B}(X)$ satisfying

$$r_{\varphi(T)\varphi(A)+\varphi(A)\varphi(T)}(x) = 0 \text{ if and only if } r_{TA+AT}(x) = 0.$$

for all $x \in X$ and $T, A \in \mathscr{B}(X)$. Note that there is a similarity in the proof of the next theorem and of the theorem 1.1 in [1]. However, their proof does not apply to our case.

Theorem 3.1 *Let X be an infinite dimensional complex Banach space and $\varphi : \mathscr{B}(X) \longrightarrow \mathscr{B}(X)$ be a surjective map. Then the following assertions are equivalent:*

1. *For every $x \in X$ and $T, A \in \mathscr{B}(X)$, we have*

$$r_{\varphi(T)\varphi(A)+\varphi(A)\varphi(T)}(x) = 0 \Longleftrightarrow r_{TA+AT}(x) = 0.$$

2. *For every $x \in X$ and $T, A \in \mathscr{B}(X)$, we have*

$$H_0(\varphi(T)\varphi(A) + \varphi(A)\varphi(T)) = H_0(TA + AT).$$

3. *There exists a function $\gamma : \mathscr{B}(X) \longrightarrow \mathbb{C}\backslash\{0\}$ such that $\varphi(T) = \gamma(T)T$ for all $T \in \mathscr{B}(X)$.*

Proof Since the equivalence $(1) \Leftrightarrow (2)$ and the implication $(3) \Rightarrow (2)$ are obvious, we only need to establish the implication $(2) \Rightarrow (3)$. Therefore, assume that

$$H_0(\varphi(A)\varphi(T) + \varphi(T)\varphi(A)) = H_0(AT + TA)$$

for all $A, T \in \mathscr{B}(X)$, and note that the proof breaks down into four steps.

Step 1: $\varphi(A) = 0 \Longleftrightarrow A = 0$ and $\varphi(0) = 0$.

Let us show that $\varphi(0) = 0$. We have:

$$H_0(\varphi(T)\varphi(0)+\varphi(0)\varphi(T)) = H_0(T0+0T) = X = H_0(\varphi(T)0+0\varphi(T))$$

for all $T \in \mathscr{B}(X)$. Lemma 2.3 and the surjectivity of φ entail that $\varphi(0)=0$.

Similarly, if $\varphi(A)=0$ then $A=0$.

Step 2: φ preserves rank-one operators in both directions.

Let $A \in \mathscr{B}(X)$ be a rank-one operator, then by Step 1 we have $\varphi(A) \neq 0$ and by Lemma 2.2

$\operatorname{codim}(H_0(TA+AT)) \leq 2$ for every $T \in \mathscr{B}(X)$, since φ is surjective and

$$H_0(TA+AT) = H_0(\varphi(T)\varphi(A)+\varphi(A)\varphi(T))$$

then:

$$\operatorname{codim}((H_0(\varphi(T)\varphi(A)+\varphi(A)\varphi(T))) \leq 2$$

and $\varphi(A)$ is a rank-one operator.

Let $\varphi(A)$ be a rank-one operator. Similarly, as above, we establish that A is a rank-one operator.

Step 3: For every rank-one operator $A \in \mathscr{B}(X)$, there is a nonzero scalar $\alpha \in \mathbb{C}$ such that $\varphi(A) = \alpha A$.

Let $x \in X$ and $f \in X^*$. By Step 2, there exist $y \in X$ and $g \in X^*$ such that $\varphi(x \otimes f) = y \otimes g$.

Case 1: If $f(x) \neq 0$, then we have:

$$\begin{aligned}
N(f) = H_0(x \otimes f) &= H_0(2(x \otimes f)^2) \\
&= H_0(x \otimes f x \otimes f + x \otimes f x \otimes f) \\
&= H_0(\varphi(x \otimes f)\varphi(x \otimes f) + \varphi(x \otimes f)\varphi(x \otimes f)) \\
&= H_0(2(y \otimes g)^2) \\
&= H_0(2g(y) y \otimes g).
\end{aligned}$$

This shows that $g(y) \neq 0$ and $N(f) = H_0(y \otimes g) = N(g)$. Therefore, $g = \alpha f$ for a nonzero scalar $\alpha \in \mathbb{C}$. Without loss of generality, we may and shall assume that $f = g$, and $\varphi(x \otimes f) = y_{x,f} \otimes f$ for certain nonzero vector $y_{x,f} \in X$.

We claim that x and $y_{x,f}$ are linearly dependent. If not, take $h \in X^*$ such that $h(x)=1$ and $h(y_{x,f})=0$, observe that $\varphi(x \otimes h) = y_{x,h} \otimes h$, take $S = x \otimes f x \otimes h + x \otimes h x \otimes f$ and $R = y_{x,f} \otimes f y_{x,h} \otimes h + y_{x,h} \otimes h y_{x,f} \otimes f$.

We have $Sx = 2f(x)x$ and $R = f(y_{x,h}) y_{x,f} \otimes h$ is nilpotent; hence, $x \notin H_0(S) = H_0(R) = X$. This contradiction shows that $y_{x,f}$ and x are linearly dependent, and $\varphi(x \otimes f) = \alpha(x \otimes f)$ for certain nonzero $\alpha \in \mathbb{C}$.

Case 2: If $f(x) = 0$, then:

$$\begin{aligned}
X = H_0(x \otimes f) &= H_0(2(x \otimes f)^2) \\
&= H_0(x \otimes f x \otimes f + x \otimes f x \otimes f) \\
&= H_0(\varphi(x \otimes f)\varphi(x \otimes f) + \varphi(x \otimes f)\varphi(x \otimes f)) \\
&= H_0(2(y \otimes g)^2) \\
&= H_0(2g(y) y \otimes g).
\end{aligned}$$

Hence, $g(y) = 0$.

If g and f are linearly independent. Let $z \in X$ and $h \in X^*$ such that, $f(z) = h(z) = h(x) = 1$ and $g(z) = 0$. Take $R = x \otimes fz \otimes h + z \otimes hx \otimes f$ and $S = y \otimes gz \otimes h + z \otimes hy \otimes g = h(y)z \otimes g$.
Then $Rx = x$; hence, $x \notin H_0(R) = H_0(S) = X$. This contradiction asserts that $g = \alpha f$ for a nonzero scalar $\alpha \in \mathbb{C}$. Without loss of generality, we may and shall assume that $f = g$.

We claim that x and $y_{x,f}$ are linearly dependent. If not, let $z \in X$ and $h \in X^*$ such that $f(z) = h(z) = h(x) = 1$ and $h(y_{x,f}) = 0$, observe that $\varphi(x \otimes f) = y_{x,f} \otimes f$.

For $R = x \otimes fz \otimes h + z \otimes hx \otimes f$ and $S = y_{x,f} \otimes fz \otimes h + z \otimes hy_{x,f} \otimes f$, we have $Rx = x$ and $S = y_{x,f} \otimes h$ is nilpotent; hence, $x \notin H_0(R) = H_0(S) = X$. This contradiction shows that $y_{x,f}$ and x are linearly dependent, and $\varphi(x \otimes f) = \alpha(x \otimes f)$ for certain nonzero $\alpha \in \mathbb{C}$. As a consequence, see Lemma 2.3, for a finite rank operator A, there exists a nonzero scalar α such that $\varphi(A) = \alpha A$.

Step 4: $\varphi(T) = \gamma(T)T$ for all $T \in \mathscr{B}(X)$.
For every rank two operator $A \in \mathscr{B}(X)$ and every $T \in \mathscr{B}(X)\backslash\{0\}$, we have:

$$\begin{aligned} H_0(TA + AT) &= H_0(\varphi(T)\varphi(A) + \varphi(A)\varphi(T)) \\ &= H_0(\alpha\varphi(T)A + \alpha A\varphi(T)) \\ &= H_0(\alpha(\varphi(T)A + A\varphi(T))) \\ &= H_0(\varphi(T)A + A\varphi(T)). \end{aligned}$$

By Step 1 and Lemma 2.3, we see that $\varphi(T)$ and T are linearly dependent. Therefore, there exists a function $\gamma : \mathscr{B}(X) \longrightarrow \mathbb{C}\backslash\{0\}$ such that $\varphi(T) = \gamma(T)T$ for all $T \in \mathcal{B}(X)$. □

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