Operators between different weighted Fréchet and (LB)-spaces of analytic functions

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Received: 21.01.2021 • Accepted/Published Online: 04.03.2021 • Final Version: 26.03.2021

Abstract: We study some classical operators defined on the weighted Bergman Fréchet space $A^\alpha_{p+}$ (resp. weighted Bergman (LB)-space $A^\alpha_{p-}$) arising as the projective limit (resp. inductive limit) of the standard weighted Bergman spaces into the growth Fréchet space $H^\alpha_{\infty+}$ (resp. growth (LB)-space $H^\alpha_{\infty-}$), which is the projective limit (resp. inductive limit) of the growth Banach spaces. We show that, for an analytic self map $\varphi$ of the unit disc $\mathbb{D}$, the continuities of the weighted composition operator $W_{g,\varphi}$, the Volterra integral operator $T_g$, and the pointwise multiplication operator $M_g$ defined via the identical symbol function are characterized by the same condition determined by the symbol’s state of belonging to a Bloch-type space. These results have consequences related to the invertibility of $W_{g,\varphi}$ acting on a weighted Bergman Fréchet or (LB)-space. Some results concerning eigenvalues of such composition operators $C_\varphi$ are presented.

Key words: Weighted composition operator, Volterra operator, multiplication operator, Fréchet spaces, (LB)-spaces, weighted spaces of analytic functions

1. Introduction
Let $H(\mathbb{D})$ denote the Fréchet space of all analytic functions $f: \mathbb{D} \to \mathbb{C}$ equipped with the topology of uniform convergence on the compact subsets of the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\varphi$ be an analytic self map on $\mathbb{D}$, and let $g: \mathbb{D} \to \mathbb{C}$ be an analytic map. The main focus of this note is, when they are defined between projective (or inductive) limits of different well-known Banach spaces of analytic functions, to give a relation between the continuity of the Volterra integral operator

$$T_g(f)(z) = \int_0^z f(t)g'(t)dt, \quad z \in \mathbb{D},$$  \hfill (1.1)

the pointwise multiplication operator

$$M_g(f)(z) = g(z)f(z), \quad z \in \mathbb{D},$$  \hfill (1.2)

and the weighted composition operator

$$W_{g,\varphi}(f)(z) = (M_g \circ \varphi \circ f)(z) = g(z)f(\varphi(z)), \quad z \in \mathbb{D}$$  \hfill (1.3)

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2010 AMS Mathematics Subject Classification: 47B33, 46A04, 47B38, 30H30.

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in terms of conditions formulated for \( g \) and \( \varphi \). For \( 1 < p < \infty \) and \(-1 < \alpha < \infty\), the Bergman space of standard weight \( A^p_\alpha = A^p_\alpha(D) \) of the unit disc is given by

\[
A^p_\alpha := \{ f \in H(D) : \| f \|_{p,\alpha} = \left( \alpha + 1 \right) \left( \int_{D} |f(z)|^p ds_\alpha(z) \right)^{1/p} < \infty \},
\]

(1.4)

where \( ds_\alpha(z) = (1 - |z|^2)^\alpha ds(z) \), and \( ds(z) = \frac{1}{\pi} |dz| \). Each \( A^p_\alpha \) is a closed subspace of \( L^p(D, ds(z)) \) in which the polynomials are dense [18, Section 1.1]. The weighted Bergman space \( A^p_\alpha \) is a Banach space with the norm \( \| \cdot \|_{p,\alpha} \). Classical Bergman space \( H^p(\mathbb{D}) \) corresponds to the case \( \alpha = 0 \). If \( p = \infty \) we obtain the growth Banach space endowed with the norm \( \| \cdot \|_{-\infty} \). These Banach spaces, as well as their intersections and unions, play a significant role in connection with the interpolation and sampling of analytic functions. See [18, Section 4.3]. They arise as special cases of weighted Banach spaces \( H^\infty_{\alpha} \) of analytic functions on \( \mathbb{D} \), which was pioneered by the work of Shields and Williams [25], and then have been investigated by many authors, e.g. [7, 8, 23]. An analytic function \( f \) said to belong to the Bloch space \( B_\alpha \) if

\[
\| f \|_{B_\alpha} = \sup_{z \in \mathbb{D}} |f(z)|(1 - |z|^2)^\alpha < \infty.
\]

Indeed, \( \| \cdot \|_{B_\alpha} \) defined above is a seminorm. We shall use the notation \( A \lessapprox B \) if there is a constant \( c > 0 \) not depending on \( A \) or \( B \) such that \( A \leq cB \). We write \( A \asymp B \) whenever \( A \lessapprox B \) and \( B \lessapprox A \). The Bloch space \( B_\alpha \) is a Banach space when normed with \( \| f \| := |f(0)| + \| f \|_{B_\alpha} \). By [18, Proposition 1.13], given \( \alpha > 0 \) for every \( f \in H(D) \) one has

\[
\sup_{z \in \mathbb{D}} |f(z)|(1 - |z|^2)^\alpha \asymp \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2)^{\alpha+1}.
\]

(1.6)

We refer the reader to [28] for a detailed treatment of Bloch spaces. It is also possible to define these spaces with the weight \( (1 - |z|)^\alpha \) instead of \( (1 - |z|^2)^\alpha \). Since \( 1 - |z| \leq 1 - |z|^2 \leq 2(1 - |z|) \), these spaces coincide and the norms are equivalent. In this paper, the operators we shall investigate will be defined on weighted Bergman Fréchet and (LB)-spaces, which arise as intersections and unions of standard weighted Bergman spaces. For \( 1 < p < \infty \), and \( 0 < \alpha < \infty \) they are defined as follows:

\[
A^p_{\alpha,+} := \{ f \in H(D) : \left( \int_{\mathbb{D}} |f(z)|^p ds_\mu(z) \right)^{1/p} < \infty, \forall \mu > \alpha \}
\]

\[
= \bigcap_{\mu > \alpha} A^p_\mu = \bigcap_{n \in \mathbb{N}} A^p_{(\alpha + \frac{1}{n})} = \text{proj}_{n \in \mathbb{N}} A^p_{(\alpha + \frac{1}{n})},
\]

(1.7)

\[
A^p_{\alpha,-} := \{ f \in H(D) : \left( \int_{\mathbb{D}} |f(z)|^p ds_\mu(z) \right)^{1/p} < \infty, \text{ for some } \mu < \alpha \}
\]

\[
= \bigcup_{\mu < \alpha} A^p_\mu = \bigcup_{n \in \mathbb{N}} A^p_{(\alpha - \frac{1}{n})} = \text{ind}_{n \in \mathbb{N}} A^p_{(\alpha - \frac{1}{n})},
\]

(1.8)

where the inductive limit is taken over all \( n \in \mathbb{N} \) such that \( (\alpha - \frac{1}{n}) > 0 \). The paper [20] gives a description of intersections and unions of weighted Bergman spaces of order \( 0 < p < \infty \). Unlike those, we treat the space

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\(A^p_{\alpha+}\) as a Fréchet space when equipped with the locally convex topology generated by the increasing system of norms

\[
\|f\|_{p,\alpha,n} := \left( \int_D |f(z)|^p ds(\alpha + \frac{1}{n}) (z) \right)^{1/p}, \quad n \in \mathbb{N},
\]

(1.9)

for \(f \in A^p_{\alpha+}\) and each \(n \in \mathbb{N}\). We note that for \(0 < \mu < \gamma < \infty\), the natural inclusion map \(\iota_{\mu,\gamma} : A^\mu_\alpha \to A^\gamma_\alpha\) is compact. See e.g. [21, Proposition 3.1]. Hence, \(A^p_{\alpha+}\) is a Fréchet-Schwartz space. The space \(A^p_{\alpha-}\) is a complete (DFS)-space endowed with the finest locally convex topology, such that \(\iota_{\mu,\gamma}\) is continuous. It is also a regular (LB)-space, since every bounded set \(B \subseteq A^p_{\alpha-}\) is contained and bounded in the Banach space \(A^p_\mu\), for some \(0 < \mu < \alpha\). Let us also remark that, for \(\alpha > 0\) we have \(A^p_{\alpha-} \subset A^p_\alpha \subset A^p_{\alpha+}\) with continuous inclusions. Some other properties of \(A^p_{\alpha+}\) and \(A^p_{\alpha-}\) were given in author’s work [21] where these spaces were first introduced in locally convex setup. The Volterra integral operator defined between different weighted Bergman Fréchet or (LB)-spaces has been investigated by the author in [22]. Given \(0 < \alpha < \infty\),

\[
H^{\infty}_{\alpha+} := \{f \in H(D) : \sup_{z \in D} |f(z)| (1 - |z|^2)^\mu < \infty, \forall \mu > \alpha \}
\]

(1.10)

\[
= \text{proj}_{n \in \mathbb{N}} H^{\infty}_{(\alpha + \frac{1}{n})}
\]

Then \(H^{\infty}_{\alpha+}\) is a Fréchet space when endowed with the locally convex topology generated by the increasing sequence of norms

\[
\|f\|_n := \sup_{z \in D} |f(z)| (1 - |z|^2)^{(\alpha + \frac{1}{n})}, \quad n \in \mathbb{N},
\]

for \(f \in H^{\infty}_{\alpha+}\). For any pair \(0 < \mu < \alpha < \infty\), the canonical inclusion map \(\iota_{\mu,\alpha} : H^{\infty}_\mu \to H^{\infty}_\alpha\) is compact [10, Theorem 3.3]. Hence, both \(H^{\infty}_{\alpha+}\), and \(H^{\infty}_{\alpha-}\) are Schwartz spaces. The regular (LB)-space \(H^{\infty}_{\alpha-}\) is endowed with the finest locally convex topology making \(\iota_{\mu,\alpha}\) continuous. Several important properties of growth Fréchet and (LB)-spaces can be found in [2, 11, 12]. The Volterra integral operator acting on a growth Fréchet or (LB)-space has been investigated by Bonet [9] in terms of continuity, compactness, and spectrum. For a study of weighted composition operators acting on these spaces, see [17].

In Section 2, we first deal with operators defined from \(A^p_{\alpha+}\) (resp. \(A^p_{\alpha-}\)) into \(H^{\infty}_{\beta+}\) (resp. \(H^{\infty}_{\beta-}\)). If we pick the symbol function \(g : \mathbb{D} \to \mathbb{C}\) in such a way that it belongs to the Bloch-type space \(B_\tau\), for every positive \(\tau > \beta + 1 - (2 + \alpha)/p\), we show that the continuity of Volterra integral operator \(T_g : A^p_{\alpha+} \to H^{\infty}_{\beta+}\) is equivalent to the continuity of pointwise multiplication operator \(M_g : A^p_{\alpha+} \to H^{\infty}_{\beta+}\), and the continuity of the weighted composition operator \(W_{g,\varphi} : A^p_{\alpha+} \to H^{\infty}_{\beta+}\) provided that \(\varphi(0) = 0\). When we take another weighted Bergman Fréchet space (resp. (LB)-space) as the range space, we show that the symbol function \(g\) belonging to the growth Fréchet space \(H^{\infty}_{\gamma+}\), where \(\gamma = (2 + \beta)/q - (2 + \alpha)/p\), characterizes the continuity of the pointwise multiplication operator \(M_g : A^p_{\alpha+} \to A^3_{\beta+}\) as well as the Volterra integral operator \(T_g : A^p_{\alpha+} \to A^3_{\beta+}\). The same condition is also valid for the (LB)-space case. On the other hand, the continuity criterion for the weighted composition operator in between is different in this case. We give this condition in Section 3 as a
straightforward generalization of the well-known characterizations of Ćučković and Zhao [26] related to Carleson measures. Fortunately, resting on the arguments of Bourdon [13], the condition \( g \in H_\infty^\infty \), which is equivalent to continuity of pointwise multiplication and Volterra integral operators between \( A_{\alpha+}^p \) and \( A_{\beta+}^\gamma \) answers the question of invertibility for the weighted composition operator \( W_{g,\varphi} \) acting on \( A_{\alpha+}^p \) (in this case, \( \gamma = 0 \)), whenever \( \varphi \) is an automorphism of \( \mathbb{D} \). Finally, we give some results concerning the eigenvalues of composition operators \( C_\varphi \) acting on \( A_{\alpha+}^p \) or \( A_{\alpha-}^p \) in connection with their essential spectral radius defined on the Banach space \( A_\alpha^p \).

2. Continuous Volterra, multiplication, and weighted composition operators between weighted Fréchet and (LB)-spaces

Let us note that the continuity and compactness of \( W_{g,\varphi}: A_\alpha^p \to H_\beta^\infty \) was described in [24, Theorem 3.1], and in [27, Theorem 2.2] for more general weights, that is, \( W_{g,\varphi}: A_\alpha^p \to H_\alpha^\infty \). In [14] and [15] weighted composition operators \( W_{g,\varphi}: X \to H_v^\infty \) are investigated in a uniform approach covering a large family of Banach spaces of analytic functions concerning the space \( X \). Before we start our discussion on operators between Fréchet or (LB)-spaces, we need to prove the following result concerning related Banach spaces.

**Proposition 2.1** Let \( g \) be an analytic function. Let \( \varphi \) be an analytic self map on \( \mathbb{D} \) satisfying \( \varphi(0) = 0 \). Given \( 1 \leq p < \infty \) and \(-1 < \alpha, \beta < \infty\), let \( \gamma := \beta + 1 - \frac{2+\alpha}{p} \) be nonnegative. Then, the following statements are equivalent.

(1) The symbol \( g \) belongs to the Bloch space \( B_\gamma \).

(2) The Volterra operator \( T_g: A_\alpha^p \to H_\beta^\infty \) is continuous.

(3) The pointwise multiplication operator \( M_g: A_\alpha^p \to H_\beta^\infty \) is continuous.

(4) The weighted composition operator \( W_{g,\varphi}: A_\alpha^p \to H_\beta^\infty \) is continuous.

**Proof** (1) \( \Rightarrow \) (2). Note that for \( 1 < p < \infty \) for any \( f \in H(\mathbb{D}) \), we have (see e.g. [18, p. 39])

\[
|f(z)|^p(1 - |z|^2)^t \lesssim \int_\mathbb{D} |f(w)|^p(1 - |w|^2)^{t-2}ds(w), \quad t \in \mathbb{R}.
\]  

(2.1)

We also mention that (see e.g. [5, Lemma 2]) for every \( f \in H(\mathbb{D}) \) we have

\[
\int_\mathbb{D} |f(z)|^p(1 - |z|^2)^\alpha ds(z) \lesssim |f(0)|^p + \int_\mathbb{D} |f'(z)|^p(1 - |z|^2)^{p+\alpha}ds(z).
\]  

(2.2)
Now let \( g \in B_\gamma \). Then, for any \( f \in A_\alpha^p \) by (2.1) we have
\[
\|T_g f\|_{p, -\beta}^p = \sup_{z \in \mathbb{D}} \left| \int_0^z f(\xi)g'(\xi) d\xi \right|^p (1 - |z|^2)^{p\beta} \\
\lesssim \int_\mathbb{D} \left| \int_0^w f(\xi)g'(\xi) d\xi \right|^p (1 - |w|^2)^{p\beta - 2} ds(w) \\
\lesssim |f(0)|^p + \int_\mathbb{D} \left( \int_0^w f(\xi)g'(\xi) d\xi \right)^p (1 - |w|^2)^{p\beta - 2 + p} ds(w) \\
= |f(0)|^p + \int_\mathbb{D} |f(w)g'(w)|^p (1 - |w|^2)^{p\beta - 2 + p} ds(w) \\
\leq |f(0)|^p + \int_\mathbb{D} |f(w)|^p |g'(w)|^p (1 - |w|^2)^{p\beta - 2 + p + \alpha - \alpha} ds(w) \\
\leq |f(0)|^p + \sup_{w \in \mathbb{D}} |g'(w)|^p (1 - |w|^2)^{p(\beta + 1) - (2 + \alpha)} \int_\mathbb{D} |f(w)|^p (1 - |w|^2)^{\alpha} ds(w) \\
= |f(0)|^p + \frac{1}{\alpha + 1} \|g\|_{B_\alpha}^p \|f\|_{p, \alpha}^p < \infty,
\]
where the second inequality is due to (2.2). Hence \( T_g : A_\alpha^p \to H_\beta^\infty \) is continuous.

(2) \( \Rightarrow \) (1). For \( w \in \mathbb{D} \), let us pick
\[
f_w(z) := \left( \frac{1 - |w|^2}{1 - wz} \right)^{\frac{2+\alpha}{p}}.
\]
A canonical calculation yields (see e.g. [28, p. 52]) \( \|f_w\|_{p, \alpha}^p = 1 \). Since \( T_g : A_\alpha^p \to H_\beta^\infty \) is continuous, by (1.6) we obtain
\[
\|f_w\|_{p, \alpha} \gtrsim \|T_g f_w\|_{-\beta} \simeq \|T_g f_w\|_{B_{\beta+1}}.
\]
Then, for any \( w \in \mathbb{D} \), the latter yields,
\[
1 \gtrsim \sup_{z \in \mathbb{D}} [(T_g f_w)'(z)] (1 - |z|^2)^{\beta + 1} \geq [(T_g f_w)'(w)] (1 - |w|^2)^{\beta + 1} \\
= \left| \left( \int_0^w f_w(\xi)g'(\xi) d\xi \right)^p (1 - |w|^2)^{\beta + 1} \\
= |f_w(w)g'(w)| (1 - |w|^2)^{\beta + 1} \\
= |g'(w)| \left( \frac{1 - |w|^2}{1 - wz} \right)^{\frac{2+\alpha}{p}} (1 - |w|^2)^{\beta + 1} \\
= |g'(w)| (1 - |w|^2)^{\gamma}.
\]  
(2.3)
Since \( w \in \mathbb{D} \) was arbitrary, by (2.3), \( \|g\|_{B_\alpha} \lesssim 1 \). This proves (1).

(1) \( \Leftrightarrow \) (3). Follows by (1.6) and the previous result in [24, Corollary 3.3].
(1) $\Rightarrow$ (4). We make use of the following well-known estimate. For any $f \in A^p_\alpha$,

$$|f(z)| \lesssim \frac{\|f\|_{p,\alpha}}{(1 - |z|^2)^{1 + \frac{2 + \alpha}{p}}}, \quad \forall z \in \mathbb{D}. \quad (2.4)$$

Note that, by Schwartz’s lemma, one has $|\varphi(z)| \leq |z|$. Hence,

$$\sup_{z \in \mathbb{D}} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} < \infty. \quad (2.5)$$

Let $g \in B_{\gamma}$. Then, by (1.6), for any $f \in A^p_\alpha$ we have

$$\|W_{g,\varphi}f\|_{-\beta} = \sup_{z \in \mathbb{D}} |W_{g,\varphi}f(z)|(1 - |z|^2)^{\beta}$$

$$= \sup_{z \in \mathbb{D}} |g(z)f(\varphi(z))|(1 - |z|^2)^{\beta}$$

$$\lesssim \sup_{z \in \mathbb{D}} |g(z)||f(\varphi(z))|(1 - |z|^2)^{\gamma - 1 + \frac{2 + \alpha}{p}}$$

$$\leq \|g\|_{B_{\gamma}} \sup_{z \in \mathbb{D}} |f(\varphi(z))|(1 - |z|^2)^{\frac{2 + \alpha}{p}}$$

$$\leq \|g\|_{B_{\gamma}} \|f\|_{p,\alpha} \sup_{z \in \mathbb{D}} \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\frac{2 + \alpha}{p}} < \infty,$$

where the second inequality is due to (2.4), and the last one is by (2.5).

(4) $\Rightarrow$ (1). Let $W_{g,\varphi} : A^p_\alpha \to H^\infty_{\beta}$ be continuous. Given $w = \varphi(z_0) \in \mathbb{D}$ for a fixed $z_0 \in \mathbb{D}$, let us define

$$h_w(z) := \left( \frac{1 - |w|^2}{1 - |\varphi(z)|^2} \right)^{\frac{2 + \alpha}{p}},$$

for which $\|h_w\|_{p,\alpha} = 1$. Then, continuity of $W_{g,\varphi} : A^p_\alpha \to H^\infty_{\beta}$ and (1.6) imply

$$1 = \|h_w\|_{p,\alpha} \geq \|W_{g,\varphi}h_w\|_{-\beta} = \sup_{z \in \mathbb{D}} |g(z)h_w(\varphi(z))|(1 - |z|^2)^{\beta}$$

$$\lesssim \sup_{z \in \mathbb{D}} |g(z)| |h_w(\varphi(z))|(1 - |z|^2)^{\gamma - 1 + \frac{2 + \alpha}{p}}$$

$$= |g(z_0)|(1 - |z_0|^2)^{\gamma - 1} h_w(\varphi(z_0))(1 - |z|^2)^{\frac{2 + \alpha}{p}}$$

$$= |g(z_0)|(1 - |z_0|^2)^{\gamma - 1} \left( \frac{1 - |w|^2}{1 - |\varphi(z)|^2} \right) \left( 1 - |z|^2 \right)^{\frac{2 + \alpha}{p}}$$

$$\geq |g(z_0)|(1 - |z_0|^2)^{\gamma - 1} \left( \frac{1 - |z_0|^2}{1 - |\varphi(z_0)|^2} \right)^{\frac{2 + \alpha}{p}} \sim |g(z_0)|(1 - |z_0|^2)^{\gamma - 1},$$

for an arbitrary $z_0 \in \mathbb{D}$, by (2.5). Hence, $g \in B_{\gamma}$.  

The following result is well-known. For a proof, see e.g. [3, Lemma 25].
Lemma 2.2 Let \( E = \text{proj}_m E_m \) and \( F = \text{proj}_n F_n \) be Fréchet spaces such that \( E \) (resp. \( F \)) is the intersection of the sequence of Banach spaces \( E_m \) (resp. \( F_n \)), \( E \) is dense in \( E_m \) and \( E_{m+1} \subset E_m \) with continuous inclusion for each \( m \) (resp. \( F \) is dense in \( F_n \) and \( F_{n+1} \subset F_n \) with continuous inclusion for each \( n \)). Let \( T : E \to F \) be a linear operator. Then

(i) \( T \) is continuous if and only if for each \( n \), there is \( m \) such that \( T \) has a unique continuous linear extension \( T_{m,n} : E_m \to F_n \).

(ii) Assume \( T \) is continuous. Then, \( T \) is bounded if and only if there is \( m \) such that for each \( n \), \( T \) has a unique continuous linear extension \( T_{m,n} : E_m \to F_n \).

The following lemma for (LB)-spaces is also known. A proof can be seen in [4, Lemma 4.1].

Lemma 2.3 Let \( E = \text{ind}_m E_m \) and \( F = \text{ind}_n F_n \) be (LB)-spaces such that \( E \) (resp. \( F \)) is the union of the sequence of Banach spaces \( E_m \) (resp. \( F_n \)). Let \( T : E \to F \) be a linear operator. Then

(i) \( T \) is continuous if and only if, for all \( m \in \mathbb{N} \), there exists \( n \in \mathbb{N} \) such that \( T(E_m) \subset F_n \) and \( T : E_m \to F_n \) is continuous.

(ii) Let \( T \) be continuous and let \( F \) be regular. Then, \( T \) is bounded if and only if there exists \( n \in \mathbb{N} \) such that for all \( m \), \( T(E_m) \subset F_n \) and \( T : E_m \to F_n \) is continuous.

With the help of Lemma 2.2 and Lemma 2.3 we extend Proposition 2.1 to the setup of Fréchet and (LB)-spaces.

Proposition 2.4 Given \( 1 < p < \infty \) and \( 0 < \alpha, \beta < \infty \), let \( \gamma := \beta + 1 - \frac{2+\alpha}{p} \) be non-negative. Let \( \varphi \) be an analytic self map on \( \mathbb{D} \) satisfying \( \varphi(0) = 0 \). Then, the following statements are equivalent.

1. The symbol \( g \in H(\mathbb{D}) \) satisfies
   \[
g \in \bigcap_{\tau > \gamma} B_{\tau}. \tag{2.6}\]

2. The Volterra operator \( T_g : A^p_{\alpha+} \to H^\infty_{\beta+} \) is continuous.

3. The Volterra operator \( T_g : A^p_{\alpha-} \to H^\infty_{\beta-} \) is continuous.

4. The pointwise multiplication operator \( M_g : A^p_{\alpha+} \to H^\infty_{\beta+} \) is continuous.

5. The pointwise multiplication operator \( M_g : A^p_{\alpha-} \to H^\infty_{\beta-} \) is continuous.

6. The weighted composition operator \( W_{g,\varphi} : A^p_{\alpha+} \to H^\infty_{\beta+} \) is continuous.

7. The weighted composition operator \( W_{g,\varphi} : A^p_{\alpha-} \to H^\infty_{\beta-} \) is continuous.

Proof (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (4). By Lemma 2.2, the Volterra operator \( T_g : A^p_{\alpha+} \to H^\infty_{\beta+} \) (resp. the pointwise multiplication operator \( M_g : A^p_{\alpha+} \to H^\infty_{\beta+} \)) is continuous if and only if for every \( \varepsilon > 0 \) there exists \( \delta \in (0, \varepsilon] \)
such that \( T_g : A^p_{\alpha + \delta} \to H^\infty_{\beta + \varepsilon} \) (resp. \( M_g : A^p_{\alpha + \delta} \to H^\infty_{\beta + \varepsilon} \)) is continuous. By Proposition 2.1 this is equivalent to say that

\[
g \in B_{\gamma + \varepsilon - \delta},
\]

(2.7)

where \( \delta := \frac{2}{p} < \varepsilon \). Clearly (2.7) is equivalent to (2.6).

(1) \( \Rightarrow \) (3). Suppose (2.6) holds. Then, for every \( \varepsilon \in (0, \min\{\frac{\alpha + 1}{p}, \frac{\beta}{p - 1}\}) \) we have \( g \in B_{\gamma + \varepsilon} \). Then, given \(-1 < \mu := \alpha - p^2 \varepsilon \) pick \( \eta := \beta - (p - 1)\varepsilon \). We see that \( \gamma + \varepsilon = \eta + 1 - \frac{2 + \mu}{p} \), which yields \( g \in B_{\eta + 1 - \frac{2 + \mu}{p}} \). By Proposition 2.1 this is equivalent that \( T_g : A^p_\mu \to H^\infty_\eta \) is continuous. In the light of Lemma 2.3, \( T_g : A^p_\alpha \to H^\infty_\beta \) is continuous.

(3) \( \Rightarrow \) (1). Let \( T_g : A^p_\alpha \to H^\infty_\beta \) be continuous. Then, for every \( \varepsilon \in (0, \frac{\alpha + 1}{p}) \), there exists \( \delta \in (0, \min\{\varepsilon, \frac{\beta}{p}\}) \) such that \( T_g : A^p_{\alpha - \varepsilon} \to H^\infty_{\beta - \delta} \) is continuous. Without loss of any generality, let \( \gamma + \varepsilon = \eta + 1 - \frac{2 + \mu}{p} \), since otherwise \( g \) is constant so there is nothing to prove. By Proposition 2.1 this is equivalent that \( g \in B_{\gamma + \frac{\beta}{p} - \delta} \subseteq B_{\gamma + \varepsilon - \delta} \). Hence \( g \) satisfies (2.7), equivalently (2.6).

(1) \( \iff \) (5). Identical to (1) \( \iff \) (3).

(1) \( \iff \) (6). Suppose that \( \varphi \) satisfies (2.5). The symbol function \( g \) satisfying (2.6) is equivalent to say that for every \( \varepsilon > 0 \), there exists \( \tilde{\delta} \in (0, \varepsilon] \) such that \( g \in B_{\gamma + \varepsilon - \tilde{\delta}} \). Equivalently, by Proposition 2.1, the weighted composition operator \( W_{g, \varphi} : A^p_{\alpha + \delta} \to H^\infty_{\beta + \varepsilon} \) is continuous, for \( \delta = \frac{\tilde{\delta}}{p} \). By Lemma 2.2, this is equivalent to say that \( W_{g, \varphi} : A^p_\alpha \to H^\infty_\beta \) is continuous.

(1) \( \Rightarrow \) (7). If \( g \) satisfies (2.6), for every \( \varepsilon \in (0, \frac{2 + \alpha}{2p}) \) one has \( g \in B_{\gamma + \varepsilon} \subseteq B_{\beta + 1 - \varepsilon} \), since \( \frac{2 + \alpha}{p} > 2\varepsilon \). Then, by (1.6) \( g \in H^\infty_{\beta - \varepsilon} \) and the rest follows very similar to Fréchet case.

(7) \( \Rightarrow \) (1). The weighted composition operator \( W_{g, \varphi} : A^p_{\alpha - \varepsilon} \to H^\infty_{\beta - \varepsilon} \) is continuous if and only if, for every \( \varepsilon \in (0, \alpha + 1) \), there exists \( \delta \in (0, \min\{\varepsilon, \beta\}] \) such that \( W_{g, \varphi} : A^p_{\alpha - \varepsilon} \to H^\infty_{\beta - \delta} \) is continuous if and only if \( g \in B_{\gamma + \frac{\beta}{p} - \delta} \subseteq B_{\gamma + \varepsilon - \delta} \), by Proposition 2.1. This is equivalent to (2.7) hence to (2.6).

Proposition 2.5 Let \( 1 < p \leq q < \infty \), and \( 0 < \alpha, \beta < \infty \). Let \( \gamma := \frac{2 + \beta}{q} - \frac{2 + \alpha}{p} \) be non-negative. Then, for an analytic map \( g : \mathbb{D} \to \mathbb{C} \), the following statements are equivalent.

(1) The symbol \( g \) belongs to the growth Fréchet space \( H^\infty_{\gamma +} \).

(2) The pointwise multiplication operator \( M_g : A^p_\alpha \to A^q_\gamma \) is continuous.

(3) The pointwise multiplication operator \( M_g : A^p_\gamma \to A^q_\beta \) is continuous.

(4) The Volterra operator \( T_g : A^p_\alpha \to A^q_\beta \) is continuous.

(5) The Volterra operator \( T_g : A^p_\alpha \to A^q_\beta \) is continuous.

Proof (1) \( \Rightarrow \) (2). Let \( M_g : A^p_\alpha \to A^q_\gamma \) be continuous. Then, for every \( \varepsilon > 0 \) given \( \mu := \beta + q\varepsilon \) there exists
\[ \alpha < \eta < \alpha + \varepsilon \] such that \( M_g : A_p^\eta \rightarrow A_p^\mu \) is continuous. Hence, for every \( z \in \mathbb{D} \)

\[
|g(z)|(1 - |z|^2)^{\gamma+\varepsilon} = |g(z)|(1 - |z|^2)^{\frac{2q - 2q}{q}} < |g(z)|(1 - |z|^2)^{\frac{2q - 2q}{q}} < \infty.
\]

So by [26, Theorem 9], \( g \in H^\infty_{\gamma+\varepsilon} \). Hence, \( g \in H^\infty_{\gamma+\varepsilon} \).

(2) \Rightarrow (1). Let \( g \in H^\infty_{\gamma+\varepsilon} \). Then, for every \( \varepsilon > 0 \) there exists \( \delta \in (0,\varepsilon) \) such that \( g \in H^\infty_{\gamma+\varepsilon-\delta} \).

Given \( \mu := \beta + q\varepsilon \), define \( \eta := \alpha + p\delta \). Observe that for every \( z \in \mathbb{D} \),

\[
|g(z)|(1 - |z|^2)^{\frac{2q - 2q}{q}} = |g(z)|(1 - |z|^2)^{\frac{2q - 2q}{q}} < |g(z)|(1 - |z|^2)^{\frac{2q - 2q}{q}} < \infty.
\]

So \( g \in H^\infty_{\frac{2q}{q} - \frac{2q}{q}} \). By [26, Theorem 9], \( M_g : A_p^\mu \rightarrow A_p^\alpha \) is continuous. Therefore, \( M_g : A_{\alpha+}^p \rightarrow A_{\beta+}^q \) is continuous.

(3) \Rightarrow (1). Let \( M_g : A_{p-}^\alpha \rightarrow A_{\eta-}^q \) be continuous. Then, for every \( \varepsilon > 0 \), given \(-1 < \mu := \alpha - p\varepsilon \) there exists \(-1 < - \beta - p\varepsilon < \eta < \beta \) such that \( M_g : A_{\mu}^p \rightarrow A_{\eta}^q \) is continuous. Then, for every \( z \in \mathbb{D} \),

\[
|g(z)|(1 - |z|^2)^{\gamma+\varepsilon} < |g(z)|(1 - |z|^2)^{\frac{2q - 2q}{q}} < \infty.
\]

So, by [26, Theorem 9], \( g \) belongs to \( H^\infty_{\gamma+\varepsilon} \) and hence to \( H^\infty_{\gamma+\varepsilon} \).

(1) \Rightarrow (3). Suppose that \( g \in H^\infty_{\gamma+\varepsilon} \). Let \( x = \min\{\alpha + 1, (\beta + 1)\} \). Then, for every \( \varepsilon \in (0, x) \) we have \( g \in H^\infty_{\gamma+\varepsilon} \). Given \(-1 < \mu := \alpha - \frac{pq}{q+1} \varepsilon \), pick \(-1 < \eta := \beta - \frac{q}{q+1} \varepsilon \). Then, for every \( z \in \mathbb{D} \),

\[
|g(z)|(1 - |z|^2)^{\frac{2q - 2q}{q}} = |g(z)|(1 - |z|^2)^{\frac{2q - 2q}{q}} < |g(z)|(1 - |z|^2)^{\frac{2q - 2q}{q}} < \infty.
\]

Hence, \( g \in H^\infty_{\frac{2q}{q} - \frac{2q}{q}} \). By [26, Theorem 9], \( M_g : A_{\mu}^p \rightarrow A_{\eta}^q \) is continuous. Therefore, \( M_g : A_{\alpha-}^p \rightarrow A_{\beta-}^q \) is continuous.

(1) \Leftrightarrow (4) \Leftrightarrow (5). Follows by (1.6), Proposition 2.5, and [22, Proposition 2.2]. \( \square \)

Proposition 2.5 will help us characterize the invertibility of a weighted composition operator acting on a weighed Bergman Fréchet or a weighted Bergman (LB)-space. See Proposition 3.7. The following statement is derived from [26, Theorem 11] via Lemma 2.2. Its (LB)-space version can be produced in an analogue way.

**Proposition 2.6** Let \( g \) be an analytic function on \( \mathbb{D} \). Let \( 1 \leq q < p < \infty \), and \( \alpha, \beta > 0 \). Then, the following statements are equivalent.

1. The multiplication operator \( M_g : A_p^\alpha \rightarrow A_p^\beta \) is continuous.

2. For every \( \mu > \beta \), there exists \( \nu \in (\alpha, \alpha + \mu - \beta) \) such that \( g \in A_{\nu}^p \), where \( \frac{1}{s} = \frac{1}{q} - \frac{1}{p} \) and \( \eta = s \left( \frac{\bar{q}}{q} - \frac{\bar{p}}{p} \right) \).

3. Weighted composition operators between weighted Bergman Fréchet and (LB)-spaces

**3.1. Continuous weighted composition operators between different weighted Bergman Fréchet or (LB)-spaces**

An operator \( T \) on a Fréchet space \( X \) into itself is called bounded (resp. compact) if there exists a neighborhood \( U \) of the origin of \( X \) such that \( TU \) is a bounded (resp. relatively compact) set in \( X \). The following result is a
consequence of [26, Theorem 1] along with Lemma 2.2 and Lemma 2.3.

**Proposition 3.1** Let $1 < p \leq q < \infty$ and $0 < \alpha, \beta < \infty$. Let $g: \mathbb{D} \to \mathbb{C}$ be an analytic function and let \( \varphi: \mathbb{D} \to \mathbb{D} \) be an analytic self map. Then,

1. The weighted composition operator \( W_{g,\varphi}: A^p_{\alpha+} \to A^q_{\beta+} \) is continuous if and only if for every \( \mu > \beta \) there exists \( \eta \in (\alpha, \alpha + \mu - \beta) \) such that

\[
\sup_{z \in \mathbb{D}} \int_{\varphi(\mathbb{D})} \left( \frac{1 - |z|^2}{|1 - \bar{\varphi}(w)|^2} \right)^{\frac{2+\mu}{p}} |g(w)|^q ds_\mu(w) < \infty. \tag{3.1}
\]

2. The weighted composition operator \( W_{g,\varphi}: A^p_{\alpha-} \to A^q_{\beta-} \) is continuous if and only if for every \( \zeta \in (0, \alpha) \) there exists \( \theta \in [\zeta, \alpha) \) such that

\[
\sup_{z \in \mathbb{D}} \int_{\varphi(\mathbb{D})} \left( \frac{1 - |z|^2}{|1 - \bar{\varphi}(w)|^2} \right)^{\frac{2+\mu}{p}} |g(w)|^q ds_\theta(w) < \infty. \tag{3.2}
\]

Similarly, we obtain the following proposition via Lemma 2.2 and [26, Theorem 3]. An (LB)-space version can be easily derived.

**Proposition 3.2** Let $1 < q < p < \infty$ and $0 < \alpha, \beta < \infty$. Let $g: \mathbb{D} \to \mathbb{C}$ be an analytic function and let \( \varphi: \mathbb{D} \to \mathbb{D} \) be an analytic self map. Then, the following statements are equivalent.

1. The weighted composition operator \( W_{g,\varphi}: A^p_{\alpha+} \to A^q_{\beta+} \) is continuous.

2. For every \( \mu > \beta \) there exists \( \nu \in (\alpha, \alpha + \mu - \beta) \) such that for \( s := \frac{p}{p-q} \) we have

\[
\int_{\varphi(\mathbb{D})} \left( \frac{1 - |z|^2}{|1 - \bar{\varphi}(w)|^{1+2\nu}} \right)^{\frac{2+\nu}{p}} |g(w)|^q ds_\mu(w) \in A^s_\nu.
\]

**Lemma 3.3** (i) Let \( E = \text{proj}_m E_m \) and \( F = \text{proj}_n F_n \) be Fréchet spaces such that \( E \) (resp. \( F \)) is the intersection of the sequence of Banach spaces \( E_m \) (resp. \( F_n \)), \( E \) is dense in \( E_m \) and \( E_{m+1} \subset E_m \) with continuous inclusion for each \( m \) (resp. \( F \) is dense in \( F_n \) and \( F_{n+1} \subset F_n \) with continuous inclusion for each \( n \)). Let \( T: E \to F \) be a linear operator. Assume \( T \) is continuous. Then \( T \) is bounded if and only if there is \( m \) such that for each \( n \), \( T \) has a unique continuous linear extension \( T_{m,n}: E_m \to F_n \).

(ii) Let \( X = \text{ind} X_n \) and \( Y = \text{ind} Y_m \) be two \((LB)\)-spaces which are increasing unions of Banach spaces \( X = \bigcup_{n=1}^\infty X_n \) and \( Y = \bigcup_{m=1}^\infty Y_m \). Let \( T: X \to Y \) be a continuous linear map. Assume that \( Y \) is a regular \((LB)\)-space. Then, \( T \) is bounded if and only if there exists \( m \in \mathbb{N} \) such that \( T(X_n) \subset Y_m \) and \( T: X_n \to Y_m \) is continuous for all \( n \geq m \).

The following proposition is a consequence of [26, Corollary 1] and Lemma 3.3.

**Proposition 3.4** Let $1 < p \leq q < \infty$ and $0 < \alpha, \beta < \infty$. Let $g: \mathbb{D} \to \mathbb{C}$ be an analytic function and let \( \varphi: \mathbb{D} \to \mathbb{D} \) be an analytic self map. Then,
(1) The weighted composition operator \( W_{g,\varphi} : A^p_{\alpha+} \to A^q_{\beta+} \) is compact if and only if it is continuous and there exists \( \mu > \alpha \) such that for each \( \eta \in (\alpha, \mu] \) we have

\[
\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} \left( \frac{1 - |z|^2}{|1 - \varphi(w)|^2} \right)^{\frac{2\alpha + q - p}{p}} |g(w)|^q ds_\eta(w) < \infty.
\]  

(3.3)

(2) The weighted composition operator \( W_{g,\varphi} : A^p_{\alpha-} \to A^q_{\beta-} \) is compact if and only if it is continuous and there exists \( \zeta \in (0, \alpha) \) such that for each \( \theta \in [\zeta, \alpha) \) we have

\[
\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} \left( \frac{1 - |z|^2}{|1 - \varphi(w)|^2} \right)^{\frac{2\zeta + q - p}{p}} |g(w)|^q ds_\zeta(w) < \infty.
\]  

(3.4)

**Proof**

(1) Given \( \alpha, \beta > 0 \), let \( W_{g,\varphi} : A^p_{\alpha+} \to A^q_{\beta+} \) be bounded. Since \( A^p_{\alpha+} \) is a Schwartz space, this is equivalent to assume that \( W_{g,\varphi} \) is compact. Lemma 3.3(i) applied to \( A^p_{\alpha+} \) this is equivalent that there exists \( \mu > \alpha \) such that for all \( \beta < \eta < \beta + \mu - \alpha \), the weighted composition operator \( W_{g,\varphi} : A^p_{\mu} \to A^q_{\eta} \) is continuous. This is equivalent, by [26, Theorem 1], that (3.3) holds.

(2) Very similar to part (1) if we apply Lemma 3.3(ii) and [26, Theorem 1].

\( \square \)

### 3.2. Invertible weighted composition operators acting on a weighted Bergman Fréchet or (LB)-space

The characterizations of invertible weighted composition operators on the Fréchet space \( A^p_{\alpha+} \) and on the (LB)-space \( A^p_{\alpha-} \) are consequences of the following results by Bourdon [13, Theorem 2.2; Corollary 2.3]. These arguments were also used to characterize invertible weighted composition operators acting on the growth Fréchet space \( H^\infty_{\alpha+} \) and the growth (LB)-space \( H^\infty_{\alpha-} \) in [17, Proposition 4].

**Theorem 3.5** Suppose that \( E \) is a space of analytic functions on \( \mathbb{D} \) such that

(i) \( W_{g,\varphi} \) maps \( E \) to \( E \).

(ii) \( E \) contains a nonzero constant function.

(iii) \( E \) contains a function of the form \( z \to z + c \) for some constant \( c \).

(iv) There is a dense subset \( S \) of the unit circle such that, for each point in \( S \), there is a function in \( E \) that does not extend analytically to a neighborhood of that point.

If \( W_{g,\varphi} : E \to E \) is invertible, then \( \varphi \) is an automorphism of \( \mathbb{D} \).

**Theorem 3.6** If \( E, g \) and \( \varphi \) satisfy the hypotheses of Lemma 3.5, and for each \( f \in E \) we have \( f \circ h \in E \) for all automorphism \( h \) of \( \mathbb{D} \), then \( W_{g,\varphi} \) is invertible on \( E \) if and only if \( \varphi \) is an automorphism of \( \mathbb{D} \) and both \( g \) and \( 1/g \) map \( E \) into \( E \).
Whenever it is continuous, that is, (3.1) or (3.2) is satisfied, $W_{g,\varphi}$ fulfills hypothesis (i) of Theorem 3.5. Hypotheses (ii) and (iii) are verified by both $A_{\alpha+}^p$ and $A_{\alpha-}^p$, since they contain the constants and polynomials. For hypothesis (iv), let us consider the function $f_{w,s} : \mathbb{D} \to \mathbb{C}$ given by $f_{w,s} := \frac{1}{(w-s)^2}$, for $w \in \partial \mathbb{D}$ and $s > 0$. It is easy to see that $f_{w,s} \in A_{\alpha+}^p$ and $f_{w,s} \in A_{\alpha-}^p$. However, in any neighborhood $U$ of $w$, we see that $f_{a,s} \notin H(\mathbb{D})$ for any $a \in U$. So it does not extend analytically to any neighborhood of $w$ (cf. [17, Remark 2]).

**Proposition 3.7** Let $g, \varphi \in H(\mathbb{D})$ and $\varphi(D) \subset \mathbb{D}$. Let $1 < p < \infty$, and $0 < \alpha < \infty$. Then, the following statements are equivalent.

1. $g \in H_{0+}^\infty$, and $1/g \in H_{0+}^\infty$.
2. The weighted composition operator $W_{g,\varphi} : A_{\alpha+}^p \to A_{\alpha+}^p$ is invertible.
3. The weighted composition operator $W_{g,\varphi} : A_{\alpha-}^p \to A_{\alpha-}^p$ is invertible.

**Proof** Since both $A_{\alpha+}^p$ and $A_{\alpha-}^p$ satisfy all hypotheses of Theorem 3.5, we apply Theorem 3.6 to reach that $W_{g,\varphi}$ is invertible if and only if $\varphi$ is an automorphism of $\mathbb{D}$ and $M_g$ and $M_{1/g}$ are continuous on $A_{\alpha+}^p$ (resp. $A_{\alpha-}^p$). Hence the conclusion follows from proposition 2.5. \qed

### 3.3. Some results on eigenvalues of composition operators acting on a weighted Bergman Fréchet or (LB)-space

For $T \in \mathcal{L}(E)$, the resolvent set $\rho(T; E)$ of $T$ consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(E)$. The set $\sigma(T; E) := \mathbb{C} \setminus \rho(T)$ is called the spectrum of $T$. The point spectrum $\sigma_{pt}(T; X)$ of $T$ consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. The essential norm $\|T\|_{e,X}$ of an operator $T$ on a Banach space $X$ is the distance of the operator to the set of compact operators on $X$. The essential spectral radius is given by $r_e(T; X) = \lim_n \|T^n\|^{1/n}_{e,X}$. The following lemma is well known. For a proof, see [19, Lemma 2.4] and [17, Lemma 3.4].

**Lemma 3.8** Let $X \subset H(\mathbb{D})$ be a continuously included subspace of holomorphic functions containing the polynomials. Let $\varphi, g \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi(0) = 0$. Suppose that $\varphi$ is not a constant function. Then,

$$\sigma_{pt}(W_{g,\varphi}; X) \subseteq \{g(0)\varphi'(0)^j\}_{j=0}^\infty.$$ 


The following lemma is due to [19, Lemma 2.3].

**Lemma 3.9** Let $X \subset H(\mathbb{D})$ be a continuously included subspace of holomorphic functions containing the polynomials. Let $g, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$, $g \neq 0$, and $\varphi(0) = 0$. Then,

1. $g(0) \in \sigma(W_{g,\varphi}; X)$.
2. For every $j \in \mathbb{N}$ we have $g(0)\varphi'(0)^j \in \sigma(W_{g,\varphi}; X)$.

**Proposition 3.10** Let $1 < p < \infty$, and $0 < \alpha < \infty$. Suppose that $\varphi \in H(\mathbb{D}), \varphi(\mathbb{D}) \subset \mathbb{D}$, $0 < |\varphi'(0)| < 1$, and $\varphi$ is not a rotation. Then,
(1) The point spectrum of the composition operator \( C \varphi : A^p_{\alpha +} \to A^p_{\alpha +} \) satisfies the inclusions
\[
\{ \varphi'(0)^j \}_{j=0}^{\infty} \setminus \overline{B}(0, r_e(C \varphi, A^p_{\alpha})) \subset \sigma_{pt}(C \varphi, A^p_{\alpha +}) \subset \{ \varphi'(0)^j \}_{j=0}^{\infty}.
\]

(2) The point spectrum of the composition operator \( C \varphi : A^p_{\alpha -} \to A^p_{\alpha -} \) satisfies the inclusions
\[
\{ \varphi'(0)^j \}_{j=0}^{\infty} \setminus \overline{B}(0, r_e(C \varphi, A^p_{\alpha})) \subset \sigma_{pt}(C \varphi, A^p_{\alpha -}) \subset \{ \varphi'(0)^j \}_{j=0}^{\infty}.
\]

Proof

(1) By Lemma 3.8 we immediately obtain \( \sigma_{pt}(C \varphi, A^p_{\alpha +}) \subset \{ \varphi'(0)^j \}_{j=0}^{\infty} \). So the inclusion on the right hand side follows. For the other inclusion, first let us note that the essential spectral radius \( r_e(C \varphi; A^p_{\alpha}) < 1 \), by [6, Theorem 2.8]. In the light of that, we are allowed to fix a \( j \in \mathbb{N} \) such that \( |\varphi'(0)^j| > r_e(C \varphi; A^p_{\alpha}) \) so that \( \varphi'(0)^j \notin \sigma_{ess}(C \varphi; A^p_{\alpha}) \). Then, by Lemma 3.9 we obtain \( \varphi'(0)^j \in \sigma(C \varphi; A^p_{\alpha}) \). If we apply [1, Theorem 7.44] this implies \( \varphi'(0)^j \in \sigma_{pt}(C \varphi; A^p_{\alpha}) \). This means there exists \( f_0 \in A^p_{\alpha} \) such that \( C \varphi f_0 = \varphi'(0)^j f_0 \), in \( A^p_{\alpha} \). But since \( A^p_{\alpha} \subset A^p_{\alpha +} \), the latter holds also in \( A^p_{\alpha +} \). Therefore \( \varphi'(0)^j \in \sigma_{pt}(C \varphi; A^p_{\alpha +}) \), as well.

(2) Similar to part (1), the right hand side inclusion follows immediately by Lemma 3.8. For the other inclusion, fix \( 0 < \beta < \alpha < \beta + 1 < \infty \) so that by [16, Proposition 3.6; 3.8],
\[
r_e(C \varphi; A^p_{\alpha}) \leq \lim_{n \to \infty} \left( \limsup_{s \to 1, |z| \geq s} \left( \frac{1 - |z|^2}{1 - |\varphi_n(z)|^2} \right) \right)^{\beta + 1/n} \leq \lim_{n \to \infty} \left( \limsup_{s \to 1, |z| \geq s} \left( \frac{1 - |z|^2}{1 - |\varphi_n(z)|^2} \right) \right)^{\alpha + 2/n} \leq r_e(C \varphi; A^p_{\alpha}).
\]

Then let us first fix \( j \in \mathbb{N} \) such that \( |\varphi'(0)^j| > r_e(C \varphi; A^p_{\alpha}) \). Then we find \( \beta < \alpha \) satisfying \( |\varphi'(0)^j| > r_e(C \varphi; A^p_{\beta}) \) to immediately apply Lemma 3.9 and [1, Theorem 7.44] to get \( |\varphi'(0)^j| \in \sigma(C \varphi; A^p_{\beta}) \) and \( |\varphi'(0)^j| \in \sigma_{pt}(C \varphi; A^p_{\beta}) \), respectively. That implies there exists \( g_0 \in A^p_{\beta} \) such that \( C \varphi g_0 = |\varphi'(0)^j| g_0 \) in \( A^p_{\beta} \). But since \( A^p_{\beta} \subset A^p_{\alpha -} \), the same holds in \( A^p_{\alpha -} \) as well. Therefore \( |\varphi'(0)^j| \in \sigma_{pt}(C \varphi; A^p_{\alpha -}) \).

\[
\square
\]

Acknowledgment

This article was completed during the author’s stay at Instituto Universitario de Matemática Pura y Aplicada (IUMPA), Universidad Politécnica de Valencia funded by The Scientific and Technological Research Council of Turkey International Postdoctoral Research Fellowship (TÜBİTAK 2219) with grant number 1059B191800828. The author would like to acknowledge his mentor Prof. José Bonet for his generous support, and his colleagues at IUMPA for their kind hospitality.
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