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Operators between different weighted Fréchet and (LB)-spaces of analytic functions

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Abstract: We study some classical operators defined on the weighted Bergman Fréchet space $A^p_{\alpha+}$ (resp. weighted Bergman (LB)-space $A^p_{\alpha-}$) arising as the projective limit (resp. inductive limit) of the standard weighted Bergman spaces into the growth Fréchet space $H^\infty_{\alpha+}$ (resp. growth (LB)-space $H^\infty_{\alpha-}$), which is the projective limit (resp. inductive limit) of the growth Banach spaces. We show that, for an analytic self map φ of the unit disc $\mathbb{D}$, the continuities of the weighted composition operator $W_{g,\varphi}$, the Volterra integral operator T_g, and the pointwise multiplication operator M_g defined via the identical symbol function are characterized by the same condition determined by the symbol's state of belonging to a Bloch-type space. These results have consequences related to the invertibility of $W_{g,\varphi}$ acting on a weighted Bergman Fréchet or (LB)-space. Some results concerning eigenvalues of such composition operators C_φ are presented.

Key words: Weighted composition operator, Volterra operator, multiplication operator, Fréchet spaces, (LB)-spaces, weighted spaces of analytic functions

1. Introduction

Let $H(\mathbb{D})$ denote the Fréchet space of all analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ equipped with the topology of uniform convergence on the compact subsets of the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let φ be an analytic self map on $\mathbb{D}$, and let $g\colon \mathbb{D} \to \mathbb{C}$ be an analytic map. The main focus of this note is, when they are defined between projective (or inductive) limits of different well-known Banach spaces of analytic functions, to give a relation between the continuity of the Volterra integral operator

$$T_g(f)(z) = \int_0^z f(t)g'(t)\mathrm{d}t, \quad z \in \mathbb{D}, \tag{1.1}$$

the pointwise multiplication operator

$$M_g(f)(z) = g(z)f(z), \quad z \in \mathbb{D}, \tag{1.2}$$

and the weighted composition operator

$$W_{g,\varphi}(f)(z) = (M_g \circ \varphi \circ f)(z) = g(z)f(\varphi(z)), \quad z \in \mathbb{D} \tag{1.3}$$

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in terms of conditions formulated for g and φ. For $1 < p < \infty$ and $-1 < \alpha < \infty$, the *Bergman space of standard weight* $A^p_\alpha = A^p_\alpha(\mathbb{D})$ of the unit disc is given by

$$A^p_\alpha := \{f \in H(\mathbb{D}) : \|f\|_{p,\alpha} = \left((\alpha+1)\int_{\mathbb{D}} |f(z)|^p \mathrm{d}s_\alpha(z)\right)^{1/p} < \infty\}, \tag{1.4}$$

where $\mathrm{d}s_\alpha(z) = (1-|z|^2)^\alpha \mathrm{d}s(z)$, and $\mathrm{d}s(z) = \frac{1}{\pi}\mathrm{d}x\mathrm{d}y$. Each A^p_α is a closed subspace of $L^p(\mathbb{D}, \mathrm{d}s(z))$ in which the polynomials are dense [18, Section 1.1]. The weighted Bergman space A^p_α is a Banach space with the norm $\|\cdot\|_{p,\alpha}$. Classical Bergman space $A^p(\mathbb{D})$ corresponds to the case $\alpha = 0$. If $p = \infty$ we obtain the growth Banach space

$$H^\infty_\alpha := \{f \in H(\mathbb{D}) \colon \|f\|_{-\alpha} := \sup_{z\in\mathbb{D}} |f(z)|(1-|z|^2)^\alpha < \infty\}, \tag{1.5}$$

endowed with the norm $\|\cdot\|_{-\alpha}$. These Banach spaces, as well as their intersections and unions, play a significant role in connection with the interpolation and sampling of analytic functions. See [18, Section 4.3]. They arise as special cases of weighted Banach spaces H^∞_v of analytic functions on $\mathbb{D}$, which was pioneered by the work of Shields and Williams [25], and then have been investigated by many authors, e.g. [7, 8, 23]. An analytic function f said to belong to the Bloch space $\mathcal{B}_\alpha$ if

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^\alpha < \infty.$$

Indeed, $\|\cdot\|_{\mathcal{B}_\alpha}$ defined above is a seminorm. We shall use the notation $A \lesssim B$ if there is a constant $c > 0$ not depending on A or B such that $A \leq cB$. We write $A \asymp B$ whenever $A \lesssim B$ and $B \lesssim A$. The Bloch space $\mathcal{B}_\alpha$ is a Banach space when normed with $\|f\| := |f(0)| + \|f\|_{\mathcal{B}_\alpha}$. By [18, Proposition 1.13], given $\alpha > 0$ for every $f \in H(\mathbb{D})$ one has

$$\sup_{z\in\mathbb{D}} |f(z)|(1-|z|^2)^\alpha \asymp \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^{\alpha+1}. \tag{1.6}$$

We refer the reader to [28] for a detailed treatment of Bloch spaces. It is also possible to define these spaces with the weight $(1-|z|)^\alpha$ instead of $(1-|z|^2)^\alpha$. Since $1-|z| \leq 1-|z|^2 \leq 2(1-|z|)$, these spaces coincide and the norms are equivalent. In this paper, the operators we shall investigate will be defined on weighted Bergman Fréchet and (LB)-spaces, which arise as intersections and unions of standard weighted Bergman spaces. For $1 < p < \infty$, and $0 < \alpha < \infty$ they are defined as follows:

$$\begin{aligned} A^p_{\alpha+} &:= \{f \in H(\mathbb{D}) : \left(\int_{\mathbb{D}} |f(z)|^p \mathrm{d}s_\mu(z)\right)^{1/p} < \infty, \forall \mu > \alpha\} \\ &= \bigcap_{\mu>\alpha} A^p_\mu = \bigcap_{n\in\mathbb{N}} A^p_{(\alpha+\frac{1}{n})} = \operatorname*{proj}_{n\in\mathbb{N}} A^p_{(\alpha+\frac{1}{n})}, \end{aligned} \tag{1.7}$$

$$\begin{aligned} A^p_{\alpha-} &:= \{f \in H(\mathbb{D}) : \left(\int_{\mathbb{D}} |f(z)|^p \mathrm{d}s_\mu(z)\right)^{1/p} < \infty, \text{ for some } \mu < \alpha\} \\ &= \bigcup_{\mu<\alpha} A^p_\mu = \bigcup_{n\in\mathbb{N}} A^p_{(\alpha-\frac{1}{n})} = \operatorname*{ind}_{n\in\mathbb{N}} A^p_{(\alpha-\frac{1}{n})}, \end{aligned} \tag{1.8}$$

where the inductive limit is taken over all $n \in \mathbb{N}$ such that $(\alpha - \frac{1}{n}) > 0$. The paper [20] gives a description of intersections and unions of weighted Bergman spaces of order $0 < p < \infty$. Unlike those, we treat the space

$A^p_{\alpha+}$ as a Fréchet space when equipped with the locally convex topology generated by the increasing system of norms

$$|||f|||_{p,\alpha,n} := \left(\int_{\mathbb{D}} |f(z)|^p \mathrm{d}s_{(\alpha+\frac{1}{n})}(z)\right)^{1/p}, \quad n \in \mathbb{N}, \tag{1.9}$$

for $f \in A^p_{\alpha+}$ and each $n \in \mathbb{N}$. We note that for $0 < \mu < \gamma < \infty$, the natural inclusion map $\iota_{\mu,\gamma}\colon A^p_\mu \to A^p_\gamma$ is compact. See e.g. [21, Proposition 3.1]. Hence, $A^p_{\alpha+}$ is a *Fréchet-Schwartz* space. The space $A^p_{\alpha-}$ is a complete (DFS)-space endowed with the finest locally convex topology, such that $\iota_{\mu,\gamma}$ is continuous. It is also a *regular* (LB)-space, since every bounded set $B \subseteq A^p_{\alpha-}$ is contained and bounded in the Banach space A^p_μ, for some $0 < \mu < \alpha$. Let us also remark that, for $\alpha > 0$ we have $A^p_{\alpha-} \subset A^p_\alpha \subset A^p_{\alpha+}$ with continuous inclusions. Some other properties of $A^p_{\alpha+}$ and $A^p_{\alpha-}$ were given in author's work [21] where these spaces were first introduced in locally convex setup. The Volterra integral operator defined between different weighted Bergman Fréchet or (LB)-spaces has been investigated by the author in [22]. Given $0 < \alpha < \infty$,

$$\begin{aligned} H^\infty_{\alpha+} &:= \{f \in H(\mathbb{D})\colon \sup_{z\in\mathbb{D}} |f(z)|(1-|z|^2)^\mu < \infty, \forall \mu > \alpha\} \\ &= \operatorname*{proj}_{n\in\mathbb{N}} H^\infty_{(\alpha+\frac{1}{n})} \end{aligned} \tag{1.10}$$

$$\begin{aligned} H^\infty_{\alpha-} &:= \{f \in H(\mathbb{D})\colon \sup_{z\in\mathbb{D}} |f(z)|(1-|z|^2)^\mu < \infty, \text{ for some } \mu < \alpha\} \\ &= \operatorname*{ind}_{n\in\mathbb{N}} H^\infty_{(\alpha-\frac{1}{n})}. \end{aligned} \tag{1.11}$$

Then $H^\infty_{\alpha+}$ is a Fréchet space when endowed with the locally convex topology generated by the increasing sequence of norms

$$|||f|||_n := \sup_{z\in\mathbb{D}} |f(z)|(1-|z|^2)^{(\alpha+\frac{1}{n})}, \quad n \in \mathbb{N},$$

for $f \in H^\infty_{\alpha+}$. For any pair $0 < \mu < \alpha < \infty$, the canonical inclusion map $\iota_{\mu,\alpha}\colon H^\infty_\mu \to H^\infty_\alpha$ is compact [10, Theorem 3.3]. Hence, both $H^\infty_{\alpha+}$, and $H^\infty_{\alpha-}$ are Schwartz spaces. The regular (LB)-space $H^\infty_{\alpha-}$ is endowed with the finest locally convex topology making $\iota_{\mu,\alpha}$ continuous. Several important properties of growth Fréchet and (LB)-spaces can be found in [2, 11, 12]. The Volterra integral operator acting on a growth Fréchet or (LB)-space has been investigated by Bonet [9] in terms of continuity, compactness, and spectrum. For a study of weighted composition operators acting on these spaces, see [17].

In Section 2, we first deal with operators defined from $A^p_{\alpha+}$ (resp. $A^p_{\alpha-}$) into $H^\infty_{\beta+}$ (resp. $H^\infty_{\beta-}$). If we pick the symbol function $g\colon \mathbb{D} \to \mathbb{C}$ in such a way that it belongs to the Bloch-type space $\mathcal{B}_\tau$, for every positive $\tau > \beta + 1 - (2+\alpha)/p$, we show that the continuity of Volterra integral operator $T_g\colon A^p_{\alpha+} \to H^\infty_{\beta+}$ is equivalent to the continuity of pointwise multiplication operator $M_g\colon A^p_{\alpha+} \to H^\infty_{\beta+}$, and the continuity of the weighted composition operator $W_{g,\varphi}\colon A^p_{\alpha+} \to H^\infty_{\beta+}$ provided that $\varphi(0) = 0$. When we take another weighted Bergman Fréchet space (resp. (LB)-space) as the range space, we show that the symbol function g belonging to the growth Fréchet space $H^\infty_{\gamma+}$, where $\gamma = (2+\beta)/q - (2+\alpha)/p$, characterizes the continuity of the pointwise multiplication operator $M_g\colon A^p_{\alpha+} \to A^q_{\beta+}$ as well as the Volterra integral operator $T_g\colon A^p_{\alpha+} \to A^q_{\beta+}$. The same condition is also valid for the (LB)-space case. On the other hand, the continuity criterion for the weighted composition operator in between is different in this case. We give this condition in Section 3 as a straightforward generalization of the well-known characterizations of Čučković and Zhao [26] related to Carleson measures. Fortunately, resting on the arguments of Bourdon [13], the condition $g \in H^\infty_{\gamma+}$, which is equivalent to continuity of pointwise multiplication and Volterra integral operators between $A^p_{\alpha+}$ and $A^q_{\beta+}$ answers the question of invertibility for the weighted composition operator $W_{g,\varphi}$ acting on $A^p_{\alpha+}$ (in this case, $\gamma = 0$), whenever φ is an automorphism of $\mathbb{D}$. Finally, we give some results concerning the eigenvalues of composition operators C_φ acting on $A^p_{\alpha+}$ or $A^p_{\alpha-}$ in connection with their essential spectral radius defined on the Banach space A^p_α.

2. Continuous Volterra, multiplication, and weighted composition operators between weighted Fréchet and (LB)-spaces

Let us note that the continuity and compactness of $W_{g,\varphi}\colon A^p_\alpha \to H^\infty_\beta$ was described in [24, Theorem 3.1], and in [27, Theorem 2.2] for more general weights, that is, $W_{g,\varphi}\colon A^p_w \to H^\infty_v$. In [14] and [15] weighted composition operators $W_{g,\varphi}\colon X \to H^\infty_v$ are investigated in a uniform approach covering a large family of Banach spaces of analytic functions concerning the space X. Before we start our discussion on operators between Fréchet or (LB)-spaces, we need to prove the following result concerning related Banach spaces.

Proposition 2.1 *Let g be an analytic function. Let φ be an analytic self map on $\mathbb{D}$ satisfying $\varphi(0) = 0$. Given $1 \le p < \infty$ and $-1 < \alpha, \beta < \infty$, let $\gamma := \beta + 1 - \frac{2+\alpha}{p}$ be nonnegative. Then, the following statements are equivalent.*

(1) *The symbol g belongs to the Bloch space $\mathcal{B}_\gamma$.*

(2) *The Volterra operator $T_g\colon A^p_\alpha \to H^\infty_\beta$ is continuous.*

(3) *The pointwise multiplication operator $M_g\colon A^p_\alpha \to H^\infty_\beta$ is continuous.*

(4) *The weighted composition operator $W_{g,\varphi}\colon A^p_\alpha \to H^\infty_\beta$ is continuous.*

Proof (1) $\Rightarrow$ (2). Note that for $1 < p < \infty$ for any $f \in H(\mathbb{D})$, we have (see e.g. [18, p. 39])

$$|f(z)|^p(1-|z|^2)^t \lesssim \int_{\mathbb{D}} |f(w)|^p (1-|w|^2)^{t-2} \mathrm{d}s(w), \quad t \in \mathbb{R}. \tag{2.1}$$

We also mention that (see e.g. [5, Lemma 2]) for every $f \in H(\mathbb{D})$ we have

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \mathrm{d}s(z) \lesssim |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p+\alpha} \mathrm{d}s(z). \tag{2.2}$$

Now let $g \in \mathcal{B}_\gamma$. Then, for any $f \in A^p_\alpha$ by (2.1) we have

$$\begin{aligned}
\|T_g f\|^p_{-\beta} &= \sup_{z\in\mathbb{D}} \left| \int_0^z f(\xi)g'(\xi)\mathrm{d}\xi \right|^p (1-|z|^2)^{p\beta} \\
&\lesssim \int_{\mathbb{D}} \left| \int_0^w f(\xi)g'(\xi)\mathrm{d}\xi \right|^p (1-|w|^2)^{p\beta-2}\mathrm{d}s(w) \\
&\lesssim |f(0)|^p + \int_{\mathbb{D}} \left| \left(\int_0^w f(\xi)g'(\xi)\mathrm{d}\xi \right)' \right|^p (1-|w|^2)^{p\beta-2+p}\mathrm{d}s(w) \\
&= |f(0)|^p + \int_{\mathbb{D}} |f(w)g'(w)|^p (1-|w|^2)^{p\beta-2+p}\mathrm{d}s(w) \\
&\leq |f(0)|^p + \int_{\mathbb{D}} |f(w)|^p |g'(w)|^p (1-|w|^2)^{p\beta-2+p+\alpha-\alpha}\mathrm{d}s(w) \\
&\leq |f(0)|^p + \sup_{w\in\mathbb{D}} |g'(w)|^p (1-|w|^2)^{p(\beta+1)-(2+\alpha)} \int_{\mathbb{D}} |f(w)|^p (1-|w|^2)^{\alpha}\mathrm{d}s(w) \\
&= |f(0)|^p + \frac{1}{\alpha+1}\|g\|^p_{\mathcal{B}_\gamma}\|f\|^p_{p,\alpha} < \infty,
\end{aligned}$$

where the second inequality is due to (2.2). Hence $T_g \colon A^p_\alpha \to H^\infty_\beta$ is continuous.

(2) $\Rightarrow$ (1). For $w \in \mathbb{D}$, let us pick

$$f_w(z) := \left(\frac{1-|w|^2}{(1-\overline{w}z)^2} \right)^{\frac{2+\alpha}{p}}.$$

A canonical calculation yields (see e.g. [28, p. 52]) $\|f_w\|^p_{p,\alpha} = 1$. Since $T_g \colon A^p_\alpha \to H^\infty_\beta$ is continuous, by (1.6) we obtain

$$\|f_w\|_{p,\alpha} \gtrsim \|T_g f_w\|_{-\beta} \simeq \|T_g f_w\|_{\mathcal{B}_{\beta+1}}.$$

Then, for any $w \in \mathbb{D}$, the latter yields,

$$\begin{aligned}
1 &\gtrsim \sup_{z\in\mathbb{D}} |(T_g f_w)'(z)|(1-|z|^2)^{\beta+1} \geq |(T_g f_w)'(w)|(1-|w|^2)^{\beta+1} \\
&= \left| \left(\int_0^w f_w(\xi)g'(\xi)\mathrm{d}\xi \right)' \right| (1-|w|^2)^{\beta+1} \\
&= |f_w(w)g'(w)|(1-|w|^2)^{\beta+1} \\
&= |g'(w)| \left(\frac{1-|w|^2}{|1-\overline{w}w|^2} \right)^{\frac{2+\alpha}{p}} (1-|w|^2)^{\beta+1} \\
&= |g'(w)|(1-|w|^2)^{\gamma}. \qquad (2.3)
\end{aligned}$$

Since $w \in \mathbb{D}$ was arbitrary, by (2.3), $\|g\|_{\mathcal{B}_\gamma} \lesssim 1$. This proves (1).

(1) $\Leftrightarrow$ (3). Follows by (1.6) and the previous result in [24, Corollary 3.3].

(1) $\Rightarrow$ (4). We make use of the following well-known estimate. For any $f \in A^p_\alpha$,

$$|f(z)| \lesssim \frac{\|f\|_{p,\alpha}}{(1-|z|^2)^{\frac{2+\alpha}{p}}}, \quad \forall z \in \mathbb{D}. \tag{2.4}$$

Note that, by Schwartz's lemma, one has $|\varphi(z)| \leq |z|$. Hence,

$$\sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi(z)|^2} < \infty. \tag{2.5}$$

Let $g \in \mathcal{B}_\gamma$. Then, by (1.6), for any $f \in A^p_\alpha$ we have

$$\begin{aligned}
\|W_{g,\varphi}f\|_{-\beta} &= \sup_{z\in\mathbb{D}} |W_{g,\varphi}f(z)|(1-|z|^2)^\beta \\
&= \sup_{z\in\mathbb{D}} |g(z)f(\varphi(z))|(1-|z|^2)^\beta \\
&\asymp \sup_{z\in\mathbb{D}} |g(z)||f(\varphi(z))|(1-|z|^2)^{\gamma-1+\frac{2+\alpha}{p}} \\
&\leq \|g\|_{\mathcal{B}_\gamma} \sup_{z\in\mathbb{D}} |f(\varphi(z))|(1-|z|^2)^{\frac{2+\alpha}{p}} \\
&\leq \|g\|_{\mathcal{B}_\gamma} \|f\|_{p,\alpha} \sup_{z\in\mathbb{D}} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{2+\alpha}{p}} < \infty,
\end{aligned}$$

where the second inequality is due to (2.4), and the last one is by (2.5).

(4) $\Rightarrow$ (1). Let $W_{g,\varphi}\colon A^p_\alpha \to H^\infty_\beta$ be continuous. Given $w = \varphi(z_0) \in \mathbb{D}$ for a fixed $z_0 \in \mathbb{D}$, let us define

$$h_w(z) := \left(\frac{1-|w|^2}{(1-\overline{w}z)^2}\right)^{\frac{2+\alpha}{p}},$$

for which $\|h_w\|_{p,\alpha} = 1$. Then, continuity of $W_{g,\varphi}\colon A^p_\alpha \to H^\infty_\beta$ and (1.6) imply

$$\begin{aligned}
1 = \|h_w\|_{p,\alpha} \gtrsim \|W_{g,\varphi}h_w\|_{-\beta} &= \sup_{z\in\mathbb{D}} |g(z)\, h_w(\varphi(z))|(1-|z|^2)^\beta \\
&\asymp \sup_{z\in\mathbb{D}} |g(z)|\, |h_w(\varphi(z))|(1-|z|^2)^{\gamma-1+\frac{2+\alpha}{p}} \\
&= |g(z_0)|(1-|z_0|^2)^{\gamma-1} h_w(\varphi(z))(1-|z|^2)^{\frac{2+\alpha}{p}} \\
&= |g(z_0)|(1-|z_0|^2)^{\gamma-1} \left(\frac{1-|w|^2}{|1-\overline{w}\varphi(z)|^2}(1-|z|^2)\right)^{\frac{2+\alpha}{p}} \\
&\geq |g(z_0)|(1-|z_0|^2)^{\gamma-1} \left(\frac{1-|z_0|^2}{1-|\varphi(z_0)|^2}\right)^{\frac{2+\alpha}{p}} \asymp |g(z_0)|(1-|z_0|^2)^{\gamma-1},
\end{aligned}$$

for an arbitrary $z_0 \in \mathbb{D}$, by (2.5). Hence, $g \in \mathcal{B}_\gamma$. □

The following result is well-known. For a proof, see e.g. [3, Lemma 25].

Lemma 2.2 *Let $E = \text{proj}_m E_m$ and $F = \text{proj}_n F_n$ be Fréchet spaces such that E (resp. F) is the intersection of the sequence of Banach spaces E_m (resp. F_n), E is dense in E_m and $E_{m+1} \subset E_m$ with continuous inclusion for each m (resp. F is dense in F_n and $F_{n+1} \subset F_n$ with continuous inclusion for each n). Let $T\colon E \to F$ be a linear operator. Then*

(i) *T is continuous if and only if for each n, there is m such that T has a unique continuous linear extension $T_{m,n}\colon E_m \to F_n$.*

(ii) *Assume T is continuous. Then, T is bounded if and only if there is m such that for each n, T has a unique continuous linear extension $T_{m,n}\colon E_m \to F_n$.*

The following lemma for (LB)-spaces is also known. A proof can be seen in [4, Lemma 4.1].

Lemma 2.3 *Let $E = \text{ind}_m E_m$ and $F = \text{ind}_n F_n$ be (LB)-spaces such that E (resp. F) is the union of the sequence of Banach spaces E_m (resp. F_n). Let $T\colon E \to F$ be a linear operator. Then*

(i) *T is continuous if and only if, for all $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $T(E_m) \subset F_n$ and $T\colon E_m \to F_n$ is continuous.*

(ii) *Let T be continuous and let F be regular. Then, T is bounded if and only if there exists $n \in \mathbb{N}$ such that for all m, $T(E_m) \subset F_n$ and $T\colon E_m \to F_n$ is continuous.*

With the help of Lemma 2.2 and Lemma 2.3 we extend Proposition 2.1 to the setup of Fréchet and (LB)-spaces.

Proposition 2.4 *Given $1 < p < \infty$ and $0 < \alpha, \beta < \infty$, let $\gamma := \beta + 1 - \frac{2+\alpha}{p}$ be non-negative. Let φ be an analytic self map on $\mathbb{D}$ satisfying $\varphi(0) = 0$. Then, the following statements are equivalent.*

(1) *The symbol $g \in H(\mathbb{D})$ satisfies*

$$g \in \bigcap_{\tau > \gamma} \mathcal{B}_\tau. \tag{2.6}$$

(2) *The Volterra operator $T_g\colon A^p_{\alpha+} \to H^\infty_{\beta+}$ is continuous.*

(3) *The Volterra operator $T_g\colon A^p_{\alpha-} \to H^\infty_{\beta-}$ is continuous.*

(4) *The pointwise multiplication operator $M_g\colon A^p_{\alpha+} \to H^\infty_{\beta+}$ is continuous.*

(5) *The pointwise multiplication operator $M_g\colon A^p_{\alpha-} \to H^\infty_{\beta-}$ is continuous.*

(6) *The weighted composition operator $W_{g,\varphi}\colon A^p_{\alpha+} \to H^\infty_{\beta+}$ is continuous.*

(7) *The weighted composition operator $W_{g,\varphi}\colon A^p_{\alpha-} \to H^\infty_{\beta-}$ is continuous.*

Proof (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4). By Lemma 2.2, the Volterra operator $T_g\colon A^p_{\alpha+} \to H^\infty_{\beta+}$ (resp. the pointwise multiplication operator $M_g\colon A^p_{\alpha+} \to H^\infty_{\beta+}$) is continuous if and only if for every $\varepsilon > 0$ there exists $\tilde{\delta} \in (0, \varepsilon]$

such that $T_g\colon A^p_{\alpha+\tilde{\delta}} \to H^\infty_{\beta+\varepsilon}$ (resp. $M_g\colon A^p_{\alpha+\tilde{\delta}} \to H^\infty_{\beta+\varepsilon}$) is continuous. By Proposition 2.1 this is equivalent to say that

$$g \in \mathcal{B}_{\gamma+\varepsilon-\delta}, \tag{2.7}$$

where $\delta := \frac{\tilde{\delta}}{p} < \varepsilon$. Clearly (2.7) is equivalent to (2.6).

(1) $\Rightarrow$ (3). Suppose (2.6) holds. Then, for every $\varepsilon \in (0, \min\{\frac{\alpha+1}{p^2}, \frac{\beta}{p-1}\})$ we have $g \in \mathcal{B}_{\gamma+\varepsilon}$. Then, given $-1 < \mu := \alpha - p^2\varepsilon$ pick $\eta := \beta - (p-1)\varepsilon$. We see that $\gamma + \varepsilon = \eta + 1 - \frac{2+\mu}{p}$, which yields $g \in \mathcal{B}_{\eta+1-\frac{2+\mu}{p}}$. By Proposition 2.1 this is equivalent that $T_g\colon A^p_\mu \to H^\infty_\eta$ is continuous. In the light of Lemma 2.3, $T_g\colon A^p_{\alpha-} \to H^\infty_{\beta-}$ is continuous.

(3) $\Rightarrow$ (1). Let $T_g\colon A^p_{\alpha-} \to H^\infty_{\beta-}$ be continuous. Then, for every $\varepsilon \in (0, \frac{\alpha+1}{p})$, there exists $\delta \in (0, \min\{\varepsilon, \frac{\beta}{p}\})$ such that $T_g\colon A^p_{\alpha-\varepsilon} \to H^\infty_{\beta-\delta}$ is continuous. Without loss of any generality, let $\gamma + \frac{\varepsilon}{p} - \delta \geq 0$, since otherwise g is constant so there is nothing to prove. By Proposition 2.1 this is equivalent that $g \in \mathcal{B}_{\gamma+\frac{\varepsilon}{p}-\delta} \subseteq \mathcal{B}_{\gamma+\varepsilon-\delta}$. Hence g satisfies (2.7), equivalently (2.6).

(1) $\Leftrightarrow$ (5). Identical to (1) $\Leftrightarrow$ (3).

(1) $\Leftrightarrow$ (6). Suppose that φ satisfies (2.5). The symbol function g satisfying (2.6) is equivalent to say that for every $\varepsilon > 0$, there exists $\tilde{\delta} \in (0, \varepsilon]$ such that $g \in \mathcal{B}_{\gamma+\varepsilon-\tilde{\delta}}$. Equivalently, by Proposition 2.1, the weighted composition operator $W_{g,\varphi}\colon A^p_{\alpha+\delta} \to H^\infty_{\beta+\varepsilon}$ is continuous, for $\delta = \frac{\tilde{\delta}}{p}$. By Lemma 2.2, this is equivalent to say that $W_{g,\varphi}\colon A^p_{\alpha+} \to H^\infty_{\beta+}$ is continuous.

(1) $\Rightarrow$ (7). If g satisfies (2.6), for every $\varepsilon \in (0, \frac{2+\alpha}{2p})$ one has $g \in \mathcal{B}_{\gamma+\varepsilon} \subseteq \mathcal{B}_{\beta+1-\varepsilon}$, since $\frac{2+\alpha}{p} > 2\varepsilon$. Then, by (1.6) $g \in H^\infty_{\beta-\varepsilon}$ and the rest follows very similar to Fréchet case.

(7) $\Rightarrow$ (1). The weighted composition operator $W_{g,\varphi}\colon A^p_{\alpha-} \to H^\infty_{\beta-}$ is continuous if and only if, for every $\varepsilon \in (0, \alpha+1)$, there exists $\delta \in (0, \min\{\varepsilon, \beta\}]$ such that $W_{g,\varphi}\colon A^p_{\alpha-\varepsilon} \to H^\infty_{\beta-\delta}$ is continuous if and only if $g \in \mathcal{B}_{\gamma+\frac{\varepsilon}{p}-\delta} \subseteq \mathcal{B}_{\gamma+\varepsilon-\delta}$, by Proposition 2.1. This is equivalent to (2.7) hence to (2.6). □

Proposition 2.5 *Let $1 < p \leq q < \infty$, and $0 < \alpha, \beta < \infty$. Let $\gamma := \frac{2+\beta}{q} - \frac{2+\alpha}{p}$ be non-negative. Then, for an analytic map $g\colon \mathbb{D} \to \mathbb{C}$, the following statements are equivalent.*

(1) *The symbol g belongs to the growth Fréchet space $H^\infty_{\gamma+}$.*

(2) *The pointwise multiplication operator $M_g\colon A^p_{\alpha+} \to A^q_{\beta+}$ is continuous.*

(3) *The pointwise multiplication operator $M_g\colon A^p_{\alpha-} \to A^q_{\beta-}$ is continuous.*

(4) *The Volterra operator $T_g\colon A^p_{\alpha+} \to A^q_{\beta+}$ is continuous.*

(5) *The Volterra operator $T_g\colon A^p_{\alpha-} \to A^q_{\beta-}$ is continuous.*

Proof (1) $\Rightarrow$ (2). Let $M_g\colon A^p_{\alpha+} \to A^q_{\beta+}$ be continuous. Then, for every $\varepsilon > 0$ given $\mu := \beta + q\varepsilon$ there exists

$\alpha < \eta < \alpha + q\varepsilon$ such that $M_g \colon A^p_\eta \to A^q_\mu$ is continuous. Hence, for every $z \in \mathbb{D}$

$$|g(z)|(1-|z|^2)^{\gamma+\varepsilon} = |g(z)|(1-|z|^2)^{\frac{2+\mu}{q}-\frac{2+\alpha}{p}}$$
$$< |g(z)|(1-|z|^2)^{\frac{2+\mu}{q}-\frac{2+\eta}{p}} < \infty.$$

So by [26, Theorem 9], $g \in H^\infty_{\gamma+\varepsilon}$. Hence, $g \in H^\infty_{\gamma+}$.

(2) $\Rightarrow$ (1). Let $g \in H^\infty_{\gamma+}$. Then, for every $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon]$ such that we have $g \in H^\infty_{\gamma+\varepsilon-\delta}$. Given $\mu := \beta + q\varepsilon$, define $\eta := \alpha + p\delta$. Observe that for every $z \in \mathbb{D}$,

$$|g(z)|(1-|z|^2)^{\frac{2+\mu}{q}-\frac{2+\eta}{p}} = |g(z)|(1-|z|^2)^{\gamma+\varepsilon-\delta} < \infty.$$

So $g \in H^\infty_{\frac{2+\mu}{q}-\frac{2+\eta}{p}}$. By [26, Theorem 9], $M_g \colon A^p_\eta \to A^q_\mu$ is continous. Therefore, $M_g \colon A^p_{\alpha+} \to A^q_{\beta+}$ is continuous.

(3) $\Rightarrow$ (1). Let $M_g \colon A^p_{\alpha-} \to A^q_{\beta-}$ be continuous. Then, for every $\varepsilon > 0$, given $-1 < \mu := \alpha - p\varepsilon$ there exists $-1 < \beta - p\varepsilon < \eta < \beta$ such that $M_g \colon A^p_\mu \to A^q_\eta$ is continuous. Then, for every $z \in \mathbb{D}$,

$$|g(z)|(1-|z|^2)^{\gamma+\varepsilon} < |g(z)|(1-|z|^2)^{\frac{2+\eta}{q}-\frac{2+\mu}{p}} < \infty.$$

So, by [26, Theorem 9], g belongs to $H^\infty_{\gamma+\varepsilon}$ and hence to $H^\infty_{\gamma+}$.

(1) $\Rightarrow$ (3). Suppose that $g \in H^\infty_{\gamma+}$. Let $x = \min\{(\alpha+1)\frac{q-1}{pq}, (\beta+1)\frac{q-1}{q}\}$. Then, for every $\varepsilon \in (0, x)$ we have $g \in H^\infty_{\gamma+\varepsilon}$. Given $-1 < \mu := \alpha - \frac{pq}{q-1}\varepsilon$, pick $-1 < \eta := \beta - \frac{q}{q-1}\varepsilon$. Then, for every $z \in \mathbb{D}$,

$$|g(z)|(1-|z|^2)^{\frac{2+\eta}{q}-\frac{2+\mu}{p}} = |g(z)|(1-|z|^2)^{\gamma+\frac{q}{q-1}\varepsilon-\frac{1}{q-1}\varepsilon}$$
$$= |g(z)|(1-|z|^2)^{\gamma+\varepsilon} < \infty.$$

Hence, $g \in H^\infty_{\frac{2+\eta}{q}-\frac{2+\mu}{p}}$. By [26, Theorem 9], $M_g \colon A^p_\mu \to A^q_\eta$ is continuous. Therefore, $M_g \colon A^p_{\alpha-} \to A^q_{\beta-}$ is continuous.

(1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). Follows by (1.6), Proposition 2.5, and [22, Proposition 2.2]. □

Proposition 2.5 will help us characterize the invertibility of a weighted composition operator acting on a weighed Bergman Fréchet or a weighted Bergman (LB)-space. See Proposition 3.7. The following statement is derived from [26, Theorem 11] via Lemma 2.2. Its (LB)-space version can be produced in an analogue way.

Proposition 2.6 *Let g be an analytic function on $\mathbb{D}$. Let $1 \le q < p < \infty$, and $\alpha, \beta > 0$. Then, the following statements are equivalent.*

(1) *The multiplication operator $M_g \colon A^p_{\alpha+} \to A^q_{\beta+}$ is continuous.*

(2) *For every $\mu > \beta$, there exists $\nu \in (\alpha, \alpha + \mu - \beta)$ such that $g \in A^s_\eta$, where $\frac{1}{s} = \frac{1}{q} - \frac{1}{p}$ and $\eta = s\left(\frac{\mu}{q} - \frac{\nu}{p}\right)$.*

3. Weighted composition operators between weighted Bergman Fréchet and (LB)-spaces

3.1. Continuous weighted composition operators between different weighted Bergman Fréchet or (LB)-spaces

An operator T on a Fréchet space X into itself is called bounded (resp. compact) if there exists a neighborhood U of the origin of X such that TU is a bounded (resp. relatively compact) set in X. The following result is a consequence of [26, Theorem 1] along with Lemma 2.2 and Lemma 2.3.

Proposition 3.1 *Let* $1 < p \leq q < \infty$ *and* $0 < \alpha, \beta < \infty$. *Let* $g\colon \mathbb{D} \to \mathbb{C}$ *be an analytic function and let* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *be an analytic self map. Then,*

(1) *The weighted composition operator* $W_{g,\varphi}\colon A^p_{\alpha+} \to A^q_{\beta+}$ *is continuous if and only if for every* $\mu > \beta$ *there exists* $\eta \in (\alpha, \alpha + \mu - \beta)$ *such that*

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \left(\frac{1 - |z|^2}{|1 - \bar{z}\varphi(w)|^2} \right)^{\frac{2+\eta}{p}q} |g(w)|^q \mathrm{d}s_\mu(w) < \infty. \tag{3.1}$$

(2) *The weighted composition operator* $W_{g,\varphi}\colon A^p_{\alpha-} \to A^q_{\beta-}$ *is continuous if and only if for every* $\zeta \in (0, \alpha)$ *there exists* $\theta \in [\zeta, \alpha)$ *such that*

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \left(\frac{1 - |z|^2}{|1 - \bar{z}\varphi(w)|^2} \right)^{\frac{2+\zeta}{p}q} |g(w)|^q \mathrm{d}s_\theta(w) < \infty. \tag{3.2}$$

Similarly, we obtain the following proposition via Lemma 2.2 and [26, Theorem 3]. An (LB)-space version can be easily derived.

Proposition 3.2 *Let* $1 < q < p < \infty$ *and* $0 < \alpha, \beta < \infty$. *Let* $g\colon \mathbb{D} \to \mathbb{C}$ *be an analytic function and let* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *be an analytic self map. Then, the following statements are equivalent.*

(1) *The weighted composition operator* $W_{g,\varphi}\colon A^p_{\alpha+} \to A^q_{\beta+}$ *is continuous.*

(2) *For every* $\mu > \beta$ *there exists* $\nu \in (\alpha, \alpha + \mu - \beta)$ *such that for* $s := \frac{p}{p-q}$ *we have*

$$\int_{\mathbb{D}} \frac{(1 - |z|^2)^{2+\nu}}{|1 - \bar{z}\varphi(w)|^{4+2\nu}} |g(w)|^q \mathrm{d}s_\mu(w) \in A^s_\nu.$$

Lemma 3.3 (i) *Let* $E = \mathrm{proj}_m E_m$ *and* $F = \mathrm{proj}_n F_n$ *be Fréchet spaces such that* E *(resp.* F*) is the intersection of the sequence of Banach spaces* E_m *(resp.* F_n*),* E *is dense in* E_m *and* $E_{m+1} \subset E_m$ *with continuous inclusion for each* m *(resp.* F *is dense in* F_n *and* $F_{n+1} \subset F_n$ *with continuous inclusion for each* n*). Let* $T\colon E \to F$ *be a linear operator. Assume* T *is continuous. Then* T *is bounded if and only if there is* m *such that for each* n, T *has a unique continuous linear extension* $T_{m,n}\colon E_m \to F_n$.

(ii) *Let* $X = \mathrm{ind}\, X_n$ *and* $Y = \mathrm{ind}\, Y_m$ *be two (LB)-spaces which are increasing unions of Banach spaces* $X = \cup_{n=1}^{\infty} X_n$ *and* $Y = \cup_{m=1}^{\infty} Y_m$. *Let* $T\colon X \to Y$ *be a continuous linear map. Assume that* Y *is a regular (LB)-space. Then,* T *is bounded if and only if there exists* $m \in \mathbb{N}$ *such that* $T(X_n) \subset Y_m$ *and* $T\colon X_n \to Y_m$ *is continuous for all* $n \geq m$.

The following proposition is a consequence of [26, Corollary 1] and Lemma 3.3.

Proposition 3.4 *Let* $1 < p \leq q < \infty$ *and* $0 < \alpha, \beta < \infty$. *Let* $g\colon \mathbb{D} \to \mathbb{C}$ *be an analytic function and let* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *be an analytic self map. Then,*

(1) *The weighted composition operator* $W_{g,\varphi}\colon A^p_{\alpha+} \to A^q_{\beta+}$ *is compact if and only if it is continuous and there exists* $\mu > \alpha$ *such that for each* $\eta \in (\alpha, \mu]$ *we have*

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \left(\frac{1-|z|^2}{|1-\bar{z}\varphi(w)|^2} \right)^{\frac{2+\mu}{p}q} |g(w)|^q \mathrm{d}s_\eta(w) < \infty. \tag{3.3}$$

(2) *The weighted composition operator* $W_{g,\varphi}\colon A^p_{\alpha-} \to A^q_{\beta-}$ *is compact if and only if it is continuous and there exists* $\zeta \in (0, \alpha)$ *such that for each* $\theta \in [\zeta, \alpha)$ *we have*

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \left(\frac{1-|z|^2}{|1-\bar{z}\varphi(w)|^2} \right)^{\frac{2+\theta}{p}q} |g(w)|^q \mathrm{d}s_\zeta(w) < \infty. \tag{3.4}$$

Proof

(1) Given $\alpha, \beta > 0$, let $W_{g,\varphi}\colon A^p_{\alpha+} \to A^q_{\beta+}$ be bounded. Since $A^p_{\alpha+}$ is a Schwartz space, this is equivalent to assume that $W_{g,\varphi}$ is compact. Lemma 3.3(i) applied to $A^p_{\alpha+}$ this is equivalent that there exists $\mu > \alpha$ such that for all $\beta < \eta < \beta + \mu - \alpha$, the weighted composition operator $W_{g,\varphi}\colon A^p_\mu \to A^q_\eta$ is continuous. This is equivalent, by [26, Theorem 1], that (3.3) holds.

(2) Very similar to part (1) if we apply Lemma 3.3(ii) and [26, Theorem 1].

□

3.2. Invertible weighted composition operators acting on a weighted Bergman Fréchet or (LB)-space

The characterizations of invertible weighted composition operators on the Fréchet space $A^p_{\alpha+}$ and on the (LB)-space $A^p_{\alpha-}$ are consequences of the following results by Bourdon [13, Theorem 2.2; Corollary 2.3]. These arguments were also used to characterize invertible weighted composition operators acting on the growth Fréchet space $H^\infty_{\alpha+}$ and the growth (LB)-space $H^\infty_{\alpha-}$ in [17, Proposition 4].

Theorem 3.5 *Suppose that* E *is a space of analytic functions on* $\mathbb{D}$ *such that*

(i) $W_{g,\varphi}$ *maps* E *to* E.

(ii) E *contains a nonzero constant function.*

(iii) E *contains a function of the form* $z \to z + c$ *for some constant* c.

(iv) *There is a dense subset* S *of the unit circle such that, for each point in* S, *there is a function in* E *that does not extend analytically to a neighborhood of that point.*

If $W_{g,\varphi}\colon E \to E$ *is invertible, then* φ *is an automorphism of* $\mathbb{D}$.

Theorem 3.6 *If* E, g *and* φ *satisfy the hypotheses of Lemma 3.5, and for each* $f \in E$ *we have* $f \circ h \in E$ *for all automorphism* h *of* $\mathbb{D}$, *then* $W_{g,\varphi}$ *is invertible on* E *if and only if* φ *is an automorphism of* $\mathbb{D}$ *and both* g *and* $1/g$ *map* E *into* E.

Whenever it is continuous, that is, (3.1) or (3.2) is satisfied, $W_{g,\varphi}$ fulfills hypothesis (i) of Theorem 3.5. Hypotheses (ii) and (iii) are verified by both $A^p_{\alpha+}$ and $A^p_{\alpha-}$, since they contain the constants and polynomials. For hypothesis (iv), let us consider the function $f_{w,s}\colon \mathbb{D} \to \mathbb{C}$ given by $f_{w,s} := \frac{1}{(w-z)^s}$, for $w \in \partial\mathbb{D}$ and $s > 0$. It is easy to see that $f_{w,s} \in A^p_{\alpha+}$ and $f_{w,s} \in A^p_{\alpha-}$. However, in any neighborhood U of w, we see that $f_{a,s} \notin H(\mathbb{D})$ for any $a \in U$. So it does not extend analytically to any neighborhood of w (cf. [17, Remark 2]).

Proposition 3.7 *Let $g, \varphi \in H(\mathbb{D})$ and $\varphi(D) \subset \mathbb{D}$. Let $1 < p < \infty$, and $0 < \alpha < \infty$. Then, the following statements are equivalent.*

(1) $g \in H^\infty_{0+}$, *and* $1/g \in H^\infty_{0+}$.

(2) *The weighted composition operator* $W_{g,\varphi}\colon A^p_{\alpha+} \to A^p_{\alpha+}$ *is invertible.*

(3) *The weighted composition operator* $W_{g,\varphi}\colon A^p_{\alpha-} \to A^p_{\alpha-}$ *is invertible.*

Proof Since both $A^p_{\alpha+}$ and $A^p_{\alpha-}$ satisfy all hypotheses of Theorem 3.5, we apply Theorem 3.6 to reach that $W_{g,\varphi}$ is invertible if and only if φ is an automorphism of $\mathbb{D}$ and M_g and $M_{1/g}$ are continuous on $A^p_{\alpha+}$ (resp. $A^p_{\alpha-}$). Hence the conclusion follows from proposition 2.5. □

3.3. Some results on eigenvalues of composition operators acting on a weighted Bergman Fréchet or (LB)-space

For $T \in \mathcal{L}(E)$, the *resolvent set* $\rho(T;E)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda,T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(E)$. The set $\sigma(T;E) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. The *point spectrum* $\sigma_{\mathrm{pt}}(T;X)$ of T consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. The *essential norm* $\|T\|_{\mathrm{e},X}$ of an operator T on a Banach space X is the distance of the operator to the set of compact operators on X. The *essential spectral radius* is given by $r_{\mathrm{e}}(T;X) = \lim_n \|T^n\|^{1/n}_{\mathrm{e},X}$. The following lemma is well known. For a proof, see [19, Lemma 2.4] and [17, Lemma 3.4].

Lemma 3.8 *Let $X \subset H(\mathbb{D})$ be a continuously included subspace of holomorphic functions containing the polynomials. Let $\varphi, g \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi(0) = 0$. Suppose that φ is not a constant function. Then,*

$$\sigma_{pt}(W_{g,\varphi};X) \subseteq \{g(0)\varphi'(0)^j\}_{j=0}^{\infty}.$$

The following lemma is due to [19, Lemma 2.3].

Lemma 3.9 *Let $X \subset H(\mathbb{D})$ be a continuously included subspace of holomorphic functions containing the polynomials. Let $g, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$, $g \not\equiv 0$, and $\varphi(0) = 0$. Then,*

(i) $g(0) \in \sigma(W_{g,\varphi};X)$.

(ii) *For every $j \in \mathbb{N}$ we have $g(0)\varphi'(0)^j \in \sigma(W_{g,\varphi};X)$.*

Proposition 3.10 *Let $1 < p < \infty$, and $0 < \alpha < \infty$. Suppose that $\varphi \in H(\mathbb{D}), \varphi(\mathbb{D}) \subset \mathbb{D}$, $0 < |\varphi'(0)| < 1$, and φ is not a rotation. Then,*

(1) *The point spectrum of the composition operator* $C_\varphi\colon A^p_{\alpha+} \to A^p_{\alpha+}$ *satisfies the inclusions*

$$\{\varphi'(0)^j\}_{j=0}^{\infty} \setminus \overline{B}(0, r_e(C_\varphi, A^p_\alpha)) \subset \sigma_{pt}(C_\varphi; A^p_{\alpha+}) \subset \{\varphi'(0)^j\}_{j=0}^{\infty}.$$

(2) *The point spectrum of the composition operator* $C_\varphi\colon A^p_{\alpha-} \to A^p_{\alpha-}$ *satisfies the inclusions*

$$\{\varphi'(0)^j\}_{j=0}^{\infty} \setminus \overline{B}(0, r_e(C_\varphi, A^p_\alpha)) \subset \sigma_{pt}(C_\varphi; A^p_{\alpha-}) \subset \{\varphi'(0)^j\}_{j=0}^{\infty}.$$

Proof

(1) By Lemma 3.8 we immediately obtain $\sigma_{\mathrm{pt}}(C_\varphi; A^p_{\alpha+}) \subset \{\varphi'(0)^j\}_{j=0}^{\infty}$. So the inclusion on the right hand side follows. For the other inclusion, first let us note that the essential spectral radius $r_{\mathrm{e}}(C_\varphi; A^p_\alpha) < 1$, by [6, Theorem 2.8]. In the light of that, we are allowed to fix a $j \in \mathbb{N}$ such that $\left|\varphi'(0)^j\right| > r_{\mathrm{e}}(C_\varphi; A^p_\alpha)$ so that $\varphi'(0)^j \notin \sigma_{\mathrm{ess}}(C_\varphi; A^p_\alpha)$. Then, by Lemma 3.9 we obtain $\varphi'(0)^j \in \sigma(C_\varphi; A^p_\alpha)$. If we apply [1, Theorem 7.44] this implies $\varphi'(0)^j \in \sigma_{\mathrm{pt}}(C_\varphi; A^p_\alpha)$. This means there exists $f_0 \in A^p_\alpha$ such that $C_\varphi f_0 = \varphi'(0)^j f_0$, in A^p_α. But since $A^p_\alpha \subset A^p_{\alpha+}$, the latter holds also in $A^p_{\alpha+}$. Therefore $\varphi'(0)^j \in \sigma_{\mathrm{pt}}(C_\varphi; A^p_{\alpha+})$, as well.

(2) Similar to part (1), the right hand side inclusion follows immediately by Lemma 3.8. For the other inclusion, fix $0 < \beta < \alpha < \beta + 1 < \infty$ so that by [16, Proposition 3.6; 3.8],

$$\begin{aligned} r_{\mathrm{e}}(C_\varphi; A^p_\beta) &\le \lim_{n\to\infty} \left(\limsup_{s\to 1} \sup_{|z|\ge s} \left(\frac{1-|z|^2}{1-|\varphi_n(z)|^2} \right)^{\beta+1} \right)^{1/n} \\ &\le \lim_{n\to\infty} \left(\limsup_{s\to 1} \sup_{|z|\ge s} \left(\frac{1-|z|^2}{1-|\varphi_n(z)|^2} \right)^{\alpha+2} \right)^{1/n} \\ &\le r_{\mathrm{e}}(C_\varphi; A^p_\alpha). \end{aligned}$$

Then let us first fix $j \in \mathbb{N}$ such that $\left|\varphi'(0)^j\right| > r_{\mathrm{e}}(C_\varphi; A^p_\alpha)$. Then we find $\beta < \alpha$ satisfying $\left|\varphi'(0)^j\right| > r_{\mathrm{e}}(C_\varphi; A^p_\beta)$ to immediately apply Lemma 3.9 and [1, Theorem 7.44] to get $\left|\varphi'(0)^j\right| \in \sigma(C_\varphi; A^p_\beta)$ and $\left|\varphi'(0)^j\right| \in \sigma_{\mathrm{pt}}(C_\varphi; A^p_\beta)$, respectively. That implies there exists $g_0 \in A^p_\beta$ such that $C_\varphi g_0 = \left|\varphi'(0)^j\right| g_0$ in A^p_β. But since $A^p_\beta \subset A^p_{\alpha-}$, the same holds in $A^p_{\alpha-}$ as well. Therefore $\left|\varphi'(0)^j\right| \in \sigma_{\mathrm{pt}}(C_\varphi; A^p_{\alpha-})$.

□

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