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Volume properties and some characterizations of ellipsoids in $\mathbb{E}^{n+1}$

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Abstract: Suppose that M is a strictly convex and closed hypersurface in $\mathbb{E}^{n+1}$ with the origin o in its interior. We consider the homogeneous function g of positive degree d satisfying $M = g^{-1}(1)$. Then, for a positive number h the level hypersurface $g^{-1}(h)$ of g is a homothetic hypersurface of M with respect to the origin o. In this paper, for tangent hyperplanes Φ_h to $g^{-1}(h)$ $(0 < h < 1)$, we study the $(n+1)$-dimensional volume of the region enclosed by Φ_h and the hypersurface M, etc.. As a result, with the aid of the theorem of Blaschke and Deicke for proper affine hypersphere centered at the origin, we establish some characterizations for ellipsoids in $\mathbb{E}^{n+1}$. As a corollary, we extend Schneider's characterization for ellipsoids in $\mathbb{E}^3$. Finally, for further study, we raise a question for elliptic paraboloids which was originally conjectured by Golomb.

Key words: Ellipsoid, proper affine hypersphere, volume, cone, strictly convex, homothetic hypersurface, Gauss-Kronecker curvature

1. Introduction

We will say that a convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ is strictly convex if the hypersurface is of positive Gauss-Kronecker curvature K with respect to the inward unit normal.

Suppose that M is a strictly convex and closed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$. We assume that the origin o lies in the interior of M. Then, for any point $p \in M$ the ray $\overrightarrow{op}$ emanating from the origin o through the point p meets the hypersurface M only at the point p and passes through it transversally. We consider the family of homothetic hypersurfaces M_t of M with respect to the origin for positive number t, which is given by

$$M_t = \{tp | p \in M\}. \tag{1.1}$$

For a positive real number d, let us denote by $g = g_d$ the homogeneous function of degree d defined by $g(tp) = t^d, p \in M$ on the open set $U = \{tp | t > 0, p \in M\} = \mathbb{E}^{n+1} \setminus \{0\}$. Then for each $t > 0$ the homothetic hypersurface M_t of M is the level hypersurface $g^{-1}(t^d)$ of the homogeneous function $g = g_d$. The homogeneous function $g = g_d$ of degree d (> 0) determined by the strictly convex and closed hypersurface M is said to be the homogeneous function of M of degree d. Since M is strictly convex, for each t (> 0) the homothetic

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hypersurface M_t of M is also strictly convex because the Gauss-Kronecker curvature $K(tp)$ of M_t at tp is given by

$$K(tp) = t^{-n} K(p), \tag{1.2}$$

where $K(p)$ denotes the Gauss-Kronecker curvature of M at p.

Note that the origin o lies in the interior of M. For a fixed point $p \in M$ and a positive number h with $h < 1$, we consider the closest tangent hyperplane Φ_h of the homothetic hypersurface $g^{-1}(h) = M_t, h = t^d$ at some point $v \in M_t$ which is parallel to the tangent hyperplane Φ of M at $p \in M$. Then, the homogeneity of the function $g = g_d$ shows that we have

$$\nabla g(hp) = h^{d-1} \nabla g(p), \tag{1.3}$$

where $\nabla g(hp)$ denotes the gradient of g at hp, and so on. Hence, we get $v = tp \in M_t$ with $t = h^{1/d}$.

We denote by $A_p^*(h), V_p^*(h)$ and $C_p^*(h)$ the n-dimensional area of the section in Φ_h enclosed by $\Phi_h \cap M$, the $(n+1)$-dimensional volume of the region bounded by M and the hyperplane Φ_h and the $(n+1)$-dimensional volume of the cone with base the section in Φ_h enclosed by $\Phi_h \cap M$ and with vertex the origin, respectively (Figure 1). Then, for the outward unit normal $N(p)$ to M at the point p the height of the cone is given by $t\mathfrak{h}(p)$ with $t = h^{1/d}$, where $\mathfrak{h}(p) = \langle p, N(p) \rangle$ denotes the support function of M at p. Hence, we have

$$C_p^*(h) = \frac{t}{n+1} \mathfrak{h}(p) A_p^*(h), \quad t = h^{1/d}. \tag{1.4}$$

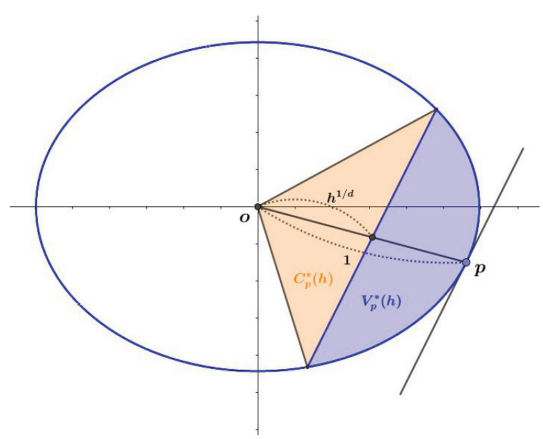

Figure 1. $V_p^*(h)$ and $C_p^*(h)$ of M with the homogeneous function $g = g_d$.

Finally, we consider the $(n+1)$-dimensional volume $I_p^*(h)$ of the ice cream cone-shaped domain which is the convex hull of the origin o and the region of M cut off by the hyperplane Φ_h (cf. [19]). Then, one obtains

$$I_p^*(h) = C_p^*(h) + V_p^*(h). \tag{1.5}$$

Now, we consider the following four conditions.

(V^*) $V_p^*(h)$ with $p \in M$ and $h \in (0,1)$ is a nonnegative function $\alpha(h)$, which depends only on h.

(A^*) $A_p^*(h)/|\nabla g(p)|$ with $p \in M$ and $h \in (0,1)$ is a nonnegative function $\beta(h)$, which depends only on h.

(C^*) $C_p^*(h)$ with $p \in M$ and $h \in (0,1)$ is a nonnegative function $\gamma(h)$, which depends only on h.

(I^*) $I_p^*(h)$ with $p \in M$ and $h \in (0,1)$ is a nonnegative function $\delta(h)$, which depends only on h.

In this paper, we study strictly convex and closed hypersurfaces in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ which satisfies one of the conditions (V^*), (A^*), (C^*), and (I^*).

As a result, first of all, we show that a strictly convex and closed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ satisfying one of the conditions (V^*), (A^*), (C^*), and (I^*) is an ellipsoid centered at the origin as follows.

Theorem A. Let M denote a strictly convex and closed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the origin in its interior and the homogeneous function $g = g_d$ of degree d. Then M satisfies one of the following conditions if and only if M is an ellipsoid in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ centered at the origin.

(1) M satisfies (V^*).

(2) M satisfies (A^*).

(3) M satisfies (C^*).

(4) M satisfies (I^*).

(5) $K(p) = \alpha\mathfrak{h}(p)^{n+2}$ for a nonzero constant α, where $K(p)$ is the Gauss-Kronecker curvature of M at p and $\mathfrak{h}(p)$ denotes the support function of M at p.

(6) $K(p)|\nabla g(p)|^{n+2}$ is a nonzero constant on M, where $\nabla g(p)$ denotes the gradient of g at p.

When $n = 1$, it was shown that a plain curve X with nonvanishing curvature satisfies the condition (5) in Theorem A if and only if X is an open arc of an ellipse or a hyperbola centered at the origin ([7]). See also [13] for the condition (6) in Theorem A. When $n = 2$, it is well-known that 2-dimensional ellipsoids satisfy the condition (5) in Theorem A ([18]).

In Theorem 6 of [2], it was shown that among the smooth ovaloids M lying inside an ellipsoid $E \subset \mathbb{E}^{n+1}$ centered at the origin, the ellipsoids centered at the origin which are homothetic to the ellipsoid E with respect to the origin are the only ones such that the volume of the $(n+1)$-dimensional compact set of smaller volume cut off from the ellipsoid E by any hyperplane tangent to M is constant.

Suppose that M is a strictly convex and closed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the origin in its interior and with the outward unit normal N. For a fixed point $p \in M$, the support function $\mathfrak{h}(p) = \langle p, N(p)\rangle$ of M at p is nothing but the distance from the origin to the tangent hyperplane Φ to M at the point p. For a constant $t \in (0, \mathfrak{h}(p)]$, we consider the hyperplane Φ_t parallel to the tangent hyperplane Φ and passing through the point $q = p - tN(p)$. Then, t is the distance from the point q to the hyperplane Φ.

We denote by $A_p(t)$, $V_p(t)$, and $C_p(t)$ the n-dimensional area of the section in Φ_t enclosed by $\Phi_t \cap M$, the $(n+1)$-dimensional volume of the region bounded by M and the hyperplane Φ_t, and the $(n+1)$-dimensional volume of the cone with base the section in Φ_t enclosed by $\Phi_t \cap M$ and with vertex the origin o, respectively. We also denote by $I_p(t)$ the $(n+1)$-dimensional volume of the ice cream cone-shaped domain which is the convex hull of the origin o and the region of M cut off by the hyperplane Φ_t. Then, we have ([10])

$$\frac{d}{dt}V_p(t) = A_p(t). \tag{1.6}$$

It follows from definitions that

$$C_p(t) = \frac{1}{n+1}A_p(t)\,(\mathfrak{h}(p) - t) \tag{1.7}$$

and

$$I_p(t) = C_p(t) + V_p(t). \tag{1.8}$$

For a constant t $(> \mathfrak{h}(p))$ such that the hyperplane Φ_t intersects M, $A_p(t)$, $V_p(t)$, $C_p(t)$, and $I_p(t)$ are also well-defined. In this case, (1.7) shows that $C_p(t) < 0$, which is (-1) times the volume of the corresponding cone with vertex the origin. Hence, (1.8) implies that $I_p(t)$ is the volume of a concave domain in $\mathbb{E}^{n+1}$ (Figure 2).

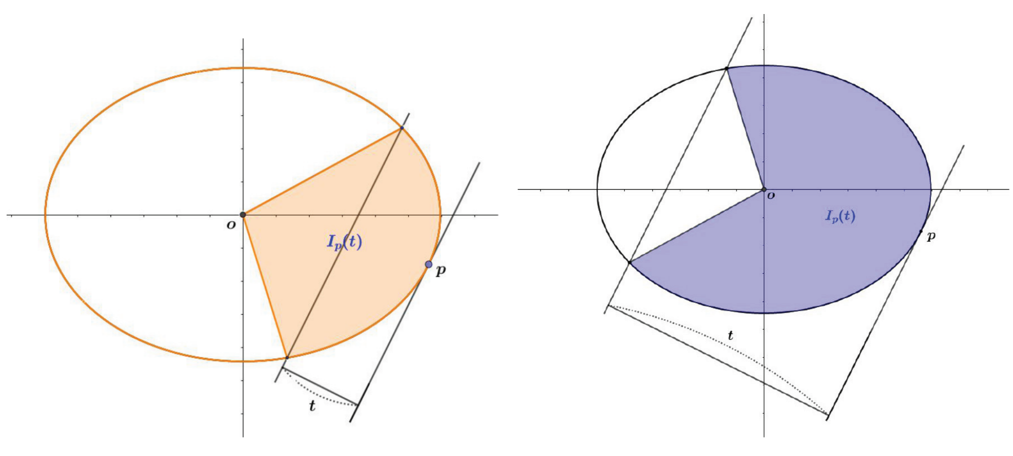

Figure 2. $I_p(t)$ with $t < \mathfrak{h}(p)$ and $I_p(t)$ with $t > \mathfrak{h}(p)$.

As a corollary of Theorem A, we prove the following characterization theorem, which is originally due to R. Schneider ([19]).

Theorem B. Suppose that the centrally symmetric convex body B centered at the origin o in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ has smooth boundary M which is of positive Gauss-Kronecker curvature. Then, for a positive constant β and a positive function ϕ defined on M, M satisfies $I_p(t) = \phi(p)t^\beta$ if and only if $\beta = 1, n = 2$ and M is a 2-dimensional ellipsoid centered at the origin in the 3-dimensional Euclidean space $\mathbb{E}^3$. In this case, we have $I_p(t) = \alpha t/\mathfrak{h}(p)$ for some positive constant α.

For a convex body $B \subset \mathbb{E}^{n+1}$ which is not necessarily centrally symmetric, with the help of Theorem A we prove the following characterization theorem for n-dimensional ellipsoids.

Theorem C. Suppose that a convex body $B \subset \mathbb{E}^{n+1}$ with the origin o in its interior has smooth boundary M with positive Gauss-Kronecker curvature and the homogeneous function $g = g_d$ of degree d. Then, for a positive function ϕ, one of $V_p(t)$, $A_p(t)/|\nabla g(p)|$, $C_p(t)$, and $I_p(t)$ is a function of the form $\phi\left(t/\mathfrak{h}(p)\right)$, where $\mathfrak{h}(p)$ is the support function of M at p if and only if M is an n-dimensional ellipsoid centered at the origin in the Euclidean space $\mathbb{E}^{n+1}$.

Note that Theorem C is an ellipsoidal analogue of the characterization theorem for n-dimensional round spheres in [9, 10].

If $B \subset \mathbb{E}^{n+1}$ is a closed convex body with smooth boundary M which is of positive Gauss-Kronecker curvature, then it is obvious that the boundary M cannot satisfy neither $V_p(t) = \phi(p)t^{\beta}$ nor $A_p(t) = \phi(p)t^{\beta}$, where β is a positive constant and $\phi(p)$ is a positive function of $p \in M$. On the other hand, every elliptic paraboloid $M \subset \mathbb{E}^{n+1}$ satisfies $V_p(t) = \phi(p)(\sqrt{t})^{n+2}$ and $A_p(t) = \psi(p)(\sqrt{t})^{n}$ for some functions ϕ and ψ on M. For this, see the proof of Theorem 5 in [10].

Let us denote by $K_p(t)$ the volume of the compact $(n+1)$-dimensional cone whose base is the region in Φ_t enclosed by $\Phi_t \cap M$ and whose vertex is the point p. Then we have

$$K_p(t) = \frac{t}{n+1} A_p(t). \tag{1.9}$$

Using (1.6) and Lemma 2.1, one obtains

Proposition D. Let M denote a strictly convex smooth hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$. Then the following conditions are equivalent.

(1) $V_p(t) = \phi(p)(\sqrt{t})^{n+2}$ for a function ϕ on M.

(2) $A_p(t) = \psi(p)(\sqrt{t})^{n}$ for a function ψ on M.

(3) $K_p(t)/V_p(t)$ is a constant.

Finally, for further study, we raise a question which was originally conjectured by M. Golomb [4]. See the last paragraph of [4], where $V_p(t)$ and $K_p(t)$ were denoted by $S(p;t)$ and $T(p;t)$, respectively. See also Question B in [10].

Question E. Let M denote a strictly convex smooth hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$. Is it an elliptic paraboloid if M satisfies one of the conditions in Proposition D?

In [11], Question E was answered affirmatively for $n = 1$.

A lot of properties of conic sections (especially, parabolas) have been proved to be characteristic ones [7, 11, 13, 15].

Some characterization theorems for hyperplanes, circular hypercylinders, hyperspheres, ellipsoids, elliptic paraboloids, and elliptic hyperboloids in the Euclidean space $\mathbb{E}^{n+1}$ were established in [5, 6, 8–10, 12, 16, 19, 20]. For some characterizations of hyperbolic space in the Minkowski space $\mathbb{E}^{n+1}_1$, we refer to [14].

Throughout this article, all objects are smooth (C^3) and connected, unless otherwise mentioned.

2. Preliminaries

In order to prove our theorems, first of all, we need the following.

Lemma 2.1. Suppose that M is a strictly convex and closed smooth hypersurface of the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the origin in its interior and g the homogeneous function of M. Then we have the following:

(1)

$$\lim_{t\to 0}\frac{1}{(\sqrt{t})^n}A_p(t)=\frac{(\sqrt{2})^n\omega_n}{\sqrt{K(p)}}, \tag{2.1}$$

(2)

$$\lim_{t\to 0}\frac{1}{(\sqrt{t})^{n+2}}V_p(t)=\frac{(\sqrt{2})^{n+2}\omega_n}{(n+2)\sqrt{K(p)}}, \tag{2.2}$$

(3)

$$\lim_{t\to 0}\frac{1}{(\sqrt{t})^n}C_p(t)=\frac{(\sqrt{2})^n\omega_n}{(n+1)\sqrt{K(p)}}\mathfrak{h}(p), \tag{2.3}$$

(4)

$$\lim_{t\to 0}\frac{1}{(\sqrt{t})^n}I_p(t)=\frac{(\sqrt{2})^n\omega_n}{(n+1)\sqrt{K(p)}}\mathfrak{h}(p), \tag{2.4}$$

where ω_n denotes the volume of the n-dimensional unit ball in the n-dimensional Euclidean space $\mathbb{E}^n$ and $\mathfrak{h}(p)$ the support function of M at $p \in M$.

Proof. For proofs of (1) and (2), see Lemma 8 of [10]. Together with (1) and (2), it follows from (1.7) and (1.8) that (3) and (4) hold. $\square$

Lemma 2.2. Let M denote a strictly convex and closed smooth hypersurface of the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the origin in its interior and $g=g_d$ the homogeneous function of M. Suppose that M satisfies one of the conditions (V^*), (A^*), (C^*), and (I^*). Then, M satisfies the following condition:

(C) $K(p)|\nabla g(p)|^{n+2}$ is a nonzero constant on M, where $K(p)$ is the Gauss-Kronecker curvature of M at p and $\nabla g(p)$ the gradient of g at p.

Proof. It follows from Lemma 8 or Lemma 9 in [6] that if M satisfies one of conditions (V^*) and (A^*), then it satisfies condition (C).

Note that the outward unit normal to the hypersurface M is given by $N = \nabla g/|\nabla g|$. For a fixed point $p \in M$ and a sufficiently small $t(> 0)$, the hyperplane parallel to the tangent hyperplane Φ of M at p and passing through the point $p - tN(p)$ is tangent to a homothetic hypersurface (say, $g^{-1}(h) = M_{h^{1/d}}$) of M at some point $v \in M_{h^{1/d}}$. Hence, the nonnegative function $h = h(t)(< 1)$ is defined on an interval $[0, \epsilon)$ with $h(0) = 1$ and the following holds:

$$V_p(t) = V_p^*(h(t)), A_p(t) = A_p^*(h(t)), C_p(t) = C_p^*(h(t)) \tag{2.5}$$

and

$$I_p(t) = I_p^*(h(t)). \tag{2.6}$$

On the other hand, since the homogeneous function g of M is of degree $d > 0$, we have $\nabla g(hp) = h^{d-1}\nabla g(p)$. Hence. we get $v = h^{1/d}p \in M_{h^{1/d}}$. It follows from the Euler Identity $\langle p, \nabla g(p)\rangle = dg(p)(= d)$ that

$$\mathfrak{h}(p) = \frac{d}{|\nabla g(p)|}. \tag{2.7}$$

This shows that

$$h(t) = \left(1 - \frac{1}{d}|\nabla g(p)|t\right)^d. \tag{2.8}$$

Now, suppose that M satisfies condition (C^*) for arbitrary positive number h with $h < 1$. Then, we get

$$C_p(t) = C_p^*(h(t)) = \gamma(h(t)), \tag{2.9}$$

where $\gamma(h)$ is a nonnegative function with $\gamma(1) = 0$. Hence, one obtains

$$\begin{aligned} \frac{1}{(\sqrt{t})^n} C_p(t) =& \frac{\gamma(h)}{(\sqrt{1 - h^{1/d}})^n} \left(\sqrt{\frac{1 - h(t)^{1/d}}{t}}\right)^n \\ =& \frac{\gamma(h)}{(\sqrt{1 - h^{1/d}})^n} \left(\frac{|\nabla g(p)|}{d}\right)^{n/2}, \end{aligned} \tag{2.10}$$

where the second equality follows from (2.8).

Let us put $\lim_{h\to 1} \gamma(h)/(\sqrt{1 - h^{1/d}})^n = \gamma$, which is independent of $p \in M$. Then it follows from (3) in Lemma 2.1, (2.7) and (2.10) that

$$K(p)|\nabla g(p)|^{n+2} = \frac{2^n \omega_n^2 d^{n+2}}{(n+1)^2\gamma^2}, \tag{2.11}$$

which is constant on the hypersurface M.

The remaining cases can be treated similarly. This completes the proof. $\square$

3. Proper affine hyperspheres

For strictly convex hypersurfaces, there are naturally two types of proper affine hyperspheres, depending on whether the affine normal points toward or away from the center. For an elliptic affine hypersphere, such as an ellipsoid, the affine normals point inward toward the center. Hyperbolic affine hyperspheres have affine normals which point away from the center. One component of a hyperboloid of two sheets is the quadric example of a hyperbolic affine hypersphere ([17]).

In order to prove Theorem A, we need some equivalent conditions for a hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ to be a proper affine hypersphere. For a function $g : U \subset \mathbb{E}^{n+1} \to \mathbb{R}$ defined on an open set $U \subset \mathbb{E}^{n+1}$, we define

$$\psi_g(p) = \det \begin{pmatrix} D^2 g(p) & (\nabla g(p))^T \\ \nabla g(p) & 0 \end{pmatrix}, \tag{3.1}$$

where $D^2 g(p)$ is the Hessian matrix of g at $p \in \mathbb{E}^{n+1}$ (cf. [3]). Then, we have the following.

Lemma 3.1. We fix a regular value k of a function $g : U \subset \mathbb{E}^{n+1} \to \mathbb{R}$. Then, the level hypersurface $M_k = g^{-1}(k)$ is a proper affine hypersphere, centered at the origin, if and only if the function g satisfies for all $p \in M_k$,

$$\psi_g(p) = a \langle p, \nabla g(p) \rangle^{n+2}, \tag{3.2}$$

where a is a nonzero constant and $\nabla g(p)$ denotes the gradient of g at p.

Proof. See the proof of Proposition 1 of [3]. $\square$

Next, we prove the following lemma, which gives a geometric meaning of $\psi_g(p)$ for $p \in M_k$.

Lemma 3.2. For a regular value k of a function $g : U \subset \mathbb{E}^{n+1} \to \mathbb{R}$, we have for all $p \in M_k$,

$$\psi_g(p) = (-1)^{n+1} K(p) |\nabla g(p)|^{n+2}, \tag{3.3}$$

where $K(p)$ denotes the Gauss-Kronecker curvature of $M_k = g^{-1}(k)$ at p.

Proof. We fix a point $p \in M_k$. By a Euclidean motion of coordinates, without loss of generality, we may assume that

$$\nabla g(p) = (0, 0, \cdots, 0, |\nabla g(p)|). \tag{3.4}$$

Hence, the level hypersurface $M_k = g^{-1}(k)$ is, at least locally, the graph of a function $f : V \subset \mathbb{E}^n \to \mathbb{R}$ which satisfies $p = (q, f(q)), q \in V$ and $f_i(q) = 0$ for $i = 1, \cdots, n$, where we denote by f_i the derivative of f with respect to x_i. Differentiating $g(x, f(x)) = k$ with respect to x_i and x_j for $i, j = 1, \cdots, n$ successively, we get

$$\begin{aligned} &g_{ij}(x, f(x)) + g_{in+1}(x, f(x)) f_j(x) \\ &+\{g_{n+1j}(x, f(x)) + g_{n+1n+1}(x, f(x)) f_j(x)\} f_i(x) + g_{n+1}(x, f(x)) f_{ij}(x) = 0, \end{aligned} \tag{3.5}$$

where g_{ij} denotes the derivative of g_i with respect to x_j, and so on.

At the point $p = (q, f(q)) \in M_k$, it follows from (3.4) and (3.5) that

$$g_{ij}(p) = -g_{n+1}(p) f_{ij}(q) = -|\nabla g(p)| f_{ij}(q). \tag{3.6}$$

Thus, we obtain

$$\begin{aligned}\psi_g(p) &= \det \begin{pmatrix} D^2 g(p) & (\nabla g(p))^T \\ \nabla g(p) & 0 \end{pmatrix} \\ &= -|\nabla g(p)|^2 \det \begin{pmatrix} g_{11}(p) & \cdots & g_{1n}(p) \\ \vdots & & \vdots \\ g_{n1}(p) & \cdots & g_{nn}(p) \end{pmatrix} \\ &= (-1)^{n+1} |\nabla g(p)|^{n+2} \det(f_{ij}(q)). \end{aligned} \tag{3.7}$$

Since $f_i(q) = 0$ for $i = 1, \cdots, n$, the Gauss-Kronecker curvature $K(p)$ of M_k at p is given by ([1] or [21, p. 93])

$$K(p) = \det(f_{ij}(q)). \tag{3.8}$$

Together with (3.7), (3.8) completes the proof. □

Combining Lemmas 3.1 and 3.2, we obtain the following which will be used in the proof of Theorem A.

Proposition 3.3. For a regular value k of a function $g : U \subset \mathbb{E}^{n+1} \to \mathbb{R}$, the level hypersurface $M_k = g^{-1}(k)$ is a proper affine hypersphere, centered at the origin, if and only if for some nonzero constant a the function g satisfies for all $p \in M_k$,

$$K(p) |\nabla g(p)|^{n+2} = a \langle p, \nabla g(p) \rangle^{n+2}, \tag{3.9}$$

where $K(p)$ denotes the Gauss-Kronecker curvature of $M_k = g^{-1}(k)$ at p.

4. Proofs

Suppose that M is a strictly convex and closed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the origin in its interior and the outward unit normal N. We denote by g the homogeneous function of M. That is, on the open set $U = \{tp | t > 0, p \in M\} = \mathbb{E}^{n+1} \setminus \{0\}$ the homogeneous function g is defined by

$$g(tp) = t^d, p \in M, \tag{4.1}$$

where d is a positive number. Since the function g is homogeneous of degree d (> 0) and the outward unit normal N is given by $N = \nabla g / |\nabla g|$, the Euler Identity shows that for all $p \in M$

$$\langle p, \nabla g(p) \rangle = d g(p) = d. \tag{4.2}$$

Together with (4.2), Proposition 3.3 implies the following.

Proposition 4.1. Suppose that M is a strictly convex and closed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the origin in its interior and $g = g_d$ the homogeneous function of M of degree d. Then, the following are equivalent.

(1) The hypersurface M is a proper affine hypersphere centered at the origin.

(2) The hypersurface M satisfies for some nonzero constant a

$$K(p)|\nabla g(p)|^{n+2} = a, \tag{4.3}$$

where $K(p)$ denotes the Gauss-Kronecker curvature of M at p.

First, we prove Theorem A as follows.

Let us denote by M a strictly convex and closed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the origin in its interior and $g = g_d$ the homogeneous function of M of degree d. Suppose that M satisfies one of (1), (2), (3), (4), and (5) in Theorem A stated in Section 1. Then, it follows from Lemma 2.2 or (2.7) that it satisfies (6) in Theorem A. Hence, Proposition 4.1 shows that M is a proper affine hypersphere centered at the origin. Since M is closed, the theorem of Blaschke and Deicke implies that M is an ellipsoid in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ centered at the origin (cf. Section 2.4 of [17]). This completes the proof of the only if part of Theorem A.

Conversely, in order to prove the if part of Theorem A, let us denote by E the ellipsoid centered at the origin defined by

$$a_1^2x_1^2 + \cdots + a_n^2x_n^2 + b^2z^2 = 1, \tag{4.4}$$

where $a_1, \ldots, a_n, b > 0$. For conveniences, we consider the homogeneous function $g = g_2$ of degree 2 of the ellipsoid M, which is given by

$$g(x_1, \ldots, x_n, z) = a_1^2x_1^2 + \cdots + a_n^2x_n^2 + b^2z^2. \tag{4.5}$$

Then, for a positive number h with $h < 1$ the homothetic hypersurface $g^{-1}(h) = M_{\sqrt{h}}$ is an ellipsoid defined by

$$a_1^2x_1^2 + \cdots + a_n^2x_n^2 + b^2z^2 = h. \tag{4.6}$$

It follows from the proof of Theorem 3 in [6] that for a positive number h with $h < 1$ the ellipsoid M satisfies (V^*) and (A^*), respectively. Furthermore, if we replace a_i with a_i/b and k with $1/b^2$ in (3.38) of [6], then we see that the ellipsoid E satisfies

$$K(p)|\nabla g(p)|^{n+2} = 2^{n+2}\frac{a_1^2\cdots a_n^2}{b^{2n+4}}, \tag{4.7}$$

which is constant on the ellipsoid E. Hence, the ellipsoid E satisfies (1), (2) and (6) in Theorem A and hence satisfies (5) in Theorem A.

Together with (4.2), (1.4) implies that

$$C_p^*(h) = 2\frac{\sqrt{h}}{n+1}\frac{A_p^*(h)}{|\nabla g(p)|}. \tag{4.8}$$

This, together with (2) in Theorem A, shows that E satisfies (3) in Theorem A. Finally, it follows from (1.5) that E satisfies (4) in Theorem A. This completes the proof of the if part of Theorem A.

Now, we prove Theorem B as follows.

We denote by $g = g_d$ the homogeneous function of the boundary M of B. First, suppose that the boundary M of B satisfies

$$I_p(t) = \phi(p)t^{\beta}, \tag{4.9}$$

where $\phi(p)$ denotes a function of $p \in M$. Then we get from (2.4) that $\beta = n/2$ and

$$\phi(p) = \frac{(\sqrt{2})^n \omega_n}{(n+1)\sqrt{K(p)}}\mathfrak{h}(p), \tag{4.10}$$

where ω_n is the volume of the n-dimensional unit ball in the n-dimensional Euclidean space $\mathbb{E}^n$.

In order to compute $I_p^*(h)$, note that

$$I_p(t) = I_p^*(h(t)), \tag{4.11}$$

where the function $h(t)$ is given by (2.8). Hence, we get

$$t = \frac{d}{|\nabla g(p)|}(1 - h^{1/d}). \tag{4.12}$$

Together with (4.9) with $\beta = n/2$, this implies

$$I_p^*(h) = \eta(p)(1 - h^{1/d})^{n/2}, \tag{4.13}$$

where we put

$$\eta(p) = \phi(p)\left(\frac{d}{|\nabla g(p)|}\right)^{n/2}. \tag{4.14}$$

Since B is centrally symmetric with respect to the origin o, for the volume V of B we obtain

$$\lim_{h\to 0} I_p^*(h) = V/2. \tag{4.15}$$

It follows from (4.13) and (4.15) that the function $\eta(p)$ is a constant $V/2$. Hence, we see that $I_p^*(h)$ is a function of h only, which is independent of the point $p \in M$. Therefore, Theorem A shows that M is an n-dimensional ellipsoid in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ centered at the origin.

For a fixed point $p \in M$, let us put $a = \mathfrak{h}(p)$. Then (4.15) is nothing but the following:

$$I_p(a) = \phi(p)(\sqrt{a})^n = V/2. \tag{4.16}$$

Since B is centrally symmetric with respect to the origin, we also have (See (1.7) and (1.8) for t $(> a)$.)

$$\lim_{t\to 2a} I_p(t) = \phi(p)(\sqrt{2a})^n = V. \tag{4.17}$$

It follows from (4.16) and (4.17) that $n = 2$; hence, $\beta = 1$ and M is a 2-dimensional ellipsoid centered at the origin in the 3-dimensional Euclidean space $\mathbb{E}^3$.

Conversely, it is straightforward to show that the 2-dimensional ellipsoid given by

$$a^2x^2 + b^2y^2 + c^2z^2 = 1 \tag{4.18}$$

satisfies

$$I_p(t) = \frac{\alpha}{\mathfrak{h}(p)} t, \tag{4.19}$$

where we put $\alpha = 2\pi/(3abc)$. This completes the proof of Theorem B.

Finally, we prove Theorem C as follows. First, suppose that M satisfies

$$V_p(t) = \phi\left(\frac{t}{\mathfrak{h}(p)}\right), \tag{4.20}$$

for a positive function ϕ. Note that it follows from (2.7) and (2.8) that

$$h(t) = \left(1 - \frac{t}{\mathfrak{h}(p)}\right)^d, \tag{4.21}$$

which implies

$$\frac{t}{\mathfrak{h}(p)} = 1 - h^{1/d}. \tag{4.22}$$

Hence, we get

$$V_p^*(h) = \phi(1 - h^{1/d}). \tag{4.23}$$

Thus, the volume function $V_p^*(h)$ does not depend on the point $p \in M$; hence, Theorem A implies that M is an n-dimensional ellipsoid. The remaining cases can be treated similarly. Therefore, the proof of the only if part of Theorem C is completed.

Conversely, suppose that M is an n-dimensional ellipsoid centered at the origin. Then Theorem A shows that $V_p^*(h) = \alpha(h)$ for some function α. Hence, we have from (4.21) that

$$V_p(t) = V_p^*(h(t)) = \alpha\left(h(t)\right) = \phi\left(\frac{t}{\mathfrak{h}(p)}\right), \tag{4.24}$$

where ϕ is the function defined by $\phi(s) = \alpha((1-s)^d)$. The remaining cases can be treated similarly. This completes the proof of the if part of Theorem C.

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