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On elements whose Moore–Penrose inverse is idempotent in a $*$-ring

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Abstract: In this paper, we investigate the elements whose Moore–Penrose inverse is idempotent in a $*$-ring. Let R be a $*$-ring and $a \in R^{\dagger}$. Firstly, we give a concise characterization for the idempotency of $a^{\dagger}$ as follows: $a \in R^{\dagger}$ and $a^{\dagger}$ is idempotent if and only if $a \in R^{\#}$ and $a^2 = aa^*a$, which connects Moore–Penrose invertibility and group invertibility. Secondly, we generalize the results of Baksalary and Trenkler from complex matrices to $*$-rings. More equivalent conditions which ensure the idempotency of $a^{\dagger}$ are given. Particularly, we provide the characterizations for both a and $a^{\dagger}$ being idempotent. Finally, the equivalent conditions under which a is EP and $a^{\dagger}$ is idempotent are investigated.

Key words: Moore–Penrose inverse, group inverse, core inverse, idempotent, EP

1. Introduction

Recall that an involution $*\colon a \mapsto a^*$ in a ring R is an antiisomorphism of degree 2, i.e. $(a^*)^* = a$, $(ab)^* = b^*a^*$, $(a+b)^* = a^* + b^*$, for arbitrary a, $b \in R$. For simplicity, we call R a $*$-ring if it has an involution $*$. Let $a \in R$. If $a = a^*$, then a is called Hermitian. The element a is called a projection if $a^2 = a = a^*$. If there exists $x \in R$ such that the following four equations hold:

$$(1)\ axa = a,\ (2)\ xax = x,\ (3)\ (ax)^* = ax,\ (4)\ (xa)^* = xa,$$

then x is called the Moore–Penrose inverse of a. If x exists, then it is unique and denoted by $a^{\dagger}$. The symbol $R^{\dagger}$ denotes the set of all Moore–Penrose invertible elements in R. If a is Moore–Penrose invertible and $aa^{\dagger} = a^{\dagger}a$, then a is called EP. Generally, x is called a $\{1\}$-inverse (i.e. inner inverse) of a if the equation (1) holds. $a\{1\}$ denotes the set of all $\{1\}$-inverses of a. If the equation (2) holds, then x is called a $\{2\}$-inverse (i.e. outer inverse) of a and $a\{2\}$ denotes the set of all $\{2\}$-inverses of a. If x satisfies equations (1) and (3), then x is called a $\{1,3\}$-inverse of a. We use $a^{(1,3)}$ to denote a $\{1,3\}$-inverse of a. And $a\{1,3\}$ denotes the set of all $\{1,3\}$-inverses of a. Similarly, if x satisfies equations (1) and (4), then x is called a $\{1,4\}$-inverse of a. We use $a^{(1,4)}$ to denote a $\{1,4\}$-inverse of a. And $a\{1,4\}$ denotes the set of all $\{1,4\}$-inverses of a. The symbols $R^{\{1,3\}}$ and $R^{\{1,4\}}$ denote the sets of all $\{1,3\}$-invertible and $\{1,4\}$-invertible elements in R, respectively. For more details, readers can refer to [9, 11, 15].

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According to Drazin [7], $a \in R$ is called Drazin invertible if there exists $x \in R$ satisfying the following equations:

$$xax = x, \quad ax = xa, \quad a^{k+1}x = a^k \text{ for some } k \in \mathbb{N}^+.$$

If x exists, then it is unique and denoted by a^D. If k is the smallest positive integer such that the above equations hold, then k is called the Drazin index of a and denoted by $\mathrm{ind}(a) = k$. In particular, x is called the group inverse of a and denoted by $a^\#$ when $k = 1$. The symbol $R^\#$ denotes the set of all group invertible elements in R.

In 2010, Baksalary and Trenkler [1] introduced the core inverse of a complex matrix. Later, Rakić et al. [16] generalized this notion to a $*$-ring and characterized it by five equations, which were reduced to three equations by Xu et al. [17] as follows. Let $a \in R$. If there exists $x \in R$ such that the following equations hold:

$$(ax)^* = ax, \quad xa^2 = a, \quad ax^2 = x,$$

then x is called the core inverse of a. It is unique if it exists and denoted by $a^{\text{\textcircled{\#}}}$. The symbol $R^{\text{\textcircled{\#}}}$ denotes the set of all core invertible elements in R.

Recall that in [4, Fact 8.7.6], Bernstein proved that $A^\dagger$ is idempotent if and only if $A^2 = AA^*A$ for any $A \in \mathbb{C}^{n\times n}$. In [2], Baksalary and Trenkler investigated the matrices whose Moore–Penrose inverse is idempotent. They gave more characterizations for the idempotency of $A^\dagger$, as well as both A and $A^\dagger$ being idempotent.

Motivated by the above work, we generalize their results from complex matrices to $*$-rings. Throughout the paper, R is a $*$-ring. Let $a \in R^\dagger$. The paper is organized as follows. In Section 2, we first give a concise characterization for the idempotency of $a^\dagger$: $a \in R^\dagger$ and $a^\dagger$ is idempotent if and only if $a \in R^\#$ and $a^2 = aa^*a$, which establishes the relationship between Moore–Penrose invertibility and group invertibility. Then, we present some equivalent conditions which ensure that $a^\dagger$ is idempotent by inner and outer inverses. In Section 3, we provide the characterizations for both a and $a^\dagger$ being idempotent. Furthermore, the equivalent conditions under which a is EP and $a^\dagger$ is idempotent are investigated.

2. Characterizations for the idempotency of the Moore–Penrose inverse

In this section, we investigate the elements whose Moore–Penrose inverse is idempotent and give several corresponding equivalent characterizations. Firstly, let us recall some auxiliary lemmas.

Lemma 2.1 [10] *Let $a \in R$. Then $a \in R^\#$ if and only if $a \in a^2R \cap Ra^2$. Moreover, if $a = a^2x = ya^2$ for some x, $y \in R$, then $a^\# = yax$.*

Lemma 2.2 [6, 12] *Let $a \in R^\dagger$. Then*

(i) $(a^\dagger)^\dagger = a$;

(ii) $(a^*)^\dagger = (a^\dagger)^*$;

(iii) $(aa^*)^\dagger = (a^*)^\dagger a^\dagger, \quad (a^*a)^\dagger = a^\dagger (a^*)^\dagger$.

Lemma 2.3 [17] *Let $a \in R$. Then $a \in R^{\text{\textcircled{\#}}}$ if and only if $a \in R^\# \cap R^{\{1,3\}}$. In this case, $a^{\text{\textcircled{\#}}} = a^\# aa^{\{1,3\}}$.*

Lemma 2.4 [8] *Let $a \in R$ and $p,\ q \in R$ be two projections. If $Ra = Rp$, then for any $x \in R$ such that $p = xa$, we have $x \in a\{1,4\}$. If $aR = qR$, then for any $y \in R$ such that $q = ay$, we have $y \in a\{1,3\}$.*

Lemma 2.5 [3] *Let $a \in R$. Then*

(i) *$Ra = Ra^*a$ if and only if $a \in R^{\{1,3\}}$;*

(ii) *$aR = aa^*R$ if and only if $a \in R^{\{1,4\}}$.*

Lemma 2.6 [15] *Let $a \in R$. Then $a \in R^\dagger$ if and only if $a \in R^{\{1,3\}} \cap R^{\{1,4\}}$. In this case, $a^\dagger = a^{(1,4)}aa^{(1,3)}$.*

Lemma 2.7 [18] *Let $a \in R$. Then the following statements are equivalent:*

(i) *$a \in R^\dagger$;*

(ii) *$a \in Raa^*a$. In this case, $a^\dagger = (xa)^*$, where $a = xaa^*a$;*

(iii) *$a \in aa^*aR$. In this case, $a^\dagger = (ay)^*$, where $a = aa^*ay$.*

In [4, Fact 8.7.6], Bernstein proved that $A^\dagger$ is idempotent if and only if $A^2 = AA^*A$ for any $A \in \mathbb{C}^{n\times n}$. Inspired by his work, we generalize the results from complex matrices to $*$-rings, and explore the relationship between group invertibility and Moore–Penrose invertibility in this case.

Theorem 2.8 *Let $a \in R$. Then the following statements are equivalent:*

(i) *$a \in R^\dagger$ and $a^\dagger$ is idempotent;*

(ii) *$a \in R^\dagger$ and $a^2 = aa^*a$;*

(iii) *$a \in R^\#$ and $a^2 = aa^*a$.*

In this case, $a^\# = (a^\dagger)^*a^\dagger(a^\dagger)^*$ and $a^\dagger = (a^\# a)^*$. Furthermore, $a^n \in R^\dagger \cap R^\#$ for any $n \in \mathbb{N}^+$ and

$$(a^n)^\dagger = (a^\dagger(a^\dagger)^*)^{n-1}a^\dagger = a^\dagger((a^\dagger)^*a^\dagger)^{n-1},$$

$$(a^n)^\# = (a^\dagger)^*(a^\dagger(a^\dagger)^*)^n = ((a^\dagger)^*a^\dagger)^n(a^\dagger)^*.$$

Therefore, $a^n \in R^{\#\!\!\!\!\bigcirc}$ and

$$(a^n)^{\#\!\!\!\!\bigcirc} = (a^\dagger)^*(a^\dagger(a^\dagger)^*)^n a^n (a^\dagger(a^\dagger)^*)^{n-1}a^\dagger = ((a^\dagger)^*a^\dagger)^n(a^\dagger)^*a^n a^\dagger((a^\dagger)^*a^\dagger)^{n-1}.$$

Proof (i) $\Rightarrow$ (ii): Since $(a^\dagger)^2 = a^\dagger$, we have

$$\begin{aligned} a^2 &= aa^\dagger aaa^\dagger a = a(a^\dagger a)(aa^\dagger)a = a(a^\dagger a)^*(aa^\dagger)^*a \\ &= a(aa^\dagger a^\dagger a)^*a = a(a(a^\dagger)^2a)^*a = a(aa^\dagger a)^*a = aa^*a. \end{aligned}$$

(ii) $\Rightarrow$ (iii): On one hand,

$$\begin{aligned} a &= aa^\dagger a = a(a^\dagger a)^* = aa^*(a^\dagger)^* = aa^*(a^\dagger aa^\dagger)^* = aa^*(aa^\dagger)^*(a^\dagger)^* \\ &= (aa^*a)a^\dagger(a^\dagger)^* = a^2a^\dagger(a^\dagger)^* \in a^2R. \end{aligned}$$

On the other hand,

$$a = aa^\dagger a = (aa^\dagger)^* a = (a^\dagger)^* a^* a = (a^\dagger aa^\dagger)^* a^* a$$
$$= (a^\dagger)^* a^\dagger aa^* a = (a^\dagger)^* a^\dagger a^2 \in Ra^2.$$

Therefore, we have $a = a^2 x$, $a = ya^2$, where $x = a^\dagger (a^\dagger)^*$, $y = (a^\dagger)^* a^\dagger$. According to Lemma 2.1, $a^\#$ exists and $a^\# = yax = (a^\dagger)^* a^\dagger aa^\dagger (a^\dagger)^* = (a^\dagger)^* a^\dagger (a^\dagger)^*$.

(iii) $\Rightarrow$ (i): Since $a \in R^\#$ and $a^2 = aa^* a$, we obtain $a = a^\# a^2 = a^\# aa^* a \in Raa^* a$. Then, according to Lemma 2.7, we have $a \in R^\dagger$ and $a^\dagger = (a^\# a)^*$. And $(a^\dagger)^2 = (a^\# a)^*(a^\# a)^* = (a^\# aa^\# a)^* = (a^\# a)^* = a^\dagger$.

Next, we will verify that in this case, $a^n \in R^\dagger \cap R^\#$ for any $n \in \mathbb{N}^+$.

Since $a \in R^\#$, we have $a^n \in R^\#$ and $(a^n)^\# = (a^\#)^n$ [7]. According to Lemma 2.1, $Ra = Ra^2$, then $Ra^n = Ra = Ra^\dagger a = Rp$. Similarly, since $aR = a^2 R$, we have $a^n R = aR = aa^\dagger R = qR$. Therefore, $p = a^\dagger a = a^\dagger a^2 a^\# = a^\dagger a(aa^\#)^n = a^\dagger a(a^\#)^n a^n = xa^n$, and $q = aa^\dagger = aa^\# aa^\dagger = (aa^\#)^n aa^\dagger = a^n (a^\#)^n aa^\dagger = a^n y$, where $x = a^\dagger a(a^\#)^n$ and $y = (a^\#)^n aa^\dagger$. According to Lemma 2.4, we can obtain $x \in a^n\{1,4\}$ and $y \in a^n\{1,3\}$. Therefore, by Lemma 2.6, $a^n \in R^\dagger$. Then, according to Lemma 2.3, $a^n \in R^{\textcircled{\#}}$.

Furthermore,

$$\begin{aligned}
(a^n)^\dagger &= (a^n)^{(1,4)} a^n (a^n)^{(1,3)} = xa^n y \\
&= a^\dagger a(a^\#)^n a^n (a^\#)^n aa^\dagger \\
&= a^\dagger a(a^n)^\# a^n (a^n)^\# aa^\dagger \\
&= a^\dagger a(a^n)^\# aa^\dagger \\
&= a^\dagger a(a^\#)^n aa^\dagger.
\end{aligned}$$

According to the above proof, $a^\# = (a^\dagger)^* a^\dagger (a^\dagger)^*$, then

$$\begin{aligned}
(a^\#)^2 &= (a^\dagger)^* a^\dagger (a^\dagger)^* (a^\dagger)^* a^\dagger (a^\dagger)^* \\
&= (a^\dagger)^* a^\dagger (a^\dagger)^* a^\dagger (a^\dagger)^* \\
&= (a^\dagger)^* (a^\dagger (a^\dagger)^*)^2, \\
(a^\#)^3 &= (a^\dagger)^* a^\dagger (a^\dagger)^* a^\dagger (a^\dagger)^* (a^\dagger)^* a^\dagger (a^\dagger)^* \\
&= (a^\dagger)^* a^\dagger (a^\dagger)^* a^\dagger (a^\dagger)^* a^\dagger (a^\dagger)^* \\
&= (a^\dagger)^* (a^\dagger (a^\dagger)^*)^3 \\
&\cdots
\end{aligned}$$

By the induction,

$$(a^n)^\# = (a^\#)^n = (a^\dagger)^* (a^\dagger (a^\dagger)^*)^n = ((a^\dagger)^* a^\dagger)^n (a^\dagger)^*.$$

Since $a^\dagger a(a^\dagger)^* = (a^\dagger a)^*(a^\dagger)^* = (a^\dagger a^\dagger a)^* = (a^\dagger a)^* = a^*(a^\dagger)^*$ and similarly, $(a^\dagger)^* aa^\dagger = (a^\dagger)^* a^*$, we have

$$\begin{aligned}(a^n)^\dagger &= a^\dagger a(a^\#)^n aa^\dagger = a^\dagger a(a^\dagger)^*(a^\dagger(a^\dagger)^*)^n aa^\dagger \\ &= a^*(a^\dagger)^*(a^\dagger(a^\dagger)^*)^n aa^\dagger = a^*(a^\#)^n aa^\dagger \\ &= a^*((a^\dagger)^* a^\dagger)^n (a^\dagger)^* aa^\dagger = a^*((a^\dagger)^* a^\dagger)^n (a^\dagger)^* a^* \\ &= (a^\dagger(a^\dagger)^*)^n a^* = (a^\dagger(a^\dagger)^*)^{n-1} a^\dagger(a^\dagger)^* a^* \\ &= (a^\dagger(a^\dagger)^*)^{n-1} a^\dagger = a^\dagger((a^\dagger)^* a^\dagger)^{n-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}(a^n)^{\textcircled{\#}} &= (a^n)^\# a^n (a^n)^\dagger \\ &= (a^\dagger)^*(a^\dagger(a^\dagger)^*)^n a^n (a^\dagger(a^\dagger)^*)^{n-1} a^\dagger \\ &= ((a^\dagger)^* a^\dagger)^n (a^\dagger)^* a^n a^\dagger((a^\dagger)^* a^\dagger)^{n-1}.\end{aligned}$$

□

However, it is worth noting that merely $a^2 = aa^*a$ cannot imply that $a \in R^\#$ or $a \in R^\dagger$. We give a counterexample in the following.

Example 2.9 *Let $R = M_2(\mathbb{R})$, where $\mathbb{R}$ denotes the set of all real numbers. We define the involution $* : \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}$ (i.e. the adjoint matrix). Let $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $a^2 = 0 = a^*a$. By Lemma 2.1, $a \notin R^\#$. And $Ra \neq Ra^*a$, then $a \notin R^\dagger$ according to Lemmas 2.5 and 2.6.*

In [2, Theorem 3.1], Baksalary and Trenkler gave four equivalent characterizations for the idempotency of $A^\dagger \in \mathbb{C}^{n\times n}$. Inspired by them, we obtain the version in $*$-rings.

Theorem 2.10 Let $a \in R^\dagger$. Then $a^\dagger$ is idempotent if and only if any of the following statements holds:

(i) $a^*a^\dagger = a^*$;

(ii) $a^\dagger a^* = a^*$;

(iii) $(aa^*)^\dagger$ is an inner inverse of a;

(iv) $(aa^*)^\dagger$ is an outer inverse of a;

(v) $(a^*a)^\dagger$ is an inner inverse of a;

(vi) $(a^*a)^\dagger$ is an outer inverse of a;

(vii) $(a^*a)^\dagger$ is an inner inverse of a^*;

(viii) $(a^*a)^\dagger$ is an outer inverse of a^*;

(ix) $(aa^*)^\dagger$ is an inner inverse of a^*;

(x) $(aa^*)^\dagger$ is an outer inverse of a^*.

Proof (i): ($\Rightarrow$) On one hand,

$$a^* = (aa^\dagger a)^* = a^*(aa^\dagger)^* = a^*aa^\dagger.$$

On the other hand, since $(a^\dagger)^2 = a^\dagger$, we have

$$a^*a^\dagger = (aa^\dagger a)^*a^\dagger = a^*(aa^\dagger)^*a^\dagger = a^*(aa^\dagger)a^\dagger = a^*a(a^\dagger)^2 = a^*aa^\dagger.$$

Therefore, $a^*a^\dagger = a^*$.

($\Leftarrow$) Since $a^*a^\dagger = a^*$, we have

$$\begin{aligned} a^\dagger &= a^\dagger aa^\dagger = a^\dagger(aa^\dagger)^* = a^\dagger(a^\dagger)^*a^* = a^\dagger(a^\dagger)^*a^*a^\dagger \\ &= a^\dagger(aa^\dagger)^*a^\dagger = a^\dagger(aa^\dagger)a^\dagger = (a^\dagger aa^\dagger)a^\dagger = (a^\dagger)^2. \end{aligned}$$

That is to say, $a^\dagger$ is idempotent.

(ii): The proof is dual to that of (i).

(iii): Since

$$\begin{aligned} a(aa^*)^\dagger a = a &\Leftrightarrow a(a^\dagger)^*a^\dagger a = a \Leftrightarrow a(a^\dagger)^*(a^\dagger a)^* = a \\ &\Leftrightarrow a(a^\dagger aa^\dagger)^* = a \Leftrightarrow a(a^\dagger)^* = a \Leftrightarrow a^\dagger a^* = a^*, \end{aligned}$$

and according to the above (ii), we can obtain that $a^\dagger$ is idempotent if and only if $(aa^*)^\dagger$ is an inner inverse of a.

(iv): ($\Rightarrow$) Since $a^\dagger$ is idempotent, $(a^\dagger)^*$ is also idempotent. Therefore, we have

$$\begin{aligned} (aa^*)^\dagger a(aa^*)^\dagger &= (a^*)^\dagger a^\dagger a(a^*)^\dagger a^\dagger = (a^\dagger)^*a^\dagger a(a^\dagger)^*a^\dagger \\ &= (a^\dagger)^*(a^\dagger a)^*(a^\dagger)^*a^\dagger = (a^\dagger aa^\dagger)^*(a^\dagger)^*a^\dagger \\ &= (a^\dagger)^*(a^\dagger)^*a^\dagger = (a^\dagger)^*a^\dagger = (a^*)^\dagger a^\dagger = (aa^*)^\dagger. \end{aligned}$$

That is to say, $(aa^*)^\dagger$ is an outer inverse of a.

($\Leftarrow$) Since $(aa^*)^\dagger a(aa^*)^\dagger = (aa^*)^\dagger$, premultiplying and postmultiplying aa^* on both sides at the same time, we can get $aa^*(aa^*)^\dagger a(aa^*)^\dagger aa^* = aa^*$. Therefore,

$$\begin{aligned} aa^* &= aa^*(a^\dagger)^*a^\dagger a(a^\dagger)^*a^\dagger aa^* = aa^*(a^\dagger)^*(a^\dagger a)^*(a^\dagger)^*a^\dagger aa^* \\ &= aa^*(a^\dagger aa^\dagger)^*(a^\dagger)^*a^\dagger aa^* = aa^*(a^\dagger)^*(a^\dagger)^*a^\dagger aa^* = a(a^\dagger a)^*(a^\dagger)^*(a^\dagger a)^*a^* \\ &= a(a^\dagger a)^*(a^\dagger)^*(aa^\dagger a)^* = aa^\dagger a(a^\dagger)^*a^* = a(a^\dagger)^*a^* = a(aa^\dagger)^* = aaa^\dagger. \end{aligned}$$

Postmultiplying a on both sides, we can get $aa^*a = a^2$. Then according to Theorem 2.8, $a^\dagger$ is idempotent.

(v): Since

$$\begin{aligned} a(a^*a)^\dagger a = a &\Leftrightarrow aa^\dagger(a^\dagger)^*a = a \Leftrightarrow (aa^\dagger)^*(a^\dagger)^*a = a \\ &\Leftrightarrow (a^\dagger aa^\dagger)^*a = a \Leftrightarrow (a^\dagger)^*a = a \Leftrightarrow a^*a^\dagger = a^*, \end{aligned}$$

and according to the above (i), we can obtain that $a^\dagger$ is idempotent if and only if $(a^*a)^\dagger$ is an inner inverse of a.

(vi): $(\Rightarrow)$ Since $a^\dagger$ is idempotent, $(a^\dagger)^*$ is also idempotent. Therefore, we have

$$\begin{aligned}(a^*a)^\dagger a(a^*a)^\dagger &= a^\dagger(a^*)^\dagger aa^\dagger(a^*)^\dagger = a^\dagger(a^\dagger)^*aa^\dagger(a^\dagger)^* \\ &= a^\dagger(a^\dagger)^*(aa^\dagger)^*(a^\dagger)^* = a^\dagger(a^\dagger)^*(a^\dagger aa^\dagger)^* \\ &= a^\dagger(a^\dagger)^*(a^\dagger)^* = a^\dagger(a^\dagger)^* = a^\dagger(a^*)^\dagger = (a^*a)^\dagger.\end{aligned}$$

That is to say, $(a^*a)^\dagger$ is an outer inverse of a.

$(\Leftarrow)$ Since $(a^*a)^\dagger a(a^*a)^\dagger = (a^*a)^\dagger$, premultiplying and postmultiplying a^*a on both sides at the same time, we can get $a^*a(a^*a)^\dagger a(a^*a)^\dagger a^*a = a^*a$. Therefore,

$$\begin{aligned}a^*a &= a^*aa^\dagger(a^\dagger)^*aa^\dagger(a^\dagger)^*a^*a = a^*(aa^\dagger)^*(a^\dagger)^*aa^\dagger(a^\dagger)^*a^*a \\ &= a^*(a^\dagger aa^\dagger)^*(aa^\dagger)^*(a^\dagger)^*a^*a = a^*(a^\dagger)^*(a^\dagger aa^\dagger)^*a^*a \\ &= a^*(a^\dagger)^*(a^\dagger)^*a^*a = (a^\dagger a)^*(aa^\dagger)^*a = a^\dagger aaa^\dagger a = a^\dagger aa.\end{aligned}$$

Premultiplying a on both sides, we can get $aa^*a = a^2$. Then according to Theorem 2.8, $a^\dagger$ is idempotent.

(vii), (viii), (ix) and (x): It is obvious that $a^\dagger$ is idempotent if and only if $(a^*)^\dagger$ is idempotent. According to the equivalence between $(a^\dagger)^2 = a^\dagger$ and (iii) [(resp. (iv), (v) and (vi)], we can obtain that $a^\dagger$ is idempotent if and only if (vii) [resp. (viii), (ix) and (x)] holds. □

The next proposition shows the properties of an element whose Moore–Penrose inverse is idempotent, which generalizes [2, Theorem 3.5]. Recall that $a \in R^\dagger$ is called star-dagger if $a^*a^\dagger = a^\dagger a^*$.

Proposition 2.11 Let $a \in R^\dagger$. If $a^\dagger$ is idempotent, then:

(i) $a \in R^{\oplus}$;

(ii) a is star-dagger;

(iii) $(a^\dagger)^* = a^\# a^*a = aa^*a^\#$;

(iv) $a^2(a^\dagger)^2 = aa^*$;

(v) $(a^\dagger)^2a^2 = a^*a$.

Proof (i): By Theorem 2.8, $a \in R^{\oplus}$.

(ii): Since $a^\dagger$ is idempotent, according to Theorem 2.10, we have $a^*a^\dagger = a^* = a^\dagger a^*$.

(iii): By Theorem 2.8, $a^\dagger = (a^\# a)^*$. Therefore, $(a^\dagger)^* = a^\# a = (a^\#)^2aa = (a^\#)^2aa^*a = a^\# a^*a$. Similarly, $(a^\dagger)^* = aa^\# = aa(a^\#)^2 = aa^*a(a^\#)^2 = aa^*a^\#$.

(iv): Since $a^\dagger$ is idempotent, according to the above (ii) and Theorem 2.10, we have

$$\begin{aligned} a^2(a^\dagger)^2 &= a^2a^\dagger = a(aa^\dagger)^* = a(a^\dagger)^*a^* \\ &= (a^\dagger)^*aa^* = (a^\dagger)^*aa^\dagger a^* = (a^\dagger)^*(aa^\dagger)^*a^* \\ &= (aa^\dagger a^\dagger)^*a^* = (aa^\dagger)^*a^* \\ &= aa^\dagger a^* = aa^*. \end{aligned}$$

(v): The proof is dual to that of (iv). □

Besides, we find that $a^2 = aa^*a$ can imply several properties.

Proposition 2.12 Let $n \in \mathbb{N}^+\backslash\{1\}$, $a \in R$ and $a^2 = aa^*a$. Then $a^n = a(a^{n-1})^*a$, a^*a^n and a^na^* are Hermitian. More generally, for any $k_1,\ k_2,\ l_1,\ l_2 \in \mathbb{N}^+$, if $k_1 + k_2 = l_1 + l_2$, then $(a^{k_1})^*a^{k_2} = (a^{l_1})^*a^{l_2}$.

Proof When $n = 2$, it is true. Suppose that the conclusion holds when $n = k$. Then when $n = k+1$, we have

$$a(a^k)^*a = a(a(a^{k-1})^*a)^*a = aa^*a^{k-1}a^*a = a^{k+1}.$$

Therefore, $a^n = a(a^{n-1})^*a$ for $n \geq 2$ holds.
Thus,

$$a^*a^n = a^*a(a^{n-1})^*a = (aa^*a)^*(a^{n-2})^*a = (a^n)^*a.$$

Similarly, a^na^* is Hermitian.
In this case, for any $k_1,\ k_2,\ l_1,\ l_2 \in \mathbb{N}^+$, if $k_1 + k_2 = l_1 + l_2$, we have

$$(a^{k_1})^*a^{k_2} = a^*a^{k_1+k_2-1} = (a^{l_1})^*a^{l_2}.$$

□

3. Characterizations for both an element and its Moore–Penrose inverse being idempotent

In [2, Theorem 3.2], Baksalary and Trenkler gave several equivalent conditions which ensure that both A and $A^\dagger \in \mathbb{C}^{n\times n}$ are idempotent. We generalize them from complex matrices to $*$-rings. The conclusions are summarized as follows. First, recall that $a \in R$ is called normal (resp. binormal) if $aa^* = a^*a$ (resp. $aa^*a^*a = a^*aaa^*$) and $a \in R^\dagger$ is called a partial isometry (resp. bi-dagger and bi-EP) if $a^\dagger = a^*$ (resp. $(a^\dagger)^2 = (a^2)^\dagger$ and $aa^\dagger a^\dagger a = a^\dagger aaa^\dagger$). The notions of bi-dagger and bi-EP are generalizations to $*$-rings of [5, Theorem 2] on EP operators in Hilbert spaces.

Theorem 3.1 Let $a \in R^\dagger$. Then the following statements are equivalent:

(i) a is idempotent and $a^\dagger$ is idempotent;

(ii) $a^\dagger$ is idempotent and a is a partial isometry;

(iii) $a^\dagger$ is idempotent and $a^\dagger$ is a partial isometry;

(iv) a is idempotent and a is a partial isometry;

(v) a is idempotent and $a^\dagger$ is a partial isometry;

(vi) $a^\dagger$ is idempotent and a is bi-dagger;

(vii) a is idempotent and a is bi-dagger.

Proof (i) $\Rightarrow$ (ii): Since a and $a^\dagger$ are both idempotent, by Theorem 2.10 (i), we have

$$a^* = a^*a^\dagger = a^*a^\dagger aa^\dagger = a^*(a^\dagger a)^*a^\dagger = (a^\dagger aa)^*a^\dagger = (a^\dagger a)^*a^\dagger = a^\dagger aa^\dagger = a^\dagger.$$

Therefore, a is a partial isometry.

(ii) $\Rightarrow$ (i): If a is a partial isometry, then $a^* = a^\dagger$. Since $a^\dagger$ is idempotent, by Theorem 2.8, we have

$$a^2 = aa^*a = aa^\dagger a = a.$$

(i) $\Leftrightarrow$ (iii): The proof is the same as that of (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (iv): The proof is the same as that of (i) $\Rightarrow$ (ii).

(iv) $\Rightarrow$ (i): Since a is idempotent and a is a partial isometry, we have

$$a^2 = a = aa^\dagger a = aa^*a.$$

Therefore, by Theorem 2.8, $a^\dagger$ is idempotent.

(i) $\Leftrightarrow$ (v): The proof is the same as that of (i) $\Leftrightarrow$ (iv).

(i) $\Rightarrow$ (vi): Since a and $a^\dagger$ is idempotent, $(a^2)^\dagger = a^\dagger = (a^\dagger)^2$.

(vi) $\Rightarrow$ (i): Since $a^\dagger$ is idempotent and a is bi-dagger, $a^\dagger = (a^\dagger)^2 = (a^2)^\dagger$. Therefore, $a = a^2$.

(i) $\Rightarrow$ (vii): The proof is the same as that of (i) $\Rightarrow$ (vi).

(vii) $\Rightarrow$ (i): Since a is idempotent and a is bi-dagger, we have $(a^\dagger)^2 = (a^2)^\dagger = a^\dagger$. Therefore, $a^\dagger$ is idempotent. □

From the above theorem, we can directly obtain the following corollary, which will be discussed later.

Corollary 3.2 Let $a \in R^\dagger$. If $a^\dagger$ is idempotent, then the following statements are equivalent:

(i) a is bi-dagger;

(ii) a is a partial isometry;

(iii) a is idempotent.

Proposition 3.3 Let $a \in R^\dagger$. If $a^\dagger$ is idempotent, then the following statements are equivalent:

(i) a is Hermitian;

(ii) a is normal;

(iii) a is binormal;

(iv) a is EP;

(v) a is bi-EP.

In this case, a is idempotent.

Proof (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii): The proof is trivial.

(iii) $\Rightarrow$ (iv): Since $a^\dagger$ is idempotent, we have $a(a^2)^*a = a^3$ by Proposition 2.12. Then $a^*a^2a^* = (a^3)^*$. Since a is bi-normal, $a(a^2)^*a = a^*a^2a^*$. Thus, $a^3 = (a^3)^*$ and then $a^3R = (a^3)^*R$. Due to Theorem 2.8, $a \in R^\#$. Then $aR = a^2a^\#R = a^2R = a^3R$. Similarly, $a^*R = (a^*)^3R$. Therefore, $aR = a^*R$. And since $a \in R^\dagger$, according to [14], we know that a is EP.

(iv) $\Rightarrow$ (v): The proof is trivial.

(v) $\Rightarrow$ (i): Since $a^\dagger$ is idempotent, we have $aa^\dagger a^\dagger a = aa^\dagger a = a$ and $a^\dagger aaa^\dagger = a^\dagger aa^*aa^\dagger = (a^\dagger a)^*a^*(aa^\dagger)^* = a^*(aa^\dagger)^* = a^*$. Due to $aa^\dagger a^\dagger a = a^\dagger aaa^\dagger$, we obtain $a = a^*$.

In this case, since a is EP, we have $a^2 = aa^\dagger a^2 = a(a^\dagger)^2a^2 = a(a^\#)^2a^2 = a$. □

Remark 3.4 Under the assumption that $a^\dagger$ is idempotent, we can obtain the equivalence between (ii), (iv) and (v) in Proposition 3.3, which is a generalization of [5, Theorem 2] on EP operators in Hilbert spaces to elements in $*$-rings. Inspired by several equivalent characterizations for EP elements when $a \in R^\# \cap R^\dagger$ presented in [13, Theorem 2.1], we obtain more equivalent conditions of EPness under a stronger condition that $a^\dagger$ is idempotent in Proposition 3.3.

Actually, in [2, Theorem 3.4], Baksalary and Trenkler pointed out that if $A \in \mathbb{C}^{n\times n}$ has an idempotent Moore–Penrose inverse, then the above three in Corollary 3.2 and five in Proposition 3.3 are consistently equivalent. But in a $*$-ring R, the two parts cannot be equivalent. In other words, both a and $a^\dagger$ being idempotent cannot imply that a is EP.

Example 3.5 Let $R = M_2(\mathbb{R}) \times M_2(\mathbb{R})$. For any given $(a,b) \in R$, take the involution to be $*\colon\ (a,b) \mapsto (b^T, a^T)$. Set $a = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. By computation, we can obtain $(a,b)(a,b)^*(a,b) = (a,b)$. Since (a,b) is idempotent, $(a,b) \in R^\#$. According to Theorem 2.8, $(a,b) \in R^\dagger$ and $(a,b)^\dagger$ is idempotent. But in this case, $(a,b)^* \neq (a,b)$. Therefore, (a,b) is not EP by Proposition 3.3.

In addition, there are several sufficient conditions which ensure the idempotency of both a and $a^\dagger$ as follows.

Proposition 3.6 Let $a \in R^\dagger$. Then the following statements are equivalent:

(i) a is idempotent and a is Hermitian;

(ii) a is idempotent and a is EP;

(iii) a is idempotent and $a^\dagger$ is Hermitian;

(iv) a is idempotent and $a^\dagger$ is EP;

(v) $a^\dagger$ is idempotent and a is Hermitian;

(vi) $a^\dagger$ is idempotent and a is EP;

(vii) $a^\dagger$ is idempotent and $a^\dagger$ is Hermitian;

(viii) $a^\dagger$ is idempotent and $a^\dagger$ is EP.

In this case, both a and $a^\dagger$ are idempotent.

Proof (i) $\Leftrightarrow$ (ii): Since a is idempotent and Hermitian, obviously, $a^\dagger = a$. Therefore, $aa^\dagger = a^\dagger a$, that is, a is EP. Conversely, if a is idempotent and EP, then $a = aa^\dagger a = a^\dagger aa = a^\dagger a$. Therefore, a is Hermitian.

(i) $\Leftrightarrow$ (iii): Since a is Hermitian if and only if $a^\dagger$ is Hermitian, the proof is trivial.

(ii) $\Leftrightarrow$ (iv): Since a is EP if and only if $a^\dagger$ is EP, the proof is trivial.

Thus, (i) – (iv) are equivalent. Similarly, (v) – (viii) are equivalent.

(i) $\Leftrightarrow$ (v): Since a is idempotent and Hermitian, we have that $a^\dagger = a$ is idempotent. Conversely, according to Proposition 3.3, if $a^\dagger$ is idempotent and a is Hermitian, then a is idempotent.

To sum up, (i) – (viii) are equivalent. In this case, both a and $a^\dagger$ are idempotent. □

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