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On some generalizations of the beta function in several variables

Mustapha RAÏSSOULI[1,2,*], **Salma EL-SOUBHY**[3]
[1]Department of Mathematics, College of Science, Qassim University, Qassim, Saudi Arabia
[2]Department of Mathematics, Science Faculty, Moulay Ismail University, Meknes, Morocco
[3]Department of Mathematics, Science Faculty, Taibah University, Al Madinah Al Munawwarah, Saudi Arabia

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Abstract: Intensive studies aiming to extend the gamma and beta functions and to establish some properties for these extensions have been recently carried out. In this paper, we first introduce a generalized gamma function in n variables. Afterwards, two generalized beta functions in several variables are introduced and their properties are discussed. Among others, we investigate recurrence relationships, Mellin transform properties, and partial differential equations involving these generalized functions. At the end, some results about partial derivatives of these extended functions are presented as well.

Key words: Beta function, gamma function, beta function in n variables, extended beta function in n variables, generalized beta function in n variables

1. Introduction

Throughout this paper, we set $\mathbb{C}_{+*} =: \big\{z \in \mathbb{C} : \ \Re e(z) > 0\big\}$. The Euler's beta and gamma functions are, respectively, defined by

$$\forall x, y \in \mathbb{C}_{+*} \qquad B(x,y) =: \int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

$$\forall x \in \mathbb{C}_{+*} \qquad \Gamma(x) =: \int_0^{+\infty} t^{x-1}e^{-t}dt.$$

Such functions have wide applications in various contexts of mathematical analysis as well as in physics like the quantum mechanics. An interesting relationship expressing a connection between B and Γ is given by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.1}$$

On the other hand, it is well-known that $\Gamma(x+1) = x\Gamma(x)$ for any $x \in \mathbb{C}_{+*}$. For further properties and applications of the beta and gamma functions, we refer the interested reader to [2–6, 9, 11–14, 17, 18] for instance.

*Correspondence: raissouli.mustapha@gmail.com
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The previous functions B and Γ have been extended in the literature, see [7, 8, 16]. Let $a \in \mathbb{C}_{+*}$. Chaudhry et al. [7] introduced the following extended beta function

$$B(x,y;a) =: \int_0^1 t^{x-1}(1-t)^{y-1}e^{-a/t(1-t)}dt \tag{1.2}$$

and the extended gamma function as well

$$\Gamma_a(x) =: \int_0^\infty t^{x-1}e^{-t}e^{-a/t}dt. \tag{1.3}$$

Another extension of $B(x,y)$ was introduced by Choi et al. in [8] as follows:

$$B(x,y;a,b) =: \int_0^1 t^{x-1}e^{-a/t}(1-t)^{y-1}e^{-b/(1-t)}dt, \tag{1.4}$$

where $a,b \in \mathbb{C}_{+*}$. It is clear that

$$B(x,y;0) =: \lim_{a\to 0} B(x,y;a) = B(x,y), \quad \Gamma_0(x) =: \lim_{a\to 0}\Gamma_a(x) = \Gamma(x)$$

for any $x,y \in \mathbb{C}_{+*}$. Obviously, $B(x,y;a,a) = B(x,y;a)$. For the properties of $B(x,y;a)$ and $\Gamma_a(x)$, see [7, 16] and for those of $B(x,y;a,b)$, see [8].

Further extensions of the beta and gamma functions have been investigated in the literature. For instance, Özergin et al. [16] introduced the generalized beta and gamma functions defined, respectively, by

$$B_p^{(c,d)}(x,y) =: \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\Big(c;d;\frac{-p}{t(1-t)}\Big)dt, \tag{1.5}$$

$$\Gamma_p^{(c,d)}(x) =: \int_0^\infty t^{x-1}{}_1F_1\Big(c;d;-t-\frac{p}{t}\Big)dt, \tag{1.6}$$

where $c,d \in \mathbb{C}$ and $p \in \mathbb{C}_{+*}$. Here the notation ${}_1F_1(a;b;z)$ refers to the confluent hypergeometric function (CHF) defined through [15]

$${}_1F_1(a;b;z) =: \sum_{m=1}^\infty \frac{(a)_m}{(b)_m}\frac{z^m}{m!}, \tag{1.7}$$

provided that this series is well-defined and convergent. As usual, the notation $(\lambda)_m$, if $\lambda \in \mathbb{C}$ is nonnegative integer, stands for the Pochhammer symbol defined by

$$(\lambda)_m = \lambda(\lambda+1)...(\lambda+m-1), \text{ with } (\lambda)_0 = 1.$$

Note that ${}_1F_1(a;b;0) = 1$. If, moreover, $a, b-a \in \mathbb{C}_{+*}$ then we have, [16]

$${}_1F_1(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 u^{a-1}(1-u)^{b-a-1}e^{zu}du. \tag{1.8}$$

Making the substitution $t = 1-u$ in this latter integral formula it is easy to check that

$${}_1F_1(a;b;z) = e^z\,{}_1F_1(b-a;b;-z). \tag{1.9}$$

The importance of CHF arises from the fact that it contributes as a good tool for solving many mathematical problems. It also appears as a solution of some partial differential equations playing an important role in various mathematical areas. See [6, 15] for instance. See also Section 4 of the current manuscript.

Remark 1.1 *The formula (1.8) brings us some interesting results when the assumptions $a > 0$ and $b - a > 0$ are satisfied. For instance, we have the following assertions:*
(i) The real-map $z \longmapsto {}_1F_1(a;b;z)$ is strictly increasing and strictly convex on $\mathbb{R}$.
(ii) It follows that ${}_1F_1(a;b;z) \geq {}_1F_1(a;b;0) = 1$ for any $z \geq 0$ and by (1.9), $0 \leq {}_1F_1(a;b;z) \leq 1$ for any $z \leq 0$.
These properties are not simple to deduce from (1.7).

The extension of the beta function from two variables to n variables was introduced in the literature [1, 2, 6]. For $n \geq 3$ integer, let E_{n-1} be the standard $(n-1)$-simplex of $\mathbb{R}^{n-1}$ defined by

$$E_{n-1} = \Big\{(t_1, ..., t_{n-1}) \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} t_i \leq 1; \;\; t_i \geq 0, \text{ for } i = 1, ..., n-1\Big\}.$$

The beta function in n variables $x_1, ..., x_n \in \mathbb{C}_{+*}$ is defined by

$$B\big(x_1, ..., x_n\big) =: \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i - 1} dt_1 ... dt_{n-1}, \tag{1.10}$$

where we set $t_n =: 1 - \sum_{i=1}^{n-1} t_i$. Throughout the following, we set $\sigma(x) =: \sum_{i=1}^{n} x_i$ for the sake of simplicity. The following formula

$$B\big(x_1, ..., x_n\big) = \frac{\prod_{i=1}^{n} \Gamma(x_i)}{\Gamma\big(\sigma(x)\big)} \tag{1.11}$$

holds for any $x_1, ..., x_n \in \mathbb{C}_{+*}$. Other properties of the beta function in n variables can be found in the literature. Among these properties we mention the following [6]:

$$B\big(x_1, ..., x_n\big) = B\big(x_{\tau(1)}, ..., x_{\tau(n)}\big), \tag{1.12}$$

where τ is any permutation of the set $\{1, 2, ..., n\}$, and

$$B\big(x_1 + 1, x_2, ..., x_n\big) + ... + B\big(x_1, ..., x_{n-1}, x_n + 1\big) = B\big(x_1, ..., x_n\big). \tag{1.13}$$

Recently, the authors have extended the previous functions $B(x, y; a)$ and $B(x, y; a, b)$ for n variables. For any $x_1, ..., x_n \in \mathbb{C}$, $a_1, ..., a_n \in \mathbb{C}_{+*}$ and $a \in \mathbb{C}_{+*}$, they defined the two following extensions

$$B(x_1, ..., x_n; a) =: \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i - 1} e^{-a/\pi(t)} dt_1 ... dt_{n-1}, \tag{1.14}$$

$$B\big(x_1, ..., x_n; a_1, ..., a_n\big) =: \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i - 1} e^{-a_i/t_i} dt_1 ... dt_{n-1}, \tag{1.15}$$

where, as before, $t_n =: 1 - \sum_{i=1}^{n-1} t_i$, and $\pi(t) =: \prod_{i=1}^{n} t_i$.

This paper will be organized as follows: in Section 2, we introduce a generalized gamma function in n variables in a simple setting. Section 3 is devoted to introduce the first generalized beta function that extends (1.5) from two variables to n variables. Section 4 displays some partial differential equations satisfied by this generalized beta function. In Section 5, another generalized beta function in n variables of the second kind is also investigated. Section 6 deals with some partial derivatives of the second generalized beta function. All the previous generalized gamma and beta functions, in definitions as well as in properties, involve the confluent hypergeometric function which plays an important place in special functions theory.

2. Generalized gamma function in n variables

In this section we will discuss some extensions of (1.3) and (1.6) from one variable to n variables. Let $x =: (x_1, ..., x_n) \in \big(\mathbb{C}_{+*}\big)^n$. We define the gamma function in n variables $x_1, ..., x_n$ as follows: $\Gamma(x) =: \prod_{i=1}^n \Gamma(x_i)$. The extended gamma function in n variables may be defined as well: for $p = (p_1, ..., p_n) \in \big(\mathbb{C}_{+*}\big)^n$ we set

$$\Gamma_p(x) =: \int_{(0,\infty)^n} \prod_{i=1}^n t_i^{x_i-1} e^{-t_i} e^{-p_i/t_i} dt = \prod_{i=1}^n \Gamma_{p_i}(x_i), \tag{2.1}$$

where $dt =: dt_1...dt_n$ and $\Gamma_{p_i}(x_i)$ is defined by (1.3). We also introduce the following definition.

Definition 2.1 *Let $x =: (x_1, ..., x_n) \in \big(\mathbb{C}_{+*}\big)^n$, $\alpha =: (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n, \beta =: (\beta_1, ..., \beta_n) \in \mathbb{C}^n$ and $p = (p_1, ..., p_n) \in \big(\mathbb{C}_{+*}\big)^n$. The generalized gamma function in n variables is defined by*

$$\Gamma_p^{(\alpha,\beta)}(x) =: \prod_{i=1}^n \Gamma_{p_i}^{(\alpha_i,\beta_i)}(x_i) = \int_{(0,\infty)^n} \prod_{i=1}^n t_i^{x_i-1} {}_1F_1\Big(\alpha_i; \beta_i; -t_i - \frac{p_i}{t_i}\Big) dt, \tag{2.2}$$

where $dt =: dt_1...dt_n$ and $\Gamma_{p_i}^{(\alpha_i,\beta_i)}(x_i)$ is defined following (1.6).

Clearly, if $\alpha = \beta$ then (2.2) coincides with (2.1). Otherwise, $\Gamma_0(x) = \Gamma(x)$ for any $x = (x_1, ..., x_n) \in \big(\mathbb{C}_{+*}\big)^n$, and we also set

$$\Gamma^{(\alpha,\beta)}(x) =: \prod_{i=1}^n \Gamma^{(\alpha_i,\beta_i)}(x_i) = \int_{(0,\infty)^n} \prod_{i=1}^n t_i^{x_i-1} {}_1F_1\Big(\alpha_i; \beta_i; -t_i\Big) dt = \Gamma_0^{(\alpha,\beta)}(x). \tag{2.3}$$

The previous definition means that the generalized gamma function in n variables is defined as the product, in the habitual way, of the n components of the generalized gamma functions in one variable. Therefore, the properties of $\Gamma_p^{(\alpha,\beta)}(x)$ can be immediately deduced from those of $\Gamma_{p_i}^{(\alpha_i,\beta_i)}(x_i)$ for $i = 1, ..., n$. As an example, the following relationship

$$\Gamma_p^{(\alpha,\beta)}(x) = \Gamma_p^{(\alpha,\beta)}(-x) \prod_{i=1}^n p_i^{x_i} \tag{2.4}$$

holds for any $x =: (x_1, ..., x_n) \in \big(\mathbb{C}_{+*}\big)^n$, $\alpha =: (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n, \beta =: (\beta_1, ..., \beta_n) \in \mathbb{C}^n$ and $p = (p_1, ..., p_n) \in \big(\mathbb{C}_{+*}\big)^n$.

Remark 2.2 *(i) The generalized gamma function previously defined will appear when studying the properties of the generalized beta functions in n variables of the first kind and the second kind introduced in Sections 3 and 5, respectively.*

(ii) We left to the reader the routine task for formulating other relationships for $\Gamma_p^{(\alpha,\beta)}(x)$, as done in (2.4).

3. Generalized beta function of the first kind

We preserve the same notations as in the previous sections. In the ongoing section we will introduce generalized beta functions in n variables of the first kind as recited in the following.

Definition 3.1 *Let $x =: (x_1, ..., x_n) \in \left(\mathbb{C}_{+*}\right)^n$; $c, d, r \in \mathbb{C}$ and $q \in \mathbb{C}_{+*}$. The generalized beta function, of the first kind, is defined by:*

$$B_r^{(c,d)}(x;q) =: \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i-1} {}_1F_1\left(c; d; -\frac{q}{\pi(t)} - r\frac{\pi(t)}{q}\right) dt, \tag{3.1}$$

where we set $dt =: dt_1...dt_{n-1}$ and $\pi(t) =: \prod_{i=1}^n t_i$, with $t_n =: 1 - \sum_{i=1}^{n-1} t_i$.

It is clear that if $n = 2$ and $r = 0$ then (3.1) yields (1.5). Furthermore, if $c = d$ and $r = 0$ then (3.1) coincides with (1.14). Remark that (3.1) presents a singularity at $q = 0$ if $r \neq 0$. The first properties of $B_r^{(c,d)}(x;q)$, analogous to (1.12) and (1.13), are embodied in the following result.

Proposition 3.2 *Let x, c, d, r, and q be as in the previous definition.*
(i) The following relationship holds:

$$B_r^{(c,d)}(x;q) = B_r^{(c,d)}(x^*;q),$$

where we set $x^* =: (x_{\tau(1)}, ..., x_{\tau(n)})$ for any permutation τ of the set $\{1, 2, ..., n\}$.
(ii) We have

$$\sum_{j=1}^{n} B_r^{(c,d)}(x + e_j; q) = B_r^{(c,d)}(x;q),$$

where $(e_1, ..., e_n)$ refers to the canonical basis of $\mathbb{R}^n$.

Proof (i) Let $y =: (x_1, ..., x_j, ..., x_k, ..., x_n)$ and $z =: (x_1, ..., x_k, ..., x_j, ..., x_n)$. It is enough to show that $B_r^{(c,d)}(y;q) = B_r^{(c,d)}(z;q)$ for any j, k such that $1 \leq j < k \leq n$. We consider the following change of variables

$$t_1 = u_1, ..., t_j = u_k, ..., t_k = u_j, ..., t_n = u_n$$

First, it is obvious that $\pi(t) =: t_1...t_n = u_1...u_n =: \pi(u)$. Furthermore, it is clear that $(t_1, ..., t_n) \in E_{n-1}$ if and only if $(u_1, ..., u_n) \in E_{n-1}$. Moreover, it is easy to see that the absolute value of the Jacobian J of the previous transformation $(t_1, ..., t_n) \longmapsto (u_1, ..., u_n)$ is given by $|J| = 1$. By (3.1), with the standard rules of Calculus, we get

$$B_r^{(c,d)}(y;q) = \int_{E_{n-1}} \prod_{i=1,i\neq j,k}^{n} t_i^{x_i-1}\, t_j^{x_j-1} t_k^{x_k-1}\, {}_1F_1\left(c;d;-\frac{q}{\pi(t)} - r\frac{\pi(t)}{q}\right)\, dt$$

$$= \int_{E_{n-1}} \prod_{i=1,i\neq j,k}^{n} u_i^{x_i-1}\, u_j^{x_k-1} u_k^{x_j-1}\, {}_1F_1\left(c;d;-\frac{q}{\pi(u)} - r\frac{\pi(u)}{q}\right)\, du = B_r^{(c,d)}(z;q)$$

and hence the desired result.
(ii) By (3.1) we have

$$\sum_{j=1}^{n} B_r^{(c,d)}(x+e_j;q) = \int_{E_{n-1}} \sum_{j=1}^{n} \prod_{i=1,i\neq j}^{n} t_i^{x_i-1}\, t_j^{x_j}\, {}_1F_1\left(c;d;-\frac{q}{\pi(t)} - r\frac{\pi(t)}{q}\right) dt,$$

or, equivalently,

$$\sum_{j=1}^{n} B_r^{(c,d)}(x+e_j;q) = \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i-1} \left(\sum_{j=1}^{n} t_j\right)\, {}_1F_1\left(c;d;-\frac{q}{\pi(t)} - r\frac{\pi(t)}{q}\right) dt.$$

Since $\sum_{j=1}^{n} t_j = 1$ we then get the desired equality, so the proof is completed. $\square$

For $r = 0$, we set throughout the following

$$B^{(c,d)}(x;q) =: B_0^{(c,d)}(x;q) =: \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i-1}\, {}_1F_1\Big(c;d;-\frac{q}{\pi(t)}\Big), \tag{3.2}$$

where the singularity at $q = 0$ has been escaped. As first property of $B^{(c,d)}(x;q)$ we have the following result.

Theorem 3.3 *Let x, c, d, and q be as above. Assume that $c, d-c \in \mathbb{C}_{+*}$. Then there holds*

$$B^{(c,d)}(x;q) = \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int_0^1 u^{c-1}(1-u)^{d-c-1} B\big(x;qu\big) du, \tag{3.3}$$

where $B(x;qu)$ is defined by (1.14).

Proof By (3.2) and (1.8), we have

$$B^{(c,d)}(x;q) = \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i-1}\, {}_1F_1\Big(c;d;-\frac{q}{\pi(t)}\Big) dt$$

$$= \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int_{E_{n-1}} \left\{\prod_{i=1}^{n} t_i^{x_i-1} \int_0^1 u^{c-1}(1-u)^{d-c-1} e^{-\frac{qu}{\pi(t)}} du\right\} dt,$$

where, as before, $dt = dt_1...dt_{n-1}$ and $\pi(t) = t_1...t_n$. By virtue of the uniform convergence of the involved integrals we can interchange their orders for obtaining

$$B^{(c,d)}(x;q) = \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int_0^1 u^{c-1}(1-u)^{d-c-1} \left(\int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i-1} e^{-\frac{qu}{\pi(t)}} dt\right) du,$$

which, when combined with (1.14), immediately implies (3.3). □

We have the following result as well.

Theorem 3.4 *Let c, d, q, r be as in Definition 3.1. Assume that $c, d-c \in \mathbb{C}_{+*}$. Then the following relationship holds:*

$$B_r^{(c,d)}(x;q) = \sum_{m=0}^{\infty} \frac{(c)_m}{(d)_m} \frac{(-r/q)^m}{m!} B^{(c+m,d+m)}(x+me;q), \tag{3.4}$$

where $B^{(c+m,d+m)}(x+me;q)$ is defined following (3.2) and $e =: (1,1,...,1)$.

Proof By (1.8) we have

$$ {}_1F_1\Big(c;d;-\frac{q}{\pi(t)} - r\frac{\pi(t)}{q}\Big) = \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int_0^1 u^{c-1}(1-u)^{d-c-1} e^{-uq/\pi(t)} e^{-ur\pi(t)/q} dt.$$

This, with the expansion series

$$e^{-ur\pi(t)/q} = \sum_{m=0}^{\infty} \frac{(-r/q)^m}{m!} u^m \big(\pi(t)\big)^m,$$

and the uniform convergence of this latter power series, implies that

$$ {}_1F_1\Big(c;d;-\frac{q}{\pi(t)} - r\frac{\pi(t)}{q}\Big) = \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \sum_{m=0}^{\infty} \frac{(-r/q)^m}{m!} \big(\pi(t)\big)^m \int_0^1 u^{c+m-1}(1-u)^{d-c-1} e^{-uq/\pi(t)} du. \tag{3.5}$$

Again by (1.8) we can write

$$\int_0^1 u^{c+m-1}(1-u)^{d-c-1} e^{-uq/\pi(t)} du = \frac{\Gamma(c+m)\Gamma(d-c)}{\Gamma(d+m)} {}_1F_1\Big(c+m; d+m; -\frac{q}{\pi(t)}\Big).$$

On the other hand, it is not hard to check that $\Gamma(c+m) = (c)_m \Gamma(c)$. Substituting these in (3.5), we deduce that

$$ {}_1F_1\Big(c;d;-\frac{q}{\pi(t)} - r\frac{\pi(t)}{q}\Big) = \sum_{m=0}^{\infty} \frac{(c)_m}{(d)_m} \frac{(-r/q)^m}{m!} \big(\pi(t)\big)^m {}_1F_1\Big(c+m; d+m; -\frac{q}{\pi(t)}\Big).$$

Multiplying this equality by $\prod_{i=1}^n t_i^{x_i-1}$ and then integrating over E_{n-1} with respect to $(t_1,...,t_{n-1})$, we get, by virtue of the uniform convergence of the involved series sum,

$$B_r^{(c,d)}(x;q) = \sum_{m=0}^{\infty} \frac{(c)_m}{(d)_m} \frac{(-r/q)^m}{m!} \int_{E_{n-1}} \prod_{i=1}^n t_i^{x_i+m-1} {}_1F_1\Big(c+m; d+m; -\frac{q}{\pi(t)}\Big).$$

This, when combined with (3.2), yields the desired result, so the proof is completed. □

The preceding theorem has many interesting consequences. Particulary, we mention the following corollaries.

Corollary 3.5 *Assume that $q, r > 0$, $d > c > 0$, and $x \in (0,\infty)^n$. If, moreover, the sequence $\big((d)_m/(c)_m\big)_m$ is upper bounded then we have*

$$B_r^{(c,d)}(x;q) \le \sup_{m\ge 0} \frac{(d)_m}{(c)_m} \, B^{(c,d)}(x;q) \, {}_1F_1\Big(c;d;\frac{r}{q}\Big).$$

Proof By (3.3) we immediately deduce that $B^{(c,d)}(x;q) \geq 0$ for any $d > c > 0$ and $x \in (0,\infty)^n$. Furthermore, we can write

$$B^{(c+m,d+m)}(x+me;q) = \frac{\Gamma(d+m)}{\Gamma(c+m)\Gamma(d-c)} \int_0^1 u^{c+m-1}(1-u)^{d-c-1} B\big(x+me;qu\big)du.$$

This, with the fact that $\Gamma(d+m) = (d)_m \Gamma(d)$, $u^m \leq 1$ for any $u \in [0,1]$ and $B(x+me;q) \leq B(x;q)$ for any $m \geq 0$, implies that

$$B^{(c+m,d+m)}(x+me;q) \leq \frac{(d)_m}{(c)_m} \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int_0^1 u^{c-1}(1-u)^{d-c-1} B\big(x;qu\big)du,$$

which, with (3.3) again, yields

$$B^{(c+m,d+m)}(x+me;q) \leq \sup_{m\geq 0} \frac{(d)_m}{(c)_m}\, B^{(c,d)}(x;q). \tag{3.6}$$

Taking the absolute value of (3.4) side by side, with the standard triangular inequality, and using (3.6) with the help of (1.7), we deduce the desired inequality. □

Corollary 3.6 *Let c, d, q, r be as in Definition 3.1. Then, for any $k = 0, 1, \ldots$, we have*

$$\frac{\partial^k}{\partial r^k} B_r^{(c,d)}(x;q)|_{r=0} = \frac{(c)_k}{(d)_k} \frac{(-1)^k}{q^k} B^{(c+k,d+k)}(x+ke;q). \tag{3.7}$$

Proof Expansion (3.4) can be considered to be a Taylor series of $B_r^{(c,d)}(x;q)$ in a neighborhood of $r = 0$. The desired result immediately follows. □

We now state the following result.

Theorem 3.7 *Let c, d, q, r be as in Definition 3.1. Assume that $|r| < |q|^2$. Then the following formula*

$$\frac{B_r^{(c,d)}(x;q)}{B(x)} = \sum_{m,k=0}^{\infty} \binom{m}{k} \frac{(c)_m}{(d)_m} \frac{(-q)^m}{m!} \Big(\frac{r}{q^2}\Big)^k \frac{\big(1-\sigma(x)-2nk\big)_{nm}}{\prod_{i=1}^n (1-x_i-2k)_m}, \tag{3.8}$$

holds for any nonintegers $x_1, \ldots, x_n \in \mathbb{C}_{+*}$ such that $\sigma(x)$ is noninteger.

In particular, if $r = 0$, then one has

$$\frac{B^{(c,d)}(x;q)}{B(x)} = \sum_{m=0}^{\infty} \frac{(c)_m}{(d)_m} \frac{(-q)^m}{m!} \frac{\big(1-\sigma(x)\big)_{nm}}{\prod_{i=1}^n (1-x_i)_m}. \tag{3.9}$$

Proof By (3.1) and (1.7), and a simple algebraic operation, we have

$$B^{(c,d)}(x;q) = \int_{E_{n-1}} \prod_{i=1}^n t_i^{x_i-1} \sum_{m=0}^{\infty} \frac{(c)_m}{(d)_m} \frac{(-q)^m}{m!} \frac{1}{(\pi(t))^m} \left(1 + r\Big(\frac{\pi(t)}{q}\Big)^2\right)^m dt. \tag{3.10}$$

Since $\pi(t) =: t_1...t_n < 1$ for any $(t_1,...,t_{n-1}) \in E_{n-1}$ then the condition $|r| < |q|^2$ leads to the expansion series

$$\left(1 + r\Big(\frac{\pi(t)}{q}\Big)^2\right)^m = \sum_{k=0}^{\infty} \binom{m}{k} \Big(\frac{r}{q^2}\Big)^k \big(\pi(t)\big)^{2k}.$$

Substituting this in (3.10) and interchanging the order of the integral with the series sum (which is uniformly convergent), we get

$$B_r^{(c,d)}(x;q) = \sum_{m,k=0}^{\infty} \binom{m}{k} \frac{(c)_m}{(d)_m} \frac{(-q)^m}{m!} \Big(\frac{r}{q^2}\Big)^k \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i+2k-m-1} dt. \tag{3.11}$$

According to (1.10) and (1.11), we have

$$\int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i+2k-m-1} dt = B\big(x_1 + 2k - m, ..., x_n + 2k - m\big) = \frac{\prod_{i=1}^n \Gamma(x_i + 2k - m)}{\Gamma\big(\sigma(x) + 2nk - nm\big)},$$

provided that $x_, ..., x_n$ and $\sigma(x)$ are noninteger complex numbers. Otherwise, it is not hard to check that, for any non-integer $x \in \mathbb{C}$, we have

$$\Gamma(x - m) = \frac{(-1)^m}{(1 - x)_m} \Gamma(x).$$

Applying this formula we obtain

$$\prod_{i=1}^{n} \Gamma(x_i + 2k - m) = \prod_{i=1}^{n} \frac{(-1)^m}{(1 - x_i - 2k)_m} \Gamma(x_i) = (-1)^{nm} \frac{\prod_{i=1}^n \Gamma(x_i)}{\prod_{i=1}^n (1 - x_i - 2k)_m}$$

$$\Gamma\big(\sigma(x) + 2nk - nm\big) = \frac{(-1)^{nm}}{\big(1 - \sigma(x) - 2nk\big)_{nm}} \Gamma\big(\sigma(x)\big).$$

Substituting these in (3.11) and then in (3.10), and using (1.11) again, we get (3.8) after a simple manipulation. Taking $r = 0$ in (3.8) we deduce (3.9) after simple manipulations, so the proof is completed. □

Remark 3.8 *We can write (3.9) in the following form:*

$$\frac{B^{(c,d)}(x;q)}{B(x)} = \sum_{m=0}^{\infty} \frac{(c)_m}{(d)_m} \frac{(-q)^m}{m!} \prod_{i=1}^{n} \frac{\big(1 - \sigma(x) + (i-1)m\big)_m}{(1 - x_i)_m}.$$

Indeed, it is not hard to check that $(x)_{nm} = \prod_{i=1}^n \big(x + (i-1)m\big)_m$, *which when substituted in (3.9), yields the desired equality.*

From Theorem 3.7 we may deduce many interesting consequences. In particular, we cite the following corollaries.

Corollary 3.9 *With the same hypotheses as previous, for any* $j = 0, 1, ...$ *we have*

$$\frac{d}{dr^j} B_r^{(c,d)}(x;q)\ |_{r=0} = \sum_{m}^{\infty} (m - j + 1)_j \frac{(c)_m}{(d)_m} \frac{(-q)^{m-2j}}{m!} \frac{\big(1 - \sigma(x) - 2nj\big)_{nm}}{\prod_{i=1}^n (1 - x_i - 2j)_m} B(x).$$

Proof The expansion (3.8) when considered to be a Taylor series of $B_r^{(c,d)}(x;q)$ at $r=0$ immediately yields the desired result. The details are simple and therefore omitted here. □

Note that $B^{(c,d)}(x;q)$ has no singularity at $q=0$ and so (3.9) presents a Taylor expansion of $B^{(c,d)}(x;q)$ at $q=0$. Then we can immediately deduce the following corollary.

Corollary 3.10 *With the same hypotheses as previous, for any $j=0,1,...$ we have*

$$\frac{d}{dq^j}B^{(c,d)}(x;q)\mid_{q=0}=B(x)(-1)^j\frac{(c)_j}{(d)_j}\prod_{i=1}^{n}\frac{\Big(1-\sigma(x)+(i-1)j\Big)_j}{(1-x_i)_j}. \tag{3.12}$$

Remark 3.11 *Following (3.8), $B_r^{(c,d)}(x;q)$ presents a singularity at $q=0$ provided that $r\neq 0$. Therefore, an analog of (3.12) for $B_r^{(c,d)}(x;q)$ when $r\neq 0$ does not hold.*

The following result gives a relationship between $B_r^{(c,d)}(x;q)$, $\Gamma_r^{(c,d)}(u)$ and the usual beta function in n variables.

Theorem 3.12 *Let $x\in\big(\mathbb{C}_{+*}\big)^n$, $c,d,r\in\mathbb{C}$ and $u\in\mathbb{C}_{+*}$. Then we have*

$$\int_0^\infty q^{u-1}B_r^{(c,d)}(x;q)\;dq=\Gamma_r^{(c,d)}(u)B(x+ue), \tag{3.13}$$

where we set, as above $e=:(1,1,...,1)$, and $B(x+ue)=B(x_1+u,...,x_n+u)$.

Proof Multiplying (3.1) by q^{u-1} and then integrating with respect to $q\in(0,\infty)$ we get

$$\int_0^\infty q^{u-1}B_r^{(c,d)}(x;q)\;dq=\int_0^\infty\left(\int_{E_{n-1}}\prod_{i=1}^{n}t_i^{x_i-1}q^{u-1}{}_1F_1\Big(c;d;-\frac{q}{\pi(t)}-r\frac{\pi(t)}{q}\Big)dt\right)dq.$$

By virtue of the uniform convergence of the involved integrals we can interchange their orders for obtaining

$$\int_0^\infty q^{u-1}B_r^{(c,d)}(x;q)\;dq=\int_{E_{n-1}}\prod_{i=1}^{n}t_i^{x_i-1}\left(\int_0^\infty q^{u-1}{}_1F_1\Big(c;d;-\frac{q}{\pi(t)}-r\frac{\pi(t)}{q}\Big)dq\right)dt. \tag{3.14}$$

Now, for the right integral over $q\in(0,\infty)$ we use the change of variables $q=\pi(t)s$ and we get, after simple algebraic operations,

$$\int_0^\infty q^{u-1}{}_1F_1\Big(c;d;-\frac{q}{\pi(t)}-r\frac{\pi(t)}{q}\Big)dq=\big(\pi(t)\big)^u\int_0^\infty s^{u-1}{}_1F_1\Big(c;d;-s-\frac{r}{s}\Big)ds,$$

which, with (1.6), implies that

$$\int_0^\infty q^{u-1}{}_1F_1\Big(c;d;-\frac{q}{\pi(t)}-r\frac{\pi(t)}{q}\Big)dq=\big(\pi(t)\big)^u\Gamma_r^{(c,d)}(u).$$

Substituting this in (3.14) and taking into account that $\pi(t) =: t_1 t_2 ... t_n$ we get, with the help of (1.10),

$$\int_0^{\infty} q^{u-1} B_r^{(c,d)}(x;q)\, dq = \Gamma_r^{(c,d)}(u) \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i+u-1} dt = \Gamma_r^{(c,d)}(u) B\big(x_1+u, ..., x_n+u\big).$$

The proof is finished. □

Taking $u = 1$ in (3.13), we immediately obtain the following formula

$$\int_0^{\infty} B_r^{(c,d)}(x;q) dq = \Gamma_r^{(c,d)}(1) B(x+e).$$

In particular, if $n = 2$ and $r = 0$, we get the following corollary which gives a relationship expressing a connection between (1.5) and (1.6).

Corollary 3.13 *Assume that $n = 2$. Then the following relationship*

$$\int_0^{\infty} B_p^{(c,d)}(x,y) dp = \Gamma_0^{(c,d)}(1) B(x+1, y+1)$$

holds for any $c, d \in \mathbb{C}$ and $x, y \in \mathbb{C}_{+*}$.

With the aim to state another interesting result, we need to recall the following lemma about the so-called Mellin transform representation and its inverse, see [10] for instance.

Lemma 3.14 *Let $f : (0,\infty) \longrightarrow \mathbb{R}$. Assume that the following integral*

$$g(s) =: \int_0^{\infty} f(x) x^{s-1} dx$$

exists. Then we have

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s) x^{-s} ds.$$

By using the previous lemma, the following result may be stated.

Theorem 3.15 *Let $x \in \big(\mathbb{C}_{+*}\big)^n$, $c, d, r \in \mathbb{C}$ and $q \in \mathbb{C}_{+*}$. Then we have*

$$B_r^{(c,d)}(x;q) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} q^{-u} B(x+ue) \Gamma_r^{(c,d)}(u) du. \tag{3.15}$$

Proof Applying the previous lemma to (3.13), with appropriate manipulations, we immediately deduce the desired result. The details are simple and therefore omitted here for the reader. □

4. Partial differential equations involving $B_r^{(c,d)}(x;q)$

Since all involved sum series and integrals in this paper are uniformly convergent we can therefore differentiate under the sum sign and under the integral sign. As an example, we can check that

$$\frac{d}{dz}{}_1F_1(a;b;z) = \frac{a}{b}{}_1F_1(a+1;b+1;z), \tag{4.1}$$

which is a well-known result in the literature, see [15] for instance.

We have the following results as well.

Proposition 4.1 *Let $B_r^{(c,d)}(x;q)$ be the generalized beta function of the first kind. For any integer $k=1,\ldots$ we have*

$$\frac{\partial^k}{\partial r^k}B_r^{(c,d)}(x;q) = \frac{(c)_k}{(d)_k}\frac{(-1)^k}{q^k}B_r^{(c+k,d+k)}(x+ke;q) \tag{4.2}$$

Proof By differentiating (3.1) with respect to r, and with the help of (4.1), we obtain

$$\frac{\partial}{\partial r}B_r^{(c,d)}(x;q) = \int_{E_{n-1}}\prod_{i=1}^{n}t_i^{x_i-1}\frac{c}{d}{}_1F_1\left(c+1;d+1;-\frac{q}{\pi(t)}-r\frac{\pi(t)}{q}\right)\left(-\frac{\pi(t)}{q}\right)dt$$

or, equivalently,

$$\frac{\partial}{\partial r}B_r^{(c,d)}(x;q) = -\frac{c}{dq}B_r^{(c+1,d+1)}(x+e;q). \tag{4.3}$$

Hence, the desired result for $k=1$. We then deduce (4.2) by a simple mathematical induction. $\square$

Remark 4.2 *(i) Assume that $c, d-c, q>0$, $r\geq 0$, and $x\in(0,\infty)^n$. Then (4.2), with $k=1$ and $k=2$, implies that the real-map $r\longmapsto B_r^{(c,d)}(x;q)$ is strictly decreasing and strictly convex. This can be also deduced from Remark 1.1, since $r\longmapsto -\frac{q}{\pi(t)}-r\frac{\pi(t)}{q}$ is a strictly decreasing linear affine function.*
(ii) From (4.2), we immediately get again (3.7).

Proposition 4.3 *The generalized beta function $B_r^{(c,d)}(x;q)$ satisfies the following:*

$$q^2d\frac{\partial}{\partial q}B_r^{(c,d)}(x;q) + q^2cB_r^{(c+1,d+1)}(x-e;q) = rcB_r^{(c+1,d+1)}(x+e;q). \tag{4.4}$$

In particular, if $r=0$ we get

$$d\frac{\partial}{\partial q}B^{(c,d)}(x;q) + cB^{(c+1,d+1)}(x-e;q) = 0. \tag{4.5}$$

Proof By differentiating (3.1) with respect to q, again with the help of (4.1), we get

$$\frac{\partial}{\partial q}B_r^{(c,d)}(x;q) = \frac{c}{d}\int_{E_{n-1}}\prod_{i=1}^{n}t_i^{x_i-1}{}_1F_1\left(c+1;d+1;-\frac{q}{\pi(t)}-r\frac{\pi(t)}{q}\right)\left(-\frac{1}{\pi(t)}+r\frac{\pi(t)}{q^2}\right)dt$$

or, equivalently,

$$\frac{\partial}{\partial q}B_r^{(c,d)}(x;q) = -\frac{c}{d}B_r^{(c+1,d+1)}(x-e;q) + \frac{rc}{q^2 d}B_r^{(c+1,d+1)}(x+e;q),$$

whence the desired result, after simple algebraic operations. □

Remark 4.4 *(i) By a mathematical induction, we can generalize (4.5) as follows: for $k \geq 1$ one has*

$$\frac{\partial^k}{\partial q^k}B^{(c,d)}(x;q) = (-1)^k \frac{(c)_k}{(d)_k}B^{(c+k,d+k)}(x-ke;q).$$

(ii) From this latter equality we can deduce again (3.12).

We have the following result as well.

Proposition 4.5 *The generalized beta function $B_r^{(c,d)}(x;q)$ satisfies:*

$$q^3\frac{\partial^2}{\partial r\partial q}B_r^{(c,d)}(x;q) + r\frac{(c)_2}{(d)_2}B_r^{(c+2,d+2)}(x+2e;q) = q\frac{c}{d}B_r^{(c+1,d+1)}(x+e;q) + q^2\frac{(c)_2}{(d)_2}B_r^{(c+2,d+2)}(x;q). \tag{4.6}$$

Proof Differentiating (4.4) with respect to r we get

$$q^2 d\frac{\partial^2}{\partial r\partial q}B_r^{(c,d)}(x;q) + q^2 c\frac{\partial}{\partial r}B_r^{(c+1,d+1)}(x-e;q) = cB_r^{(c+1,d+1)}(x+e;q) + rc\frac{\partial}{\partial r}B_r^{(c+1,d+1)}(x+e;q). \tag{4.7}$$

In another part, (4.3) gives

$$\frac{\partial}{\partial r}B_r^{(c+1,d+1)}(x-e;q) = -\frac{c+1}{q(d+1)}B_r^{(c+2,d+2)}(x;q),$$

$$\frac{\partial}{\partial r}B_r^{(c+1,d+1)}(x+e;q) = -\frac{c+1}{q(d+1)}B_r^{(c+2,d+2)}(x+2e;q).$$

Substituting these in (4.8), we get

$$q^2 d\frac{\partial^2}{\partial r\partial q}B_r^{(c,d)}(x;q) - q\frac{c(c+1)}{d+1}B_r^{(c+2,d+2)}(x;q) = cB_r^{(c+1,d+1)}(x+e;q) - \frac{rc(c+1)}{q(d+1)}B_r^{(c+2,d+2)}(x+2e;q).$$

Hence, (4.6) after simple algebraic operations. The proof is finished. □

In order to give more results we need the next lemma.

Lemma 4.6 *Let $u =: {}_1F_1\big(c;d;-\frac{q}{s}\big)$, with $s \neq 0$ is fixed. Then we have:*

(i) u is a solution of

$$qs\frac{d^2u}{dq^2} + (ds+q)\frac{du}{dq} + cu = 0. \tag{4.8}$$

(ii) $w := {}_1F_1\big(c;d;-\frac{q}{\pi(t)}\big)$ is a solution of

$$q\pi(t)\frac{d^2w}{dq^2} + (d\pi(t)+q)\frac{dw}{dq} + cw = 0.$$

Proof (i) If we set $v =: {}_1F_1\big(c; d; z\big)$, it is well-known that [15] v is a solution of the differential equation

$$z\frac{d^2v}{dz^2} + (d-z)\frac{dv}{dz} - cv = 0.$$

Now, setting $z = -q/s$ and using the classical chain rule we get the desired result after simple algebraic operations.
(ii) Setting $s = \pi(t)$, the desired result follows from (4.8). □

Now, we can state the following result.

Theorem 4.7 *The generalized beta function* $B^{(c,d)}(x;q)$ *satisfies*

$$q\frac{d^2}{dq^2}B^{(c,d)}(x+3e;q) - d\frac{d}{dq}B^{(c,d)}(x+2e;q) - q\frac{d}{dq}B^{(c,d)}(x+e;q) + cB^{(c,d)}(x;q) = 0. \tag{4.9}$$

Proof Let us set $w =: {}_1F_1\Big(c; d; -\dfrac{q}{\pi(t)}\Big)$, with $\pi(t) =: t_1...t_n$. Differentiating (3.2), once and twice, with respect to q we get

$$\frac{d}{dq}B^{(c,d)}(x;q) = -\int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i-2}\,\frac{dw}{dq}dt, \tag{4.10}$$

$$\frac{d^2}{dq^2}B^{(c,d)}(x;q) = \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i-3}\,\frac{d^2w}{dq^2}dt. \tag{4.11}$$

According to (4.11), we can write

$$\frac{d^2}{dq^2}B^{(c,d)}(x+3e;q) = \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i-1}\pi(t)\frac{d^2w}{dq^2}dt. \tag{4.12}$$

Now, according to (4.10), we can write

$$\frac{d}{dq}B^{(c,d)}(x+2e;q) = -\int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i-1}\pi(t)\frac{dw}{dq}dt, \tag{4.13}$$

and

$$\frac{d}{dq}B^{(c,d)}(x+e;q) = -\int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i-1}\frac{dw}{dq}dt. \tag{4.14}$$

We can also write

$$B^{(c,d)}(x;q) = \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i-1}\;w\;dt. \tag{4.15}$$

Now, multiplying (4.12), (4.13), (4.14), and (4.15) by $q, -d, -q$, and c, respectively, and summing we get

$$\mathcal{M} = \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i-1}\left\{q\pi(t)\frac{d^2w}{dq^2} + \big(d\pi(t)+q\big)\frac{dw}{dq} + cw\right\}dt,$$

where $\mathcal{M}$ denotes the left hand-side of (4.9). Thanks to Lemma 4.6 (ii), the desired result (4.9) follows, so the proof is completed. □

We also need the following lemma.

Lemma 4.8 *Let* $u =: {}_1F_1\Big(c; d; -\dfrac{q}{s} - r\dfrac{s}{q}\Big)$. *Then we have:*

$$q(q^2 + rs^2)\frac{\partial^2 u}{\partial r^2} + s(q^2 + dqs + rs^2)\frac{\partial u}{\partial r} + cs^3 u = 0.$$

Proof As already pointed before, $v =: {}_1F_1\big(c; d; z\big)$ is a solution of the differential equation

$$z\frac{d^2 v}{dz^2} + (d - z)\frac{dv}{dz} - cv = 0.$$

Now, setting $z = -q/s - rs/q$ and differentiating with respect to r by using the classical chain rule we deduce the desired result after simple algebraic operations. The details are simple and therefore omitted here. □

Now, we are in the position to state the following result.

Theorem 4.9 *The generalized beta function of the first kind satisfies*

$$\begin{aligned} q^5\frac{\partial^2}{\partial r^2}B_r^{(c,d)}(x-2e;q) + q^3 r\frac{\partial^2}{\partial r^2}B_r^{(c,d)}(x;q) - q^3\frac{\partial}{\partial r}B_r^{(c,d)}(x;q) \\ - dq^2\frac{\partial}{\partial r}B_r^{(c,d)}(x+e;q) - rq\frac{\partial}{\partial r}B_r^{(c,d)}(x+2e;q) + cB_r^{(c,d)}(x+3e;q) = 0. \end{aligned} \tag{4.16}$$

Proof For the sake of simplicity, we set

$$B_r^{(c,d)}(x;q) = \int_{E_{n-1}}\prod_{i=1}^{n} t_i^{x_i-1}\theta\, dt, \text{ with } \theta =: {}_1F_1\Big(c; d; -\frac{q}{\pi(t)} - r\frac{\pi(t)}{q}\Big).$$

Differentiating $B_r^{(c,d)}(x;q)$ with respect to r, once and twice, we get

$$\frac{\partial}{\partial r}B_r^{(c,d)}(x;q) = -\frac{1}{q}\int_{E_{n-1}}\prod_{i=1}^{n} t_i^{x_i}\frac{\partial\theta}{\partial r}dt, \tag{4.17}$$

$$\frac{\partial^2}{\partial r^2}B_r^{(c,d)}(x;q) = \frac{1}{q^2}\int_{E_{n-1}}\prod_{i=1}^{n} t_i^{x_i+1}\frac{\partial^2\theta}{\partial r^2}dt. \tag{4.18}$$

From (4.18), we can write

$$\frac{\partial^2}{\partial r^2}B_r^{(c,d)}(x-2e;q) = \frac{1}{q^2}\int_{E_{n-1}}\prod_{i=1}^{n} t_i^{x_i-1}\frac{\partial^2\theta}{\partial r^2}dt \tag{4.19}$$

and

$$\frac{\partial^2}{\partial r^2}B_r^{(c,d)}(x;q) = \frac{1}{q^2}\int_{E_{n-1}}\prod_{i=1}^{n} t_i^{x_i-1}\big(\pi(t)\big)^2\frac{\partial^2\theta}{\partial r^2}dt. \tag{4.20}$$

In another part, by (4.17) we can write

$$\frac{\partial}{\partial r}B_r^{(c,d)}(x;q) = -\frac{1}{q}\int_{E_{n-1}}\prod_{i=1}^{n} t_i^{x_i-1}\Big(\pi(t)\Big)\frac{\partial\theta}{\partial r}dt, \tag{4.21}$$

$$\frac{\partial}{\partial r}B_r^{(c,d)}(x+e;q) = -\frac{1}{q}\int_{E_{n-1}}\prod_{i=1}^{n} t_i^{x_i-1}\Big(\pi(t)\Big)^2\frac{\partial\theta}{\partial r}dt \tag{4.22}$$

and

$$\frac{\partial}{\partial r}B_r^{(c,d)}(x+2e;q) = -\frac{1}{q}\int_{E_{n-1}}\prod_{i=1}^{n} t_i^{x_i-1}\Big(\pi(t)\Big)^3\frac{\partial\theta}{\partial r}dt. \tag{4.23}$$

We can also write

$$B_r^{(c,d)}(x+3e;q) = \int_{E_{n-1}}\prod_{i=1}^{n} t_i^{x_i-1}\Big(\pi(t)\Big)^3\theta\ dt. \tag{4.24}$$

Now, multiplying (4.19), (4.20), (4.21), (4.22), (4.23), and (4.24), respectively, by $q^5, q^3r, -q^3, -q^2d, -qr$, and c, and summing, we get

$$\mathcal{L} = \int_{E_{n-1}}\prod_{i=1}^{n} t_i^{x_i-1}\left(q(q^2+rs^2)\frac{\partial^2\theta}{\partial r^2} + s(q^2+dqs+rs^2)\frac{\partial\theta}{\partial r} + cs^3\theta\right)dt,$$

where $s =: \pi(t)$ and $\mathcal{L}$ denotes the left side of (4.16). According to Lemma 4.8, we then obtain (4.16), so the proof is completed. □

Finally, we will state a result about the partial derivatives of $B_r^{(c,d)}(x;q)$ as a function in the multivariate $x =: (x_1, ..., x_n)$. As previous, we set

$$\theta(t) =: {}_1F_1\left(c; d; -\frac{q}{\pi(t)} - r\frac{\pi(t)}{q}\right).$$

We have the following result.

Proposition 4.10 *Let $c, d, q, r \in \mathbb{C}$ and $x \in (0,\infty)^n$. For any $j, k = 1, ..., n$, we have the following equalities*

$$\frac{\partial^k}{\partial x_j^k}B_r^{(c,d)}(x;q) = \int_{E_{n-1}}\prod_{i=1}^{n} t_i^{x_i-1}\big(\log\ t_j\big)^k\ \theta(t)\ dt.$$

$$\frac{\partial^n}{\partial x_1 ... \partial x_n}B_r^{(c,d)}(x;q) = \int_{E_{n-1}}\prod_{i=1}^{n}\left(t_i^{x_i-1}\log\ t_i\right)\theta(t)\ dt.$$

Proof It is straightforward and the details are therefore omitted here. □

5. Generalized beta function of the second kind

This section deals with another generalized beta function in n variables. In what follows, we need to lighten the writing by putting

$$x =: (x_1, ..., x_n), a =: (a_1, ..., a_n), \alpha =: (\alpha_1, ..., \alpha_n), \beta =: (\beta_1, ..., \beta_n), p =: (p_1, ..., p_n).$$

Another central definition is given in what follows.

Definition 5.1 *Let $x \in (\mathbb{C}_{+*})^n$, $\alpha, \beta \in \mathbb{C}^n$ and $a, p \in \big(\mathbb{C}_{+*}\big)^n$. The generalized beta function, of the second kind, is defined by:*

$$B_p^{(\alpha,\ \beta)}\big(x; a\big) = \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i - 1} {}_1F_1\left(\alpha_i; \beta_i; -\frac{a_i}{t_i} - p_i \frac{t_i}{a_i}\right) dt, \tag{5.1}$$

where we set $dt =: dt_1 ... dt_{n-1}$ and $t_n =: 1 - \sum_{i=1}^{n-1} t_i$.

The basic properties of $B_p^{(\alpha,\beta)}(x; a)$, analogous to those of Proposition 3.2, are embodied in the following result.

Proposition 5.2 *Let x, α, β, a, p be as in the previous definition.*
(i) We have the following relationship

$$B_p^{(\alpha,\beta)}(x; a) = B_{p^*}^{(\alpha^*,\beta^*)}(x^*; a^*),$$

where τ is any permutation of the set $\{1, 2, ..., n\}$ and $p^* =: (p_{\tau(1)}, ..., p_{\tau(n)})$, with similar settings for $\alpha^*, \beta^*, x^*, a^*$.
(ii) The following relation holds:

$$\sum_{i=1}^{n} B_p^{(\alpha,\beta)}(x + e_i; a) = B_p^{(\alpha,\beta)}(x; a),$$

where $(e_1, ..., e_n)$ refers to the canonical basis of $\mathbb{R}^n$.

Proof Similar to that of Proposition 3.2. The details are omitted here as an interesting exercise for the reader. □

Following the uniform convergence of the series (1.7), we can interchange in (5.1) the sum series and the integral. This shows that (5.1) is well-defined, i.e. the involved integral is convergent. Furthermore, such integral is uniformly convergent in any compact set included in the interior of E_{n-1}. This implies that we can take limits and differentiation under the integral sign of (5.1). In particular, we have

$$\lim_{p \to 0} B_p^{(\alpha,\ \beta)}\big(x; a\big) = B^{(\alpha,\ \beta)}\big(x; a\big), \tag{5.2}$$

where

$$B^{(\alpha,\ \beta)}\big(x; a\big) =: \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i - 1} {}_1F_1\Big(\alpha_i; \beta_i; -\frac{a_i}{t_i}\Big) dt \tag{5.3}$$

It is easy to see that if $n = 2$ and $\alpha = \beta$ then (5.3) yields (1.4). Otherwise, the following result justifies that $B_p^{(\alpha;\ \beta)}\big(x; a\big)$ is a generalization of the extended beta function $B(x; a)$ in n variables defined by (1.15).

Proposition 5.3 *For any* $\alpha \in \mathbb{C}^n$ *and* $x, a \in \left(\mathbb{C}_{+*}\right)^n$ *we have*

$$B(x;a) = \lim_{p\to 0} B_p^{(\alpha;\,\alpha)}(x;a) =: B^{(\alpha,\,\alpha)}\big(x;a\big).$$

Proof By (1.7), with the help of (5.2) and (5.3), we have for any $i = 1, 2, ..., n$,

$$ {}_1F_1\Big(\alpha_i;\alpha_i;-\frac{a_i}{t_i}\Big) = \sum_{m=0}^{\infty} \frac{(-a_i/t_i)^m}{m!} = e^{-a_i/t_i}.$$

This, with (1.15) and (5.1), yields the desired result. □

We have the following result as well.

Theorem 5.4 *Let* x, a, p, α, β *be as above. Assume that* $|p_i| < |a_i|^2$ *for any* $i = 1, ..., n$. *Then the following relation*

$$\frac{B_p^{(\alpha,\,\beta)}\big(x;a\big)}{B(x)} = \sum_{\mathbf{m},\mathbf{k}=0}^{\infty} \frac{(\alpha_i)_{m_i}}{(\beta_i)_{m_i}} \frac{(-1)^{m_i}}{m_i!} \binom{m_i}{k_i} p_i^{k_i}(a_i)^{m_i-2k_i} \frac{\big(1-\sigma(x)-2\sigma(k)\big)_{\sigma(\mathbf{m})}}{\prod_{i=1}^{n}(1-x_i-k_i)_{m_i}}, \tag{5.4}$$

where under the sigma summation we set $\mathbf{m} =: (m_1, ..., m_n)$ *and* $\mathbf{k} =: (k_1, ..., k_n)$ *for the sake of simplicity. In particular, if* $p = 0$ *one has*

$$\frac{B^{(\alpha,\,\beta)}\big(x;a\big)}{B(x)} = \sum_{\mathbf{m}=0}^{\infty} \frac{(\alpha_i)_{m_i}}{(\beta_i)_{m_i}} \frac{(-a_i)^{m_i}}{m_i!} \frac{\big(1-\sigma(x)\big)_{\sigma(m)}}{\prod_{i=1}^{n}(1-x_i)_{m_i}} \tag{5.5}$$

holds true for any $x \in \left(\mathbb{C}_{+*}\right)^n$ *such that* $x_1, ..., x_n$ *and* $\sigma(x)$ *are nonintegers.*

Proof By (1.7) we can write, for any $i = 1, ..., n$,

$$ {}_1F_1\left(\alpha_i;\beta_i;-\frac{a_i}{t_i}-p_i\frac{t_i}{a_i}\right) = \sum_{m_i=0}^{\infty} \frac{(\alpha_i)_{m_i}}{(\beta_i)_{m_i}} \frac{(-1)^{m_i}}{m_i!} \Big(\frac{a_i}{t_i}\Big)^{m_i} \Big(1+p_i\frac{t_i^2}{a_i^2}\Big)^{m_i}. \tag{5.6}$$

Since $0 \le t_i \le 1$ for any $i = 1, ..., n$, the condition $|p_i| < |a_i|^2$ allows us to write the expansion series

$$\Big(1+p_i\frac{t_i^2}{a_i^2}\Big)^{m_i} = \sum_{k_i=0}^{\infty} \binom{m_i}{k_i} p_i^{k_i} \Big(\frac{t_i}{a_i}\Big)^{2k_i}.$$

Substituting these in (5.6) we get, for any $i = 1, ..., n$,

$$ {}_1F_1\left(\alpha_i;\beta_i;-\frac{a_i}{t_i}-p_i\frac{t_i}{a_i}\right) = \sum_{m_i=0,k_i=0}^{\infty} \frac{(\alpha_i)_{m_i}}{(\beta_i)_{m_i}} \frac{(-1)^{m_i}}{m_i!} \binom{m_i}{k_i} p_i^{k_i} \Big(\frac{t_i}{a_i}\Big)^{2k_i-m_i}.$$

Multiplying this latter equality by $\prod_{i=1}^{n} t_i^{x_i-1}$ and then substituting in (5.1), we get

$$B_p^{(\alpha,\,\beta)}\big(x;a\big) = \int_{E_{n-1}} \sum_{\mathbf{m},\mathbf{k}=0}^{\infty} \frac{(\alpha_i)_{m_i}}{(\beta_i)_{m_i}} \frac{(-1)^{m_i}}{m_i!} \binom{m_i}{k_i} p_i^{k_i}(a_i)^{m_i-2k_i} t_i^{x_i+2k_i-m_i-1}dt,$$

where under the sigma summation we set $\mathbf{m} =: (m_1, ..., m_n)$ and $\mathbf{k} =: (k_1, ..., k_n)$. Due to similar arguments as previous, we can interchange the orders of the integral and the sum series for obtaining

$$B_p^{(\alpha,\,\beta)}\big(x;a\big) = \sum_{\mathbf{m},\mathbf{k}=0}^{\infty} \frac{(\alpha_i)_{m_i}}{(\beta_i)_{m_i}} \frac{(-1)^{m_i}}{m_i!} \binom{m_i}{k_i} p_i^{k_i} (a_i)^{m_i-2k_i} \int_{E_{n-1}} t_i^{x_i+2k_i-m_i-1} dt$$

$$= \sum_{\mathbf{m},\mathbf{k}=0}^{\infty} \frac{(\alpha_i)_{m_i}}{(\beta_i)_{m_i}} \frac{(-1)^{m_i}}{m_i!} \binom{m_i}{k_i} p_i^{k_i} (a_i)^{m_i-2k_i} B(x_i + 2k_i - m_i).$$

By similar way as in the proof of Theorem 3.7, we have

$$B\big(x_i + 2k_i - m_i\big) = \frac{\big(1 - \sigma(x) - 2\sigma(k)\big)_{\sigma(\mathbf{m})}}{\prod_{i=1}^n (1 - x_i - k_i)_{m_i}} B(x),$$

and (5.4) is obtained. Taking $p = 0$ in (5.4) we get (5.5). The proof is finished. □

The following result is also of interest.

Theorem 5.5 *Let $x, a, p, \alpha, \beta - \alpha \in \big(\mathbb{C}_{+*}\big)^n$. Then we have*

$$B_p^{(\alpha,\,\beta)}\big(x;a\big) = \sum_{\mathbf{m}=0}^{\infty} \frac{(\alpha_i)_{m_i}}{(\beta_i)_{m_i}} \frac{\big(-p_i/a_i\big)^{m_i}}{m_i!} B^{(\alpha+\mathbf{m},\beta+\mathbf{m})}\big(x + \mathbf{m}; a\big), \tag{5.7}$$

where $B^{(\alpha+\mathbf{m},\beta+\mathbf{m})}\big(x + \mathbf{m}; a\big)$ is defined through (5.3) and $\mathbf{m} =: (m_1, ..., m_n)$.

Proof It is similar to the proof of Theorem 3.4 by the same way and using analogous arguments. We omit the details to the reader. □

The following result gives a relationship between $B_p^{(\alpha,\beta)}(x;a)$, $\Gamma_p^{(\alpha,\beta)}(u)$ and the standard beta function in n variables.

Theorem 5.6 *Let $x \in \big(\mathbb{C}_{+*}\big)^n$ and $u = (u_1, u_2, ..., u_n) \in \big(\mathbb{C}_{+*}\big)^n$. Then we have*

$$\int_{(0,\infty)^n} \prod_{i=1}^n a_i^{u_i-1} B_p^{(\alpha;\beta)}\big(x;a\big) da = \Gamma_p^{(\alpha;\beta)}(u) B(x+u), \tag{5.8}$$

where we set $da =: da_1 ... da_n$, $B(x+u) =: B(x_1 + u_1, ..., x_n + u_n)$ and $\Gamma_p^{(\alpha;\beta)}(u)$ is the generalized gamma function defined by (2.2).

Proof If we multiply (5.1) by $\prod_{i=1}^n a_i^{u_i-1}$ and we integrate over $(0,\infty)^n$ with respect to $a =: (a_1, a_2, ..., a_n)$, we obtain

$$\int_{(0,\infty)^n} \prod_{i=1}^n a_i^{u_i-1} B_p^{(\alpha;\beta)}\big(x;a\big) da = \int_{(0,\infty)^n} \left\{ \prod_{i=1}^n a_i^{u_i-1} \int_{E_{n-1}} \prod_{i=1}^n t_i^{x_i-1} {}_1F_1\left(\alpha_i; \beta_i; -\frac{a_i}{t_i} - p_i \frac{t_i}{a_i}\right) dt \right\} da,$$

with $t_n = 1-\sum_{i=1}^{n-1} t_i$. Due to an argument of uniform convergence of the involved integrals, we can interchange the order of the integrals, so

$$\int_{(0,\infty)^n} \prod_{i=1}^n a_i^{u_i-1} B_p^{(\alpha;\beta)}(x;a)da = \int_{E_{n-1}} \left\{ \prod_{i=1}^n t_i^{x_i-1} \prod_{i=1}^n \Big(\int_0^\infty a_i^{u_i-1} {}_1F_1\Big(\alpha_i;\beta_i;-\frac{a_i}{t_i}-p_i\frac{t_i}{a_i}\Big)da_i \right\} dt. \tag{5.9}$$

Making a simple change of variables, we can easily check that

$$\int_0^\infty a_i^{u_i-1} {}_1F_1\Big(\alpha_i;\beta_i;-\frac{a_i}{t_i}-p_i\frac{t_i}{a_i}\Big)da_i = t_i^{u_i}\Gamma_{p_i}^{(\alpha_i;\beta_i)}(u_i), \;\; 1 \le i \le n.$$

This, with (5.9), yields

$$\int_{(0,\infty)^n} \prod_{i=1}^n a_i^{u_i-1} B_p^{(\alpha;\beta)}(x;;a)da = \prod_{i=1}^n \Gamma_{p_i}^{(\alpha_i;\beta_i)}(u_i) \int_{E_{n-1}} \prod_{i=1}^n t_i^{x_i+u_i-1}dt = B(x+u)\prod_{i=1}^n \Gamma_{p_i}^{(\alpha_i;\beta_i)}(u_i);$$

hence, (5.8) is proven. □

Remark 5.7 *Setting $u = e =: (1, 1..., 1)$ in (5.8), we immediately obtain the following relationship.*

$$\int_{(0,\infty)^n} \prod_{i=1}^n B_p^{(\alpha;\beta)}(x;a)da = B(x+e)\prod_{i=1}^n \Gamma_{p_i}^{(\alpha_i;\beta_i)}(1).$$

6. Partial derivatives of $B_p^{(\alpha,\beta)}(x;a)$

As a function of the variable $p = (p_1, ..., p_n)$, $B_p^{(\alpha,\beta)}(x;a)$ satisfies the following result.

Proposition 6.1 *We have:*

$$\Big(\prod_{i=1}^n a_i\Big) \frac{\partial^n}{\partial p_1 ... \partial p_n} B_p^{(\alpha,\beta)}(x;a) = (-1)^n \Big(\prod_{i=1}^n \frac{\alpha_i}{\beta_i}\Big) B_p^{(\alpha+e,\beta+e)}(x+e;a). \tag{6.1}$$

Proof Let $j = 1, 2, ..., n$. Differentiating (5.1) with respect to p_j, we get

$$\frac{\partial}{\partial p_j} B_p^{(\alpha,\beta)}(x;a) = \int_{E_{n-1}} \prod_{i=1}^n t_i^{x_i-1} \prod_{i=1, i\neq j}^n {}_1F_1\Big(\alpha_i;\beta_i;-\frac{a_i}{t_i}-p_i\frac{t_i}{a_i}\Big)$$

$$\times \frac{\alpha_j}{\beta_j}\Big(-\frac{t_j}{a_j}\Big) {}_1F_1\left(\alpha_j+1;\beta_j+1;-\frac{a_j}{t_j}-p_j\frac{t_j}{a_j}\right) dt. \tag{6.2}$$

Now, differentiating (5.1) n-times with respect to $p_n, p_{n-1}, ..., p_1$ and utilizing (6.2), we get

$$\frac{\partial^n}{\partial p_1 ... \partial p_n} B_p^{(\alpha,\beta)}(x;a) = \frac{(-1)^n}{\prod_{i=1}^n a_i} \frac{\alpha_1...\alpha_n}{\beta_1...\beta_n} \int_{E_{n-1}} \prod_{i=1}^n t_i^{x_i} {}_1F_1\left(\alpha_i+1;\beta_i+1;-\frac{a_i}{t_i}-p_i\frac{t_i}{a_i}\right) dt.$$

Hence, (6.1). The proof is complete. □

The previous proposition can be generalized as follows.

Proposition 6.2 *Let $m \geq 1$ be an integer. Then we have*

$$\Big(\prod_{i=1}^{n} a_i\Big)^m \frac{\partial^{mn}}{\partial p_1^m ... \partial p_n^m} B_p^{(\alpha,\beta)}(x;a) = (-1)^{mn} \prod_{i=1}^{n} \frac{(\alpha_i)_m}{(\beta_i)_m} B_p^{(\alpha+me,\beta+me)}(x+me;a).$$

Proof It follows from (6.1) with a simple mathematical induction on $m \geq 1$. The details are straightforward and therefore omitted here. □

Now, let us set

$$\omega(t) =: \prod_{i=1}^{n} {}_1F_1\Big(\alpha_i;\beta_i;-\frac{a_i}{t_i} - p_i\frac{t_i}{a_i}\Big).$$

The function $x \longmapsto B_p^{(\alpha,\beta)}(x;a)$, i.e. with the multivariate $x =: (x_1,...,x_n) \in (0,\infty)^n$, satisfies the following result.

Proposition 6.3 *For any $j,k = 1,...,n$, we have the following*

$$\frac{\partial^k}{\partial x_j^k} B_p^{(\alpha,\beta)}(x;a) = \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i-1}\big(\log\ t_j\big)^k\ \omega(t)\ dt.$$

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} B_p^{(\alpha,\beta)}(x;a) = \int_{E_{n-1}} \prod_{i=1}^{n} \Big(t_i^{x_i-1}\log\ t_i\Big)\ \omega(t)\ dt.$$

Proof It is immediate. The details are simple and therefore omitted here. □

As a function of the multivariate $a =: (a_1,...,a_n)$, $B_p^{(\alpha,\beta)}(x;a)$ satisfies the following result.

Proposition 6.4 *The following relation holds:*

$$\frac{\partial^n}{\partial a_1 ... \partial a_n} B_p^{(\alpha,\beta)}(x;a) = \prod_{i=1}^{n}\Big(\frac{\alpha_i}{a_i^2\beta_i}\Big) \times \int_{E_{n-1}} \prod_{i=1}^{n}\Big\{t_i^{x_i-2}(p_i t_i^2 - a_i^2)\,{}_1F_1\Big(\alpha_i+1;\beta_i+1;-\frac{a_i}{t_i} - p_i\frac{t_i}{a_i}\Big)\Big\}\ dt. \quad (6.3)$$

Proof Let $j = 1,2,...,n$. By differentiating (5.1) with respect to a_j we obtain

$$\frac{\partial}{\partial a_j} B_p^{(\alpha,\beta)}(x;a) = \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i-1} \prod_{i=1,i\neq j}^{n} {}_1F_1\Big(\alpha_i;\beta_i;-\frac{a_i}{t_i} - p_i\frac{t_i}{a_i}\Big)$$
$$\times \frac{\alpha_j}{\beta_j}\Big(-\frac{1}{t_j} + p_j\frac{t_j}{a_j^2}\Big){}_1F_1\ \Big(\alpha_j+1;\beta_j+1;-\frac{a_j}{t_j} - p_j\frac{t_j}{a_j}\Big)\ dt. \quad (6.4)$$

Differentiating (5.1) n-times with respect to $a_n, a_{n-1}, ..., a_1$ and using (6.4), we get (6.3) after simple algebraic operations and manipulations. □

7. Conclusion

Special functions arise in various contexts and contribute as tools for solving many scientific problems. They attract the attention of many researchers by their nice properties and interesting applications. It has been proven in a lot of studies that special function theory is useful in theoretical point of view as well as in practical purposes. Since special functions have extensive applications in pure mathematics, as well as in some applied areas such as acoustics, fluid dynamics, heat conduction, electrical current, solutions of wave equations, and quantum mechanics, an enormous amount of effort has been devoted by many researchers to understanding some extended special functions. In this paper, we investigated a contribution in such direction by introducing some generalized gamma and beta functions with a systematic study of their properties such as recurrence relations, Mellin transform properties and partial differential equations of them. Since the univariate versions of these extended functions have already many applications in the literature, our definitions will have many potential applications in probability theory and special functions theory, and as a result, they will have many potential applications in physical and engineering problems. Furthermore, our extended functions presented here involve the confluent hypergeometric function which constitutes a primordial interest in special functions theory. Thus, a new horizon for scientific research is open and many questions are generated as subjects for future research.

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