Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 797 – 819
© TÜBİTAK
doi:10.3906/mat-2010-89

Estimations on some hybrid exponential sums related to Kloosterman sums

Yingjie CHENG[1,*], **Xiwang CAO**[1,2], **Liqin QIAN**[1], **Jinlong WAN**[1]
[1]Department of Mathematics, School of Science, Nanjing University of Aeronautics and Astronautics, Nanjing, China
[2]Laboratory of Information Security, Institute of Information Engineering, Chinese Academy of Sciences, Beijing, China

Received: 27.10.2020 • **Accepted/Published Online:** 05.02.2021 • **Final Version:** 26.03.2021

Abstract: In this paper, we revisit the bounds of the mixed exponential sums introduced by Lv and Zhang (2020). Moreover, we give some estimations for some new hybrid exponential sums related to Kloosterman sums over finite fields of odd characteristic by using the properties of Jacobi sums and Gaussian sums.

Key words: Exponential sum, Kloosterman sum, Gaussian sum, Jacobi sum

1. Introduction

As usual, let ψ be a canonical additive character of $\mathbb{F}_q$, where $q = p^s$ with p is an odd prime and s is a positive integer. For any integers m and n, the two-term exponential sum $K(m, n, k, h; q)$ is defined as follows:

$$K(m, n, k, h; q) = \sum_{x \in \mathbb{F}_q^*} \psi(mx^k + nx^h),$$

where k and h are integers with $k > h$, and $\mathbb{F}_q^* = \mathbb{F}_q \backslash \{0\}$ which is a cyclic group consisting of $q-1$ elements.

If $k = 1$ and $h = -1$, then $K(m, n, 1, -1; q)$ becomes the famous Kloosterman sum, which is defined by

$$K(\psi; a, b) = \sum_{c \in \mathbb{F}_q^*} \psi(ac + bc^{-1}).$$

It is clear that these exponential sums over finite fields are important topics because their extensive applications in finite fields [2, 12], coding theory [6, 8] and the designing of sequences [5]. Meanwhile, numerous results about the properties of $K(m, n, k, h; q)$ and $K(\psi; a, b)$ have been reported. By way of example, Zhang et al. [15] obtained an equation about $K(m, n, 3, 1; p)$ as follows:

$$\sum_{m \in \mathbb{F}_p^*} \left| \sum_{a \in \mathbb{F}_p} \psi(ma^3 + na) \right|^4 = \begin{cases} 2p^3 - p^2, & \text{if } 3 \nmid (p-1); \\ 2p^3 - 7p^2, & \text{if } 3 \mid (p-1), \end{cases}$$

where p is an odd prime and n is an integer with $(n, p) = 1$.

*Correspondence: 18972773039@163.com

2010 *AMS Mathematics Subject Classification:* 11L40, 11T23

In [7], Liu and Li used the properties of Gaussian sums to prove the following result about two-term exponential sums relate to $K(m,n,3,1;p)$ of $\mathbb{F}_p$

$$\sum_{m\in\mathbb{F}_p^*}\left|\sum_{a\in\mathbb{F}_p^*}\chi(ma^3+a)\right|^2\left|\sum_{b\in\mathbb{F}_p^*}\psi(mb^3+b)\right|^2=2p^3+E(p),$$

where χ denotes a multiplicative character of $\mathbb{F}_p$ and $E(p)$ satisfies the inequality

$$-12p^2-2p\le E(p)\le 4p^2-2p.$$

In this paper, we mainly focus on calculating the values of some hybrid exponential sums related to $K(m,n,k,h;q)$ and Kloosterman sums, which are defined as:

$$\sum_{m\in\mathbb{F}_q^*}\left|\sum_{a,b\in\mathbb{F}_q^*}\chi(a+b+m\bar{a}\bar{b})\right|^2\left|\sum_{c\in\mathbb{F}_q^*}\psi(mc+\bar{c})\right|^2, \tag{1.1}$$

$$\sum_{m\in\mathbb{F}_q^*}\left|\sum_{a,b\in\mathbb{F}_q^*}\chi(a+b+m\bar{a}\bar{b})\right|^2\left|\sum_{c\in\mathbb{F}_q^*}\psi(mc^3+c)\right|^2, \tag{1.2}$$

$$\sum_{m\in\mathbb{F}_q^*}\left|\sum_{a,b,c\in\mathbb{F}_q^*}\chi(a+b+c+m\bar{a}\bar{b}\bar{c})\right|^2\left|\sum_{d\in\mathbb{F}_q^*}\psi(md+\bar{d})\right|^2, \tag{1.3}$$

$$\sum_{m\in\mathbb{F}_q^*}\left|\sum_{a,b,c\in\mathbb{F}_q^*}\chi(a+b+c+m\bar{a}\bar{b}\bar{c})\right|^2\left|\sum_{d\in\mathbb{F}_q^*}\psi(md^4+d)\right|^2. \tag{1.4}$$

About these kinds of hybrid exponential sums, Di [3] and Lv and Zhang [11] also obtained some related contents, the first one reads:

$$\sum_{m\in\mathbb{F}_p^*}\left|\sum_{a\in\mathbb{F}_p^*}\chi(ma+a^{-1})\right|^2\left|\sum_{b\in\mathbb{F}_p^*}\psi(mb^k+nb)\right|^2=\begin{cases}2p^3+O(|k|p^2), & \text{if } 2\mid k;\\ 2p^3+O(|k|p^{\frac{5}{2}}), & \text{if } 2\nmid k,\end{cases}$$

where χ is a multiplicative character of $\mathbb{F}_p$ and ψ a canonical additive character of $\mathbb{F}_p$. Lv and Zhang [11] introduced the bounds of Summations (1.1) and (1.2) over prime fields. Summations (1.1) and (1.2) have been studied over prime fields, we use the properties of Jacobi sums and Gaussian sums to obtain the bounds of summations (1.1) and (1.2) over general finite fields of odd characteristic. Then by using a similar method, we can also obtain some estimations of summations (1.3) and (1.4), respectively. The main results of this paper are stated as the following theorems. We denote a multiplicative character χ of $\mathbb{F}_q$ as a power-d character, if there exists a multiplicative character χ_1 of $\mathbb{F}_q$ such that $\chi=\chi_1^d$ in the sequel.

Theorem 1.1 *Assume that ψ is a canonical additive character of $\mathbb{F}_q$ with $3 \mid (q-1)$. Then*

$$\sum_{a,b\in\mathbb{F}_q^*} \chi(a+b+m\bar{a}\bar{b})$$

does not vanish if χ is a power-three character, in which case we have

$$\sum_{m\in\mathbb{F}_q^*} \left|\sum_{a,b\in\mathbb{F}_q^*} \chi(a+b+m\bar{a}\bar{b})\right|^2 \left|\sum_{c\in\mathbb{F}_q^*} \psi(mc+\bar{c})\right|^2 = 3q^2(q^2-q-1)+T_1,$$

and

$$\sum_{m\in\mathbb{F}_q^*} \left|\sum_{a,b\in\mathbb{F}_q^*} \chi(a+b+m\bar{a}\bar{b})\right|^2 \left|\sum_{c\in\mathbb{F}_q^*} \psi(mc^3+c)\right|^2 = 3q^2(q^2-3q-1)+T_2,$$

where $|T_1| \le 6q^{\frac{7}{2}}$ and $|T_2| \le 6q^3+12q^{\frac{5}{2}}$.

Theorem 1.2 *Assume that ψ is a canonical additive character of $\mathbb{F}_q$ with $3 \nmid (q-1)$. Then*

$$\sum_{m\in\mathbb{F}_q^*} \left|\sum_{a,b\in\mathbb{F}_q^*} \chi(a+b+m\bar{a}\bar{b})\right|^2 \left|\sum_{c\in\mathbb{F}_q^*} \psi(mc+\bar{c})\right|^2 = q^2(q^2-q-1),$$

and

$$\sum_{m\in\mathbb{F}_q^*} \left|\sum_{a,b\in\mathbb{F}_q^*} \chi(a+b+m\bar{a}\bar{b})\right|^2 \left|\sum_{c\in\mathbb{F}_q^*} \psi(mc^3+c)\right|^2 = q^2(q^2-q-1).$$

Theorem 1.3 *Assume that ψ is a canonical additive character of $\mathbb{F}_q$ with $4 \mid (q-1)$. Then*

(i) if χ is a power-four multiplicative character of $\mathbb{F}_q$, we have

$$\sum_{m\in\mathbb{F}_q^*} \left|\sum_{a,b,c\in\mathbb{F}_q^*} \chi(a+b+c+m\bar{a}\bar{b}\bar{c})\right|^2 \left|\sum_{d\in\mathbb{F}_q^*} \psi(md+\bar{d})\right|^2 = 4q^3(q^2-q-1)+T_3,$$

where $|T_3| \le 4\sqrt{6}q^{\frac{7}{2}}$. Moreover, if $8 \mid (q-1)$, it follows that

$$\sum_{m\in\mathbb{F}_q^*} \left|\sum_{a,b,c\in\mathbb{F}_q^*} \chi(a+b+c+m\bar{a}\bar{b}\bar{c})\right|^2 \left|\sum_{d\in\mathbb{F}_q^*} \psi(md^4+d)\right|^2 = 4q^3(q^2-4q-1)+T_4,$$

and if $8 \nmid (q-1)$, it follows that

$$\sum_{m\in\mathbb{F}_q^*} \left|\sum_{a,b,c\in\mathbb{F}_q^*} \chi(a+b+c+m\bar{a}\bar{b}\bar{c})\right|^2 \left|\sum_{d\in\mathbb{F}_q^*} \psi(md^4+d)\right|^2 = 4q^3(q^2-4q-1)+T_5,$$

where $|T_4| \le (8\sqrt{3}+8\sqrt{6})(q^4+q^{\frac{7}{2}})$ *and* $|T_5| \le 8\sqrt{3}(q^4+q^{\frac{7}{2}})$*;*

(ii) if χ *is not a power-four multiplicative character, then*

$$\sum_{m\in\mathbb{F}_q^*}\left|\sum_{a,b,c\in\mathbb{F}_q^*}\chi(a+b+c+m\bar{a}\bar{b}\bar{c})\right|^2\left|\sum_{d\in\mathbb{F}_q^*}\psi(md^4+d)\right|^2=0,$$

and

$$\sum_{m\in\mathbb{F}_q^*}\left|\sum_{a,b,c\in\mathbb{F}_q^*}\chi(a+b+c+m\bar{a}\bar{b}\bar{c})\right|^2\left|\sum_{d\in\mathbb{F}_q^*}\psi(md^4+d)\right|^2=0.$$

Theorem 1.4 *Assume that* ψ *is a canonical additive character of* $\mathbb{F}_q$ *with* $4\nmid(q-1)$*. Then*

(i) if χ *is a power-two multiplicative character of* $\mathbb{F}_q$*, we get*

$$\sum_{m\in\mathbb{F}_q^*}\left|\sum_{a,b,c\in\mathbb{F}_q^*}\chi(a+b+c+m\bar{a}\bar{b}\bar{c})\right|^2\left|\sum_{d\in\mathbb{F}_q^*}\psi(md+\bar{d})\right|^2=2q^3(q^2-q-1),$$

and

$$\sum_{m\in\mathbb{F}_q^*}\left|\sum_{a,b,c\in\mathbb{F}_q^*}\chi(a+b+c+m\bar{a}\bar{b}\bar{c})\right|^2\left|\sum_{d\in\mathbb{F}_q^*}\psi(md^4+d)\right|^2=2q^3(q^2-2q-1);$$

(ii) if χ *is not a power-two multiplicative character, then*

$$\sum_{m\in\mathbb{F}_q^*}\left|\sum_{a,b,c\in\mathbb{F}_q^*}\chi(a+b+c+m\bar{a}\bar{b}\bar{c})\right|^2\left|\sum_{d\in\mathbb{F}_q^*}\psi(md+\bar{d})\right|^2=0,$$

and

$$\sum_{m\in\mathbb{F}_q^*}\left|\sum_{a,b,c\in\mathbb{F}_q^*}\chi(a+b+c+m\bar{a}\bar{b}\bar{c})\right|^2\left|\sum_{d\in\mathbb{F}_q^*}\psi(md^4+d)\right|^2=0.$$

2. Preliminaries

Suppose that $q=p^n$ for an odd prime p and a positive integer n. Let $\mathbb{F}_q$ be the finite field with q elements. Then the group of units of $\mathbb{F}_q$, denote by $\mathbb{F}_q^*$, is a cyclic group consisting of $q-1$ elements. For a fixed primitive element α of $\mathbb{F}_q$ and every integer i with $0\le i\le q-2$, the function $\chi_i(\alpha^j)=\zeta_{q-1}^{ij}$ give a multiplicative character of $\mathbb{F}_q$, where $0\le j\le q-2$ and ζ_p is a complex primitive p-th root of unity. For convenience later, χ_1 is written as χ. And we set $\chi(0)=0$. Meanwhile, the additive character ψ is a homomorphism from the finite field $\mathbb{F}_q$ to the multiplicative group $\mathbb{C}^*$ of the complex field $\mathbb{C}$, which can be defined as:

$$\psi_m(n)=\zeta_p^{\mathrm{Tr}(mn)},\ \textit{for all}\ n\in\mathbb{F}_q,$$

where Tr is a trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$ and ζ_p is a complex primitive p-th root of unity. For convenience, ψ_1 is written as ψ, which is called the canonical additive character of $\mathbb{F}_q$. And the Gaussian sum $G(\chi,\psi)$ of $\mathbb{F}_q$ is defined by

$$G(\chi,\psi)=\sum_{x\in\mathbb{F}_q^*}\chi(x)\psi_m(x).$$

In this paper, we only focus on $G(\chi,\psi_1)$, which is denoted by $G(\chi)$ for convenience. A crucial role in the present paper is played by the notion of the Jacobi sum, which is understood according to the following definition.

Definition 2.1 [8] *Let $\lambda_1,\cdots,\lambda_k$ be nontrivial multiplicative characters of $\mathbb{F}_q$. The sum*

$$J(\lambda_1,\cdots,\lambda_k)=\sum_{x_1+\cdots+x_k=1}\lambda_1(x_1)\cdots\lambda_k(x_k),$$

where $x_1,\cdots,x_k$ are elements of $\mathbb{F}_q$, which satisfying $x_1+\cdots+x_k=1$, is called a Jacobi sum of $\mathbb{F}_q$.

Then we provide some lemmas about Jacobi sums, which will be used to prove our main results in the sequel.

Lemma 2.2 [8] *Let $\lambda_1,\cdots,\lambda_k$ be nontrivial multiplicative characters of $\mathbb{F}_q$. Then*

$$J(\lambda_1,\cdots,\lambda_k)=\frac{G(\lambda_1)\cdots G(\lambda_k)}{G(\lambda_1\cdots\lambda_k)}$$

if $\lambda_1\cdots\lambda_k$ is a nontrivial multiplicative character, and otherwise,

$$J(\lambda_1,\cdots,\lambda_k)=-\frac{1}{q}G(\lambda_1)\cdots G(\lambda_k).$$

Lemma 2.3 [8] *Let $\lambda_1,\cdots,\lambda_k$ be nontrivial multiplicative characters of $\mathbb{F}_q$. Then*

$$|J(\lambda_1,\cdots,\lambda_k)|=q^{\frac{k-1}{2}},$$

if $\lambda_1\cdots\lambda_k$ is a nontrivial multiplicative character and

$$|J(\lambda_1,\cdots,\lambda_k)|=q^{\frac{k-2}{2}},$$

if $\lambda_1\cdots\lambda_k$ is a trivial multiplicative character.

3. The proofs of main results

In this section, we will prove the main results of this paper, namely, Theorems 1.1–1.4. For convenience, we denote

$$A(\chi,m,q)=\sum_{a,b\in\mathbb{F}_q^*}\chi(a+b+m\bar{a}\bar{b}),$$

$$B(\chi,m,q)=\sum_{a,b,c\in\mathbb{F}_q^*}\chi(a+b+c+m\bar{a}\bar{b}\bar{c}),$$

$$C(\psi, m, q) = \sum_{a\in\mathbb{F}_q^*} \psi(ma + \overline{a}),$$

$$D(\psi, m, q) = \sum_{a\in\mathbb{F}_q^*} \psi(ma^3 + a),$$

and

$$E(\psi, m, q) = \sum_{a\in\mathbb{F}_q^*} \psi(ma^4 + a),$$

where ψ is a canonical additive character of $\mathbb{F}_q$, χ is a nonprincipal multiplicative character in $\mathbb{F}_q$, and $m \in \mathbb{F}_q^*$.

3.1. The proofs of Theorems 1.1 and 1.2

The proofs of Theorems 1.1 and 1.2 are very complicated. Hence, we divide our proofs into the following preparatory works. For convenience, we denote

$$A_0 = \frac{1}{q} \sum_{\substack{a,b,c\in\mathbb{F}_q^*\\ c\neq 1}} \overline{\chi}(abc) \sum_{d,e,f\in\mathbb{F}_q^*} (1 + \tau(f) + \overline{\tau}(f))\psi(d(a^2bc - 1) + e(ab^2c - 1) + \overline{d}\overline{e}fm(c - 1)),$$

$$A_1 = \sum_{a,b\in\mathbb{F}_q^*} \overline{\chi}(ab) \sum_{d,e\in\mathbb{F}_q^*} \psi(d(a^2b - 1) + e(ab^2 - 1)),$$

$$A_2 = \frac{1}{q} \sum_{\substack{a,b,c\in\mathbb{F}_q^*\\ c\neq 1}} \overline{\chi}(abc) \sum_{d,e,f\in\mathbb{F}_q^*} \psi(d(a^2bc - 1) + e(ab^2c - 1) + \overline{d}\overline{e}fm(c - 1)),$$

$$A_3 = \frac{1}{q} \sum_{\substack{a,b,c\in\mathbb{F}_q^*\\ c\neq 1}} \overline{\chi}(abc) \sum_{d,e,f\in\mathbb{F}_q^*} \tau(f)\psi(d(a^2bc - 1) + e(ab^2c - 1) + \overline{d}\overline{e}fm(c - 1)),$$

$$A_4 = \frac{1}{q} \sum_{\substack{a,b,c\in\mathbb{F}_q^*\\ c\neq 1}} \overline{\chi}(abc) \sum_{d,e,f\in\mathbb{F}_q^*} \overline{\tau}(f)\psi(d(a^2bc - 1) + e(ab^2c - 1) + \overline{d}\overline{e}fm(c - 1)),$$

where χ is a nonprincipal multiplicative character in $\mathbb{F}_q$, ψ is a canonical additive character of $\mathbb{F}_q$, and $m \in \mathbb{F}_q^*$.

Lemma 3.1 *Assume that χ is a nonprincipal multiplicative character in $\mathbb{F}_q$ with $3 \mid (q-1)$, and $m \in \mathbb{F}_q^*$. If χ is not a power-three multiplicative character in $\mathbb{F}_q$, we have*

$$A(\chi, m, q) = 0.$$

If χ is a power-three multiplicative character in $\mathbb{F}_q$, then

$$|A(\chi, m, q)|^2 = 3q^2 + \frac{G^3(\tau)}{q}\overline{\tau}(m) \sum_{a,b,c\in\mathbb{F}_q^*} \overline{\chi}(abc)\overline{\tau}(c-1)\overline{\tau}(a^2bc-1)\overline{\tau}(ab^2c-1)$$
$$+ \frac{G^3(\overline{\tau})}{q}\tau(m) \sum_{a,b,c\in\mathbb{F}_q^*} \overline{\chi}(abc)\tau(c-1)\tau(a^2bc-1)\tau(ab^2c-1), \qquad (3.1)$$

where τ is a multiplicative character of $\mathbb{F}_q$ of order three.

Proof If χ is not a power-three multiplicative character in $\mathbb{F}_q$, then there exists an element $r \in \mathbb{F}_q$ such that $r^3 = 1$ and $\chi(r) \neq 1$, which implies

$$\begin{aligned} A(\chi, m, q) &= \sum_{a,b \in \mathbb{F}_q^*} \chi(ar + br + m\overline{r}^2\overline{a}\overline{b}) \\ &= \chi(r)A(\chi, m, q). \end{aligned}$$

Due to $\chi(r) \neq 1$, it follows that

$$A(\chi, m, q) = 0.$$

If χ is a power-three multiplicative character in $\mathbb{F}_q$, by $|G(\chi)| = q^{\frac{1}{2}}$, it follows that

$$\begin{aligned} |A(\chi, m, q)|^2 &= \frac{1}{q}\left|\sum_{a,b,c \in \mathbb{F}_q^*} \chi(a + b + m\overline{a}\overline{b})\overline{\chi}(c)\psi(c)\right|^2 \\ &= \frac{1}{q}\left|\sum_{a,b,c \in \mathbb{F}_q^*} \overline{\chi}(c)\psi(c(a + b + m\overline{a}\overline{b}))\right|^2 \\ &= \frac{1}{q}\left|\sum_{a,b,c \in \mathbb{F}_q^*} \overline{\chi}(abc)\psi(c(a^2b + ab^2 + m))\right|^2 \\ &= \frac{1}{q}\sum_{a,b,c \in \mathbb{F}_q^*} \overline{\chi}(abc) \sum_{d,e,f \in \mathbb{F}_q^*} \psi(df(a^2bc - 1) + ef(ab^2c - 1) + f\overline{de}m(c - 1)) \\ &= \frac{1}{q}\sum_{a,b,c \in \mathbb{F}_q^*} \overline{\chi}(abc) \sum_{d,e,f \in \mathbb{F}_q^*} \psi(d(a^2bc - 1) + e(ab^2c - 1) + \overline{d}\overline{e}f^3m(c - 1)). \end{aligned}$$

According to $3 \mid (q-1)$, we can obtain the following equation

$$\sum_{a \in \mathbb{F}_q^*} \chi(a^3m) = (1 + \tau(a) + \overline{\tau}(a))\chi(am), \tag{3.2}$$

which implies that

$$\begin{aligned} |A(\chi, m, q)|^2 &= \frac{q-1}{q}\sum_{a,b \in \mathbb{F}_q^*} \overline{\chi}(ab) \sum_{d,e \in \mathbb{F}_q^*} \psi(d(a^2b - 1) + e(ab^2 - 1)) + A_0 \\ &= \frac{q-1}{q}A_1 + A_2 + A_3 + A_4. \end{aligned}$$

From the properties of the nontrivial additive character, we have the following equation

$$\sum_{m \in \mathbb{F}_q} \chi(mn) = \begin{cases} q, & \text{if } n = 0; \\ 0, & \text{if } n \neq 0, \end{cases} \tag{3.3}$$

which yields

$$\begin{aligned}
A_2 &= -\frac{1}{q}\sum_{\substack{a,b,c\in\mathbb{F}_q^*\\ c\neq 1}}\overline{\chi}(abc)\sum_{d,e\in\mathbb{F}_q^*}\psi(d(a^2bc-1)+e(ab^2c-1))\\
&= -\frac{1}{q}\sum_{a,b,c\in\mathbb{F}_q^*}\overline{\chi}(abc)\sum_{d,e\in\mathbb{F}_q^*}\psi(d(a^2bc-1)+e(ab^2c-1))+\frac{1}{q}A_1\\
&= -q\sum_{\substack{a,b,c\in\mathbb{F}_q^*\\ a^2bc=ab^2c=1}}\overline{\chi}(abc)+2\sum_{\substack{a,b,c\in\mathbb{F}_q^*\\ a^2bc=1}}\overline{\chi}(abc)+\frac{1}{q}A_1\\
&= \frac{1}{q}A_1.
\end{aligned}$$

Hence, $|A(\chi,m,q)|^2$ can be written as

$$|A(\chi,m,q)|^2 = A_1+A_3+A_4.$$

Similarly, we can also get

$$\begin{aligned}
A_1 &= q^2\sum_{\substack{a,b\in\mathbb{F}_q^*\\ a^2b=ab^2=1}}\overline{\chi}(ab)+2q\sum_{\substack{a,b\in\mathbb{F}_q^*\\ a^2b=1}}\overline{\chi}(ab)\\
&= q^2\sum_{\substack{a\in\mathbb{F}_q^*\\ a^3=1}}\chi(a).
\end{aligned}$$

So that $A_1=3q^2$. Finally, from the definition of Gaussian sums, it follows that

$$\begin{aligned}
A_3 &= \frac{1}{q}\sum_{a,b,c\in\mathbb{F}_q^*}\overline{\chi}(abc)\sum_{d,e,f\in\mathbb{F}_q^*}\tau(def\overline{m}\overline{(c-1)})\psi(d(a^2bc-1)+e(ab^2c-1)+f)\\
&= \frac{G^3(\tau)}{q}\overline{\tau}(m)\sum_{a,b,c\in\mathbb{F}_q^*}\overline{\chi}(abc)\overline{\tau}(c-1)\overline{\tau}(a^2bc-1)\overline{\tau}(ab^2c-1).
\end{aligned}$$

Analogue to the calculation of A_3, we also have

$$A_4 = \frac{G^3(\overline{\tau})}{q}\tau(m)\sum_{a,b,c\in\mathbb{F}_q^*}\overline{\chi}(abc)\tau(c-1)\tau(a^2bc-1)\tau(ab^2c-1).$$

So we can obtain Equation (3.1). □

Lemma 3.2 *Suppose that χ is a nonprincipal multiplicative character in $\mathbb{F}_q$ with $3\nmid(q-1)$, and $m\in\mathbb{F}_q^*$. Then we have*

$$|A(\chi,m,q)|^2 = q^2.$$

Proof Note that $3 \nmid (q-1)$. So if f runs through $\mathbb{F}_q$, so does f^3. According to a similar method of Lemma 3.1, we have

$$\begin{aligned}|A(\chi, m, q)|^2 &= \frac{1}{q} \sum_{a,b,c \in \mathbb{F}_q^*} \overline{\chi}(abc) \sum_{d,e,f \in \mathbb{F}_q^*} \psi(d(a^2bc-1) + e(ab^2c-1) + \overline{d}\overline{e}f^3m(c-1)) \\ &= \frac{1}{q} \sum_{a,b,c \in \mathbb{F}_q^*} \overline{\chi}(abc) \sum_{d,e,f \in \mathbb{F}_q^*} \psi(d(a^2bc-1) + e(ab^2c-1) + \overline{d}\overline{e}fm(c-1)) \\ &= A_1 \\ &= q^2 \sum_{\substack{a \in \mathbb{F}_q^* \\ a^3=1}} \chi(a).\end{aligned}$$

As $3 \nmid (q-1)$, then we have

$$|A(\chi, m, q)|^2 = q^2.$$

□

Lemma 3.3 *Suppose that ψ is a canonical additive character of $\mathbb{F}_q$. Then we can obtain the following identities*

$$\sum_{m \in \mathbb{F}_q^*} |C(\psi, m, q)|^2 = q^2 - q - 1,$$

$$\sum_{m \in \mathbb{F}_q^*} |D(\psi, m, q)|^2 = \begin{cases} q^2 - 3q - 1, & \text{if } 3 \mid (q-1); \\ q^2 - q - 1, & \text{if } 3 \nmid (q-1), \end{cases}$$

and

$$\sum_{m \in \mathbb{F}_q^*} |E(\psi, m, q)|^2 = \begin{cases} q^2 - 4q - 1, & \text{if } 4 \mid (q-1); \\ q^2 - 2q - 1, & \text{if } 4 \nmid (q-1). \end{cases}$$

Proof From Equation (3.3), we can get

$$\begin{aligned}\sum_{m \in \mathbb{F}_q^*} |C(\psi, m, q)|^2 &= \sum_{a,b,m \in \mathbb{F}_q^*} \psi(m(a-b) + (\overline{a} - \overline{b})) \\ &= \sum_{a,b \in \mathbb{F}_q^*} \sum_{m \in \mathbb{F}_q} \psi(m(a-b) + (\overline{a} - \overline{b})) - \sum_{a,b \in \mathbb{F}_q^*} \psi(\overline{a} - \overline{b}) \\ &= q^2 - q - 1.\end{aligned}$$

Analogue to the calculation of $\sum_{m\in\mathbb{F}_q^*}|C(\psi,m,q)|^2$, we can also obtain

$$\begin{aligned}\sum_{m\in\mathbb{F}_q^*}|D(\psi,m,q)|^2 &= \sum_{a,b,m\in\mathbb{F}_q^*}\psi(m(a^3-b^3)+(a-b))\\ &= \sum_{a,b\in\mathbb{F}_q^*}\sum_{m\in\mathbb{F}_q}\psi(m(a^3-b^3)+(a-b)) - \sum_{a,b\in\mathbb{F}_q^*}\psi(a-b)\\ &= q\sum_{\substack{a,b\in\mathbb{F}_q^*\\ a^3=b^3}}\psi(a-b)-1\\ &= \begin{cases} q^2-3q-1, & \text{if } 3\mid(q-1)\\ q^2-q-1, & \text{if } 3\nmid(q-1).\end{cases}\end{aligned}$$

Similarly, we can also get

$$\sum_{m\in\mathbb{F}_q^*}|E(\psi,m,q)|^2 = \begin{cases} q^2-4q-1, \text{ if } \ 4\mid(q-1)\\ q^2-2q-1, \text{ if } \ 4\nmid(q-1).\end{cases}$$

□

Lemma 3.4 *Let* $3\mid(q-1)$ *and assume that* τ *is a multiplicative character of* $\mathbb{F}_q$ *of order three and* ψ *is a canonical additive character of* $\mathbb{F}_q$*. Then*

$$\left|\sum_{m\in\mathbb{F}_q^*}\tau(m)\,|C(\psi,m,q)|^2\right| = q^{\frac{3}{2}},$$

and

$$\left|\sum_{m\in\mathbb{F}_q^*}\tau(m)\,|D(\psi,m,q)|^2\right| \le q+2\sqrt{q}.$$

Proof Due to the fact that $\tau(-1) = 1$ and $\tau^2 = \overline{\tau}$, we have

$$\begin{aligned}
\sum_{m\in\mathbb{F}_q^*} \tau(m) \left|C(\psi,m,q)\right|^2 &= \sum_{a,b,m\in\mathbb{F}_q^*} \tau(m)\psi(m(a-b)+(\overline{a}-\overline{b})) \\
&= G(\tau) \sum_{a,b\in\mathbb{F}_q^*} \overline{\tau}(a-b)\psi(\overline{a}-\overline{b}) \\
&= G(\tau) \sum_{a,b\in\mathbb{F}_q^*} \overline{\tau}(ab-b)\psi(\overline{a}\overline{b}-\overline{b}) \\
&= G(\tau) \sum_{a,b\in\mathbb{F}_q^*} \tau(b)\overline{\tau}(a-1)\psi(b(\overline{a}-1)) \\
&= G^2(\tau) \sum_{a\in\mathbb{F}_q^*} \overline{\tau}(a-1)\overline{\tau}(\overline{a}-1) \\
&= G^2(\tau) \sum_{a\in\mathbb{F}_q^*} \tau(a)\tau(1-a) \\
&= G^2(\tau)J(\tau,\tau).
\end{aligned}$$

From Lemma 2.3, it follows that $|J(\tau,\tau)| = q^{\frac{1}{2}}$. Then combining with $|G(\tau)| = q^{\frac{1}{2}}$, we can easily check that

$$\left|\sum_{m\in\mathbb{F}_q^*} \tau(m) \left|C(\psi,m,q)\right|^2\right| = q^{\frac{3}{2}}.$$

Similarly, by Equation (3.2), we can also obtain

$$\begin{aligned}
\sum_{m\in\mathbb{F}_q^*} \tau(m) \left|D(\psi,m,q)\right|^2 &= \sum_{a,b,m\in\mathbb{F}_q^*} \tau(m)\psi(m(a^3-b^3)+(a-b)) \\
&= \sum_{a,b,m\in\mathbb{F}_q^*} \tau(m)\psi(mb^3(a^3-1)+b(a-1)) \\
&= G(\tau) \sum_{a,b\in\mathbb{F}_q^*} \overline{\tau}(b^3(a^3-1))\psi(b(a-1)) \\
&= G(\tau) \sum_{a,b\in\mathbb{F}_q^*} \overline{\tau}(a^3-1)\psi(b(a-1)) \\
&= -G(\tau) \sum_{a\in\mathbb{F}_q^*} \overline{\tau}(a^3-1) \\
&= -G(\tau) \sum_{a\in\mathbb{F}_q^*} (1+\tau(a)+\overline{\tau}(a))\overline{\tau}(a-1) \\
&= G(\tau) - G(\tau)J(\overline{\tau},\overline{\tau}) - G(\tau)J(\tau,\overline{\tau}).
\end{aligned}$$

Therefore, from Lemma 2.3 and $|G(\tau)| = q^{\frac{1}{2}}$, we have

$$\left|\sum_{m\in\mathbb{F}_q^*}\tau(m)\left|D(\psi,m,q)\right|^2\right| \le |G(\tau)| + |G(\tau)||J(\bar\tau,\bar\tau)| + |G(\tau)||J(\tau,\overline{\tau})|$$
$$= q + 2\sqrt{q}.$$

□

Lemma 3.5 *Assume that χ is a power-three multiplicative character in $\mathbb{F}_q$ with $3 \mid (q-1)$. Then, for any multiplicative character τ of $\mathbb{F}_q$ of order three, it follows that*

$$\left|\sum_{a,b,c\in\mathbb{F}_q^*}\overline{\chi}(abc)\tau(c-1)\tau(a^2bc-1)\tau(ab^2c-1)\right| \le 3q^{\frac{3}{2}}.$$

Proof From $3 \mid (q-1)$, it implies that there exists an element $t \in \mathbb{F}_q^*$ such that

$$1 + \tau(t) + \overline{\tau}(t) = 0.$$

Then according to Equation (3.1), we can obtain

$$|A(\chi,m,q)|^2 + \left|\sum_{a,b\in\mathbb{F}_q^*}\chi(a+b+mt\bar a\bar b)\right|^2 + \left|\sum_{a,b\in\mathbb{F}_q^*}\chi(a+b+mt^2\bar a\bar b)\right|^2 = 9q^2,$$

which yields

$$|A(\chi,m,q)|^4 + \left|\sum_{a,b\in\mathbb{F}_q^*}\chi(a+b+mt\bar a\bar b)\right|^4 + \left|\sum_{a,b\in\mathbb{F}_q^*}\chi(a+b+mt^2\bar a\bar b)\right|^4 \le 81q^4. \tag{3.4}$$

On the other hand, by Equation (3.1), it follows that

$$\begin{aligned}|A(\chi,m,q)|^4 &+ \left|\sum_{a,b\in\mathbb{F}_q^*}\chi(a+b+mt\bar a\bar b)\right|^4 + \left|\sum_{a,b\in\mathbb{F}_q^*}\chi(a+b+mt^2\bar a\bar b)\right|^4 \\ &= 27q^4 + 6q\left|\sum_{a,b,c\in\mathbb{F}_q^*}\overline{\chi}(abc)\tau(c-1)\tau(a^2bc-1)\tau(ab^2c-1)\right|^2.\end{aligned} \tag{3.5}$$

Hence, from Equations (3.4) and (3.5), we have

$$\left|\sum_{a,b,c\in\mathbb{F}_q^*}\overline{\chi}(abc)\tau(c-1)\tau(a^2bc-1)\tau(ab^2c-1)\right| \le 3q^{\frac{3}{2}}.$$

□

Now we use the lemmas above to complete the proofs of Theorems 1.1 and 1.2. Firstly, we prove Theorem 1.1.

Proof By Equation (3.1) and Lemma 3.3, we have

$$\sum_{m\in\mathbb{F}_q^*} |A(\chi,m,q)|^2 \left|\sum_{c\in\mathbb{F}_q^*} \psi(mc+\overline{c})\right|^2$$

$$= 3q^2(q^2-q-1) + \frac{G^3(\tau)}{q} \sum_{m\in\mathbb{F}_q^*} \overline{\tau}(m) \left|\sum_{d\in\mathbb{F}_q^*} \psi(md+\overline{d})\right|^2 \sum_{a,b,c\in\mathbb{F}_q^*} \overline{\chi}(abc)\overline{\tau}(c-1)\overline{\tau}(a^2bc-1)\overline{\tau}(ab^2c-1)$$

$$+ \frac{G^3(\overline{\tau})}{q} \sum_{m\in\mathbb{F}_q^*} \tau(m) \left|\sum_{d\in\mathbb{F}_q^*} \psi(md+\overline{d})\right|^2 \sum_{a,b,c\in\mathbb{F}_q^*} \overline{\chi}(abc)\tau(c-1)\tau(a^2bc-1)\tau(ab^2c-1).$$

Then from Lemmas 3.4 and 3.5, it follows that

$$\left|\sum_{m\in\mathbb{F}_q^*} \overline{\tau}(m) \left|\sum_{d\in\mathbb{F}_q^*} \psi(md+\overline{d})\right|^2\right| = q^{\frac{3}{2}},$$

and

$$\left|\sum_{a,b,c\in\mathbb{F}_q^*} \overline{\chi}(abc)\tau(c-1)\tau(a^2bc-1)\tau(ab^2c-1)\right| \le 3q^{\frac{3}{2}}.$$

And due to $|G(\tau)| = q^{\frac{1}{2}}$, it follows that

$$\sum_{m\in\mathbb{F}_q^*} |A(\chi,m,q)|^2 \left|\sum_{c\in\mathbb{F}_q^*} \psi(mc+\overline{c})\right|^2 = 3q^2(q^2-q-1) + T_1,$$

where $|T_1| \le 6q^{\frac{7}{2}}$.

Analogue to the proof above, according to Lemmas 3.1 and 3.3–3.5, we can also get

$$\sum_{m\in\mathbb{F}_q^*} |A(\chi,m,q)|^2 \left|\sum_{c\in\mathbb{F}_q^*} \psi(mc^3+c)\right|^2 = 3q^2(q^2-3q-1) + T_2,$$

where $|T_2| \le 6q^3 + 12q^{\frac{5}{2}}$.

Theorem 1.2 can be checked by Lemmas 3.2 and 3.3. □

3.2. The proofs of Theorems 1.3 and 1.4

In order to prove Theorems 1.3 and 1.4, we need the following lemmas. For convenience, we denote

$$B_0 = \frac{1}{q}\sum_{\substack{a,b,c,d\in\mathbb{F}_q^*\\ d\neq 1}}\overline{\chi}(abcd)\sum_{e,f,g,h\in\mathbb{F}_q^*}(1+\lambda(h)+\eta(h)+\overline{\lambda}(h))\psi(e(a^2bcd-1)+f(ab^2cd-1)+g(abc^2d-1)$$
$$+\overline{e}\overline{f}\overline{g}hm(d-1)),$$

$$B_1 = \sum_{a,b,c\in\mathbb{F}_q^*}\overline{\chi}(abc)\sum_{e,f,g\in\mathbb{F}_q^*}\psi(e(a^2bc-1)+f(ab^2c-1)+g(abc^2-1)),$$

$$B_2 = \frac{1}{q}\sum_{\substack{a,b,c,d\in\mathbb{F}_q^*\\ d\neq 1}}\overline{\chi}(abcd)\sum_{e,f,g,h\in\mathbb{F}_q^*}\psi(e(a^2bcd-1)+f(ab^2cd-1)+g(abc^2d-1)+\overline{e}\overline{f}\overline{g}hm(d-1)),$$

$$B_3 = \frac{1}{q}\sum_{\substack{a,b,c,d\in\mathbb{F}_q^*\\ d\neq 1}}\overline{\chi}(abcd)\sum_{e,f,g,h\in\mathbb{F}_q^*}\lambda(h)\psi(e(a^2bcd-1)+f(ab^2cd-1)+g(abc^2d-1)+\overline{e}\overline{f}\overline{g}hm(d-1)),$$

$$B_4 = \frac{1}{q}\sum_{\substack{a,b,c,d\in\mathbb{F}_q^*\\ d\neq 1}}\overline{\chi}(abcd)\sum_{e,f,g,h\in\mathbb{F}_q^*}\eta(h)\psi(e(a^2bcd-1)+f(ab^2cd-1)+g(abc^2d-1)+\overline{e}\overline{f}\overline{g}hm(d-1)),$$

$$B_5 = \frac{1}{q}\sum_{\substack{a,b,c,d\in\mathbb{F}_q^*\\ d\neq 1}}\overline{\chi}(abcd)\sum_{e,f,g,h\in\mathbb{F}_q^*}\overline{\lambda}(h)\psi(e(a^2bcd-1)+f(ab^2cd-1)+g(abc^2d-1)+\overline{e}\overline{f}\overline{g}hm(d-1)),$$

where χ is a nonprincipal multiplicative character in $\mathbb{F}_q$, ψ is a canonical additive character of $\mathbb{F}_q$, and $m\in\mathbb{F}_q^*$.

Lemma 3.6 *Assume that χ is a nonprincipal multiplicative character in $\mathbb{F}_q$ with $4\mid(q-1)$, and m is an element in $\mathbb{F}_q^*$. We denote that*

$$S_1 = \sum_{a,b,c,d\in\mathbb{F}_q^*}\overline{\chi}(abcd)\eta(d-1)\eta(a^2bcd-1)\eta(ab^2cd-1)\eta(abc^2d-1),$$

$$S_2 = \sum_{a,b,c,d\in\mathbb{F}_q^*}\overline{\chi}(abcd)\lambda(d-1)\lambda(a^2bcd-1)\lambda(ab^2cd-1)\lambda(abc^2d-1),$$

and

$$S_3 = \sum_{a,b,c,d\in\mathbb{F}_q^*}\overline{\chi}(abcd)\overline{\lambda}(d-1)\overline{\lambda}(a^2bcd-1)\overline{\lambda}(ab^2cd-1)\overline{\lambda}(abc^2d-1),$$

where λ is a multiplicative character of $\mathbb{F}_q$ of order four and η is the quadratic character of $\mathbb{F}_q$. If χ is not a power-four multiplicative character in $\mathbb{F}_q$, then we have

$$B(\chi,m,q)=0.$$

Otherwise,

$$|B(\chi,m,q)|^2 = 4q^3+\frac{G^4(\lambda)}{q}\overline{\lambda}(m)S_3+\frac{G^4(\eta)}{q}\eta(m)S_1+\frac{G^4(\overline{\lambda})}{q}\lambda(m)S_2. \tag{3.6}$$

Proof If χ is not a power-four multiplicative character of $\mathbb{F}_q$, then there exists an element $r \in \mathbb{F}_q$ such that $r^4 = 1$ and $\chi(r) \neq 1$, then

$$\begin{aligned} B(\chi, m, q) &= \sum_{a,b,c \in \mathbb{F}_q^*} \chi(ar + br + cr + m\overline{r}^3\overline{a}\overline{b}\overline{c}) \\ &= \chi(r)B(\chi, m, q). \end{aligned}$$

Due to $\chi(r) \neq 1$, it follows that

$$B(\chi, m, q) = 0.$$

If χ is a power-four multiplicative character in $\mathbb{F}_q$, then analogue to Lemma 3.1, we also have

$$|B(\chi, m, q)|^2 = \frac{1}{q} \sum_{a,b,c,d \in \mathbb{F}_q^*} \overline{\chi}(abcd) \sum_{e,f,g,h \in \mathbb{F}_q^*} \psi(e(a^2bcd - 1) + f(ab^2cd - 1) + g(abc^2d - 1) + \overline{e}\overline{f}\overline{g}h^4m(d - 1)).$$

Since $4 \mid (q - 1)$, we can obtain the following equation

$$\sum_{a \in \mathbb{F}_q^*} \chi(a^4m) = (1 + \lambda(a) + \eta(a) + \overline{\lambda}(a))\chi(am). \tag{3.7}$$

Hence we have

$$\begin{aligned} |B(\chi, m, q)|^2 &= \frac{q-1}{q} \sum_{a,b,c \in \mathbb{F}_q^*} \overline{\chi}(abc) \sum_{e,f,g \in \mathbb{F}_q^*} \psi(e(a^2bc - 1) + f(ab^2c - 1) + g(abc^2 - 1)) + B_0 \\ &= \frac{q-1}{q} B_1 + B_2 + B_3 + B_4 + B_5. \end{aligned}$$

From Equation (3.3), we have

$$\begin{aligned} B_2 &= -\frac{1}{q} \sum_{\substack{a,b,c,d \in \mathbb{F}_q^* \\ d \neq 1}} \overline{\chi}(abcd) \sum_{e,f,g \in \mathbb{F}_q^*} \psi(e(a^2bcd - 1) + f(ab^2cd - 1) + g(abc^2d - 1)) \\ &= -\frac{1}{q} \sum_{a,b,c,d \in \mathbb{F}_q^*} \overline{\chi}(abcd) \sum_{e,f,g \in \mathbb{F}_q^*} \psi(e(a^2bcd - 1) + f(ab^2cd - 1) + g(abc^2d - 1)) + \frac{1}{q}B_1 \\ &= -q^2 \sum_{\substack{a,b,c,d \in \mathbb{F}_q^* \\ a^2bcd = ab^2cd = abc^2d = 1}} \overline{\chi}(abcd) + 3q \sum_{\substack{a,b,c,d \in \mathbb{F}_q^* \\ a^2bcd = ab^2cd = 1}} \overline{\chi}(abcd) - 3 \sum_{\substack{a,b,c,d \in \mathbb{F}_q^* \\ a^2bcd = 1}} \overline{\chi}(abcd) + \frac{1}{q}B_1 \\ &= \frac{1}{q}B_1. \end{aligned}$$

Hence, $|B(\chi, m, q)|^2$ can be written as

$$|B(\chi, m, q)|^2 = B_1 + B_3 + B_4 + B_5.$$

The next thing to do in the proof is to compute B_1. Similarly to the calculation of B_2, we have

$$\begin{aligned} B_1 &= q^3 \sum_{\substack{a,b,c\in\mathbb{F}_q^*\\ a^2bc=ab^2c=abc^2=1}} \overline{\chi}(abc) - 3q^2 \sum_{\substack{a,b,c\in\mathbb{F}_q^*\\ a^2bc=ab^2c=1}} \overline{\chi}(abc) + 3q \sum_{\substack{a,b,c\in\mathbb{F}_q^*\\ a^2bc=1}} \overline{\chi}(abc) \\ &= q^3 \sum_{\substack{a\in\mathbb{F}_q^*\\ a^4=1}} \chi(a). \end{aligned}$$

So that $B_1 = 4q^3$. Finally, by the property of Gaussian sums, we can obtain

$$\begin{aligned} B_3 &= \frac{1}{q} \sum_{a,b,c,d\in\mathbb{F}_q^*} \overline{\chi}(abcd) \sum_{e,f,g,h\in\mathbb{F}_q^*} \lambda(efgh\overline{m}\overline{(d-1)})\psi(e(a^2bcd-1)+f(ab^2cd-1)+g(abc^2d-1)+h) \\ &= \frac{G^4(\lambda)}{q}\overline{\lambda}(m)S_3. \end{aligned}$$

Analogue to the calculation of B_3, we also have $B_4 = \frac{G^4(\eta)}{q}\eta(m)S_1$ and $B_5 = \frac{G^4(\overline{\lambda})}{q}\lambda(m)S_2$. Hence, the proof is completed. □

Lemma 3.7 *Suppose that χ is a nonprincipal multiplicative character in $\mathbb{F}_q$ with $4 \nmid (q-1)$, and $m \in \mathbb{F}_q^*$. If χ is not a power-two multiplicative character in $\mathbb{F}_q$, then*

$$|B(\chi, m, q)| = 0.$$

Otherwise,

$$|B(\chi,m,q)|^2 = 2q^3 + q\eta(m) \sum_{a,b,c,d\in\mathbb{F}_q^*} \overline{\chi}(abcd)\eta(d-1)\eta(a^2bcd-1)\eta(ab^2cd-1)\eta(abc^2d-1),$$

where η is the quadratic character of $\mathbb{F}_q$.

Proof Analogue to the proof of Lemma 3.6, we also have

$$|B(\chi, m, q)| = 0,$$

when χ is not a power-two multiplicative character in $\mathbb{F}_q$. And if χ is a power-two multiplicative character in $\mathbb{F}_q$, then

$$|B(\chi,m,q)|^2 = \frac{1}{q} \sum_{a,b,c,d\in\mathbb{F}_q^*} \overline{\chi}(abcd) \sum_{e,f,g,h\in\mathbb{F}_q^*} \psi(e(a^2bcd-1)+f(ab^2cd-1)+g(abc^2d-1)+\overline{e}\overline{f}\overline{g}h^4m(d-1)).$$

From $4 \nmid (q-1)$, it follows that $\gcd(4, q-1) = 2$. Therefore, if h^4 runs through $\mathbb{F}_q$, then h^2 also runs through $\mathbb{F}_q$, which yields

$$|B(\chi,m,q)|^2 = \frac{1}{q} \sum_{a,b,c,d\in\mathbb{F}_q^*} \overline{\chi}(abcd) \sum_{e,f,g,h\in\mathbb{F}_q^*} \psi(e(a^2bcd-1)+f(ab^2cd-1)+g(abc^2d-1)+\overline{e}\overline{f}\overline{g}h^2m(d-1)).$$

According to the method of Lemma 3.6, we also have

$$|B(\chi,m,q)|^2 = \frac{q-1}{q}\sum_{a,b,c\in\mathbb{F}_q^*}\overline{\chi}(abc)\sum_{e,f,g\in\mathbb{F}_q^*}\psi(e(a^2bc-1)+f(ab^2c-1)+g(abc^2-1)) + \frac{1}{q}\sum_{a,b,c,d\in\mathbb{F}_q^*}\overline{\chi}(abcd)$$

$$\sum_{e,f,g,h\in\mathbb{F}_q^*}(1+\eta(h))\psi(e(a^2bcd-1)+f(ab^2cd-1)+g(abc^2d-1)+\overline{e}\overline{f}\overline{g}h^2m(d-1))$$

$$= B_1 + B_4$$

$$= 2q^3 + q\eta(m)\sum_{a,b,c,d\in\mathbb{F}_q^*}\overline{\chi}(abcd)\eta(d-1)\eta(a^2bcd-1)\eta(ab^2cd-1)\eta(abc^2d-1).$$

□

Lemma 3.8 *Assume that λ is a multiplicative character of $\mathbb{F}_q$ of order four, η is the quadratic character of $\mathbb{F}_q$ and ψ is the canonical additive character of $\mathbb{F}_q$. Then*

$$\left|\sum_{m\in\mathbb{F}_q^*}\lambda(m)\,|C(\psi,m,q)|^2\right| = q^{\frac{3}{2}},$$

and

$$\sum_{m\in\mathbb{F}_q^*}\eta(m)\,|C(\psi,m,q)|^2 = 0.$$

Proof By the property of Gaussian sums, we have

$$\sum_{m\in\mathbb{F}_q^*}\lambda(m)\,|C(\psi,m,q)|^2 = \sum_{a,b,m\in\mathbb{F}_q^*}\lambda(m)\psi(m(a-b)+(\overline{a}-\overline{b}))$$

$$= G(\lambda)\sum_{a,b\in\mathbb{F}_q^*}\overline{\lambda}(a-b)\psi(\overline{a}-\overline{b})$$

$$= G(\lambda)\sum_{a,b\in\mathbb{F}_q^*}\overline{\lambda}(ab-b)\psi(\overline{a}\overline{b}-\overline{b})$$

$$= G(\lambda)\sum_{a,b\in\mathbb{F}_q^*}\lambda(b)\overline{\lambda}(a-1)\psi(b(\overline{a}-1))$$

$$= G^2(\lambda)\sum_{a\in\mathbb{F}_q^*}\overline{\lambda}(a-1)\overline{\lambda}(\overline{a}-1)$$

$$= G^2(\lambda)\overline{\lambda}(-1)\sum_{a\in\mathbb{F}_q^*}\lambda(a)\eta(a-1)$$

$$= G^2(\lambda)\overline{\lambda}(-1)\eta(-1)J(\lambda,\eta).$$

On the other hand, from Lemma 2.3, we have the following equation

$$|J(\lambda,\eta)| = q^{\frac{1}{2}}.$$

Combining with $|G(\lambda)| = q^{\frac{1}{2}}$, we can obtain

$$\left|\sum_{m\in\mathbb{F}_q^*} \lambda(m)\left|C(\psi,m,q)\right|^2\right| = q^{\frac{1}{2}}\left|G^2(\lambda)\right| = q^{\frac{3}{2}}.$$

Similarly, we also have

$$\begin{aligned}\sum_{m\in\mathbb{F}_q^*} \eta(m)\left|C(\psi,m,q)\right|^2 &= G(\eta)\sum_{a,b\in\mathbb{F}_q^*}\eta(a-b)\psi(\overline{a}-\overline{b})\\ &= G^2(\eta)\eta(-1)\sum_{a\in\mathbb{F}_q^*}\eta(a)\\ &= 0.\end{aligned}$$

□

Lemma 3.9 *Suppose that λ is a multiplicative character of $\mathbb{F}_q$ of order four, η is the quadratic character of $\mathbb{F}_q$ and ψ is a canonical additive character of $\mathbb{F}_q$. Then:*

(i) for the quadratic character η over $\mathbb{F}_q$, we have

$$\left|\sum_{m\in\mathbb{F}_q^*} \eta(m)\left|E(\psi,m,q)\right|^2\right| \le 2q + 2\sqrt{q},$$

if $4 \mid (q-1)$, and

$$\sum_{m\in\mathbb{F}_q^*} \eta(m)\left|E(\psi,m,q)\right|^2 = 0,$$

if $4 \nmid (q-1)$;

(ii) for any character λ over $\mathbb{F}_q$ of order four, we can obtain

$$\left|\sum_{m\in\mathbb{F}_q^*} \lambda(m)\left|E(\psi,m,q)\right|^2\right| \le 2q + 2\sqrt{q},$$

if $8 \mid (q-1)$, and

$$\sum_{m\in\mathbb{F}_q^*} \lambda(m)\left|E(\psi,m,q)\right|^2 = 0,$$

if $8 \nmid (q-1)$.

Proof (i). By the property of Gaussian sums and $\eta(a^4)=1$, it follows that

$$\begin{aligned}\sum_{m\in\mathbb{F}_q^*}\eta(m)\left|E(\psi,m,q)\right|^2&=\sum_{a,b,m\in\mathbb{F}_q^*}\eta(m)\psi(m(a^4-b^4)+(a-b))\\&=\sum_{a,b,m\in\mathbb{F}_q^*}\eta(m)\psi(mb^4(a^4-1)+b(a-1))\\&=G(\eta)\sum_{a,b\in\mathbb{F}_q^*}\eta(b^4(a^4-1))\psi(b(a-1))\\&=G(\eta)\sum_{a,b\in\mathbb{F}_q^*}\eta(a^4-1)\psi(b(a-1))\\&=-G(\eta)\sum_{a\in\mathbb{F}_q^*}\eta(a^4-1).\end{aligned}$$

If $4\mid(q-1)$, then by Equation (3.7) and $\eta(-1)=1$, we have

$$\begin{aligned}-G(\eta)\sum_{a\in\mathbb{F}_q^*}\eta(a^4-1)&=-G(\eta)\sum_{a\in\mathbb{F}_q^*}(1+\lambda(a)+\eta(a)+\overline{\lambda}(a))\eta(1-a)\\&=G(\eta)-G(\eta)J(\lambda,\eta)-G(\eta)J(\eta,\eta)-G(\eta)J(\overline{\lambda},\eta).\end{aligned}$$

And by Lemma 2.3 and $|G(\eta)|=\sqrt{q}$, we can get

$$\begin{aligned}\left|\sum_{m\in\mathbb{F}_q^*}\eta(m)\left|E(\psi,m,q)\right|^2\right|&\le\sqrt{q}+\sqrt{q}|J(\lambda,\eta)|+\sqrt{q}|J(\eta,\eta)|+\sqrt{q}|J(\overline{\lambda},\eta)|\\&\le 2q+2\sqrt{q}.\end{aligned}$$

Note that $4\nmid(q-1)$. If a^4 runs through $\mathbb{F}_q$, so does a^2. Then according to $\eta(-1)=-1$, $G^2(\eta)=\eta(-1)q$ and Lemma 2.2, we have

$$\begin{aligned}\sum_{m\in\mathbb{F}_q^*}\eta(m)\left|E(\psi,m,q)\right|^2&=-G(\eta)\sum_{a\in\mathbb{F}_q^*}\eta(a^4-1)\\&=-G(\eta)\sum_{a\in\mathbb{F}_q^*}\eta(a^2-1)\\&=G(\eta)\sum_{a\in\mathbb{F}_q^*}(1+\eta(a))\eta(1-a)\\&=-G(\eta)+G(\eta)J(\eta,\eta)\\&=-G(\eta)-\frac{1}{q}G^3(\eta)\\&=0.\end{aligned}$$

(ii). By similar method of (i), it follows that

$$\sum_{m\in\mathbb{F}_q^*}\lambda(m)\left|E(\psi,m,q)\right|^2 = -G(\overline{\lambda})\overline{\lambda}(-1)(1+\lambda(a)+\eta(a)+\overline{\lambda}(a))\overline{\lambda}(1-a)$$

$$= G(\overline{\lambda})\overline{\lambda}(-1) - G(\overline{\lambda})\overline{\lambda}(-1)J(\lambda,\overline{\lambda}) - G(\overline{\lambda})\overline{\lambda}(-1)J(\eta,\overline{\lambda}) - G(\overline{\lambda})\overline{\lambda}(-1)J(\overline{\lambda},\overline{\lambda}).$$

By Lemma 2.2 and $G(\lambda)G(\overline{\lambda})=\lambda(-1)q$, we have

$$\sum_{m\in\mathbb{F}_q^*}\lambda(m)\left|E(\psi,m,q)\right|^2 = G(\overline{\lambda})\overline{\lambda}(-1)+\frac{1}{q}G(\overline{\lambda})\overline{\lambda}(-1)G(\lambda)G(\overline{\lambda}) - G(\overline{\lambda})\overline{\lambda}(-1)\frac{G(\overline{\lambda})G(\eta)}{G(\lambda)}$$

$$- G(\overline{\lambda})\overline{\lambda}(-1)\frac{G(\overline{\lambda})G(\overline{\lambda})}{G(\eta)}$$

$$= G(\overline{\lambda})(1+\overline{\lambda}(-1)) - G(\overline{\lambda})\overline{\lambda}(-1)\frac{G^2(\overline{\lambda})G(\eta)}{G(\lambda)G(\overline{\lambda})} - G(\overline{\lambda})\overline{\lambda}(-1)\frac{G^2(\overline{\lambda})G(\eta)}{G^2(\eta)}$$

$$= G(\overline{\lambda})(1+\overline{\lambda}(-1)) - \frac{1}{q}G^3(\overline{\lambda})G(\eta)(1+\overline{\lambda}(-1)).$$

When $8\nmid(q-1)$, we have $\overline{\lambda}(-1)=-1$, which yields

$$\sum_{m\in\mathbb{F}_q^*}\lambda(m)\left|E(\psi,m,q)\right| = 0.$$

When $8\mid(q-1)$, we have $\overline{\lambda}(-1)=1$. Hence by $|G(\lambda)|=q^{\frac{1}{2}}$ it follows that

$$\left|\sum_{m\in\mathbb{F}_q^*}\lambda(m)\left|E(\psi,m,q)\right|^2\right| = \left|2G(\overline{\lambda}) - \frac{2}{q}G^3(\overline{\lambda})G(\eta)\right|$$

$$\leq 2|G(\overline{\lambda})| + \frac{2}{q}|G(\overline{\lambda})|^3|G(\eta)|$$

$$= 2\sqrt{q}+2q.$$

□

Lemma 3.10 *Assume that* χ *is a power-four multiplicative character in* $\mathbb{F}_q$ *with* $4\mid(q-1)$. *Then, according to the definitions of* S_1, S_2 *and* S_3 *in Lemma 3.6, we have*

$$|S_1|\leq 4\sqrt{3}q^2 \text{ and } |S_2|=|S_3|\leq 2\sqrt{6}q^2.$$

Proof If $4\mid(q-1)$, there exists an element $h\in\mathbb{F}_q$ such that

$$1+\lambda(h)+\eta(h)+\overline{\lambda}(h)=0.$$

Since Equation (3.6), we can obtain

$$|B(\chi,m,q)|^2 + \left|\sum_{a,b,c\in\mathbb{F}_q^*} \chi(a+b+c+mh\bar{a}\bar{b}\bar{c})\right|^2 + \left|\sum_{a,b,c\in\mathbb{F}_q^*} \chi(a+b+c+mh^2\bar{a}\bar{b}\bar{c})\right|^2$$

$$+\left|\sum_{a,b,c\in\mathbb{F}_q^*} \chi(a+b+c+mh^3\bar{a}\bar{b}\bar{c})\right|^2 = 16q^3,$$

which yields

$$|B(\chi,m,q)|^4 + \left|\sum_{a,b,c\in\mathbb{F}_q^*} \chi(a+b+c+mh\bar{a}\bar{b}\bar{c})\right|^4 + \left|\sum_{a,b,c\in\mathbb{F}_q^*} \chi(a+b+c+mh^2\bar{a}\bar{b}\bar{c})\right|^4$$

$$+\left|\sum_{a,b,c\in\mathbb{F}_q^*} \chi(a+b+c+mh^3\bar{a}\bar{b}\bar{c})\right|^4 \leq 256q^6.$$

By a direct calculation and Equation (3.6), it follows that

$$|B(\chi,m,q)|^4 + \left|\sum_{a,b,c\in\mathbb{F}_q^*} \chi(a+b+c+mh\bar{a}\bar{b}\bar{c})\right|^4 + \left|\sum_{a,b,c\in\mathbb{F}_q^*} \chi(a+b+c+mh^2\bar{a}\bar{b}\bar{c})\right|^4$$

$$+\left|\sum_{a,b,c\in\mathbb{F}_q^*} \chi(a+b+c+mh^3\bar{a}\bar{b}\bar{c})\right|^4 = 64q^6 + 4q^2|S_1|^2 + 4q^2|S_2|^2 + 4q^2|S_3|^2.$$

From the definition of S_2 and S_3, it follows that $|S_2| = |S_3|$, which yields $|S_1|^2 + 2|S_2|^2 \leq 48q^4$. Due to $|S_1|^2 \geq 0$, we have $|S_2| = |S_3| \leq 2\sqrt{6}q^2$. On the other hand, by $|S_2|^2 \geq 0$, we can also obtain $|S_1| \leq 4\sqrt{3}q^2$. □

Now we complete the proofs of our results by using the lemmas in this section. Firstly, we give the proof of Theorem 1.3.

Proof By Equation (3.6), Lemmas 3.3 and 3.8, we can obtain

$$\sum_{m\in\mathbb{F}_q^*} |B(\chi,m,q)|^2 \left|\sum_{d\in\mathbb{F}_q^*} \psi(md+\bar{d})\right|^2$$

$$= 4q^3 \sum_{m\in\mathbb{F}_q^*}\left|\sum_{e\in\mathbb{F}_q^*} \psi(me+\bar{e})\right|^2 + \frac{G^4(\lambda)}{q}\sum_{m\in\mathbb{F}_q^*}\bar{\lambda}(m)\left|\sum_{e\in\mathbb{F}_q^*} \psi(me+\bar{e})\right|^2 S_3$$

$$+\frac{G^4(\eta)}{q}\sum_{m\in\mathbb{F}_q^*}\eta(m)\left|\sum_{e\in\mathbb{F}_q^*} \psi(me+\bar{e})\right|^2 S_1 + \frac{G^4(\bar{\lambda})}{q}\sum_{m\in\mathbb{F}_q^*}\lambda(m)\left|\sum_{e\in\mathbb{F}_q^*} \psi(me+\bar{e})\right|^2 S_2$$

$$= 4q^3(q^2-q-1) + \frac{G^4(\lambda)}{q}\sum_{m\in\mathbb{F}_q^*}\bar{\lambda}(m)\left|\sum_{e\in\mathbb{F}_q^*} \psi(me+\bar{e})\right|^2 S_3 + \frac{G^4(\bar{\lambda})}{q}\sum_{m\in\mathbb{F}_q^*}\lambda(m)\left|\sum_{e\in\mathbb{F}_q^*} \psi(me+\bar{e})\right|^2 S_2.$$

Then by Lemma 3.8, we have

$$\left|\sum_{m\in\mathbb{F}_q^*}\lambda(m)\,|C(\psi,m,q)|^2\right| = q^{\frac{3}{2}}.$$

And from Lemma 3.10, we can obtain

$$\left|\sum_{a,b,c,d\in\mathbb{F}_q^*}\overline{\chi}(abcd)\lambda(d-1)\lambda(a^2bcd-1)\lambda(ab^2cd-1)\lambda(abc^2d-1)\right| \le 2\sqrt{6}q^2.$$

Therefore, due to $|G(\lambda)| = q^{\frac{1}{2}}$, we can get

$$\sum_{m\in\mathbb{F}_q^*}|B(\chi,m,q)|^2\left|\sum_{d\in\mathbb{F}_q^*}\psi(md+\overline{d})\right|^2 = 4q^3(q^2-q-1)+T_3,$$

where $|T_3| \le 4\sqrt{6}q^{\frac{7}{2}}$. Similarly, from Lemmas 3.3, 3.9 and 3.10, we can also obtain

$$\sum_{m\in\mathbb{F}_q^*}|B(\chi,m,q)|^2\left|\sum_{d\in\mathbb{F}_q^*}\psi(md^4+d)\right|^2 = 4q^3(q^2-4q-1)+T_4,$$

if $8\mid(q-1)$, and

$$\sum_{m\in\mathbb{F}_q^*}|B(\chi,m,q)|^2\left|\sum_{d\in\mathbb{F}_q^*}\psi(md^4+d)\right|^2 = 4q^3(q^2-4q-1)+T_5,$$

if $8\nmid(q-1)$, where $|T_4| \le (8\sqrt{3}+8\sqrt{6})(q^4+q^{\frac{7}{2}})$ and $|T_5| \le 8\sqrt{3}(q^4+q^{\frac{7}{2}})$. □

Moreover, we can also obtain Theorem 1.4, by similar method of the proof of Theorem 1.3.

Acknowledgment

The authors are very grateful to the anonymous reviewer and the editor for their valuable comments and suggestions to improve the quality and the presentation of this paper.

References

[1] Bourgain J. Estimates on exponential sums related to the Diffie–Hellman Distributions. Geometric and Functional Analysis 2005; 15 (1): 1-34. doi: 10.1007/s00039-005-0500-4

[2] Coulter RS. Further evaluations of some Weil sums. Acta Arithmetica 1998; 86 (3): 217-226. doi: 10.4064/aa-86-3-217-226

[3] Di H. A hybrid mean value involving two-term exponential sums and polynomial character sums. Czechoslovak Mathematical Journal 2014; 64 (139): 53–62. doi: 10.1007/s10587-014-0082-0

[4] Hurt NE. Exponential sums and coding theory: a review. Acta Applicandae Mathematicae 1997; 46 (1): 49-91. doi: 10.1023/A:1005794417363

[5] Luo GJ, Cao XW. Two constructions of asymptotically optimal codebooks via the hyper Eisenstein sum. IEEE Transactions on Information Theory 2018; 64 (10): 6948-6505. doi: 10.1109/TIT.2017.2777492

[6] Luo GJ, Cao XW. A construction of linear codes and strongly regular graphs from q-polynomials. Discrete Mathematics 2017; 340: 2262-2274. doi: 10.1016/j.disc.2017.04.016

[7] Liu WM, Li XY. A hybrid mean value involving the two-term exponential sums and two-term character sums. Journal of Applied Mathematics 2014; 5: 1-5. doi: 10.1155/2014/845845

[8] Lidl R, Niederreiter H. Finite Fields. In: Encyclopedia of Mathematics and its Applications. 2nd ed. Cambridge, UK:Cambridge University Press, 1997.

[9] Li CJ, Wu P, Liu FM. On two classes of primitive BCH codes and some related codes. IEEE Transactions on Information Theory 2018; 65 (6): 3830-3840. doi: 10.1109/TIT.2018.2883615

[10] Lv XX, Zhang WP. A new hybrid power mean involving the generalized quadratic Gauss sums and sums analogous to Kloosterman sums. Lithuanian Mathematical Journal 2017; 57 (3): 359-366. doi: 10.1007/s10986-017-9366-z

[11] Lv XX, Zhang WP. On the character sum of Polynomials and the two-term exponential sums. Acta Mathematica Sinica, English Series 2020; 36 (2): 196-206. doi: 10.1007/s10114-020-9255-y

[12] Paterson KG. Application of exponential sums in communications theory. In: Walker M (editor). Cryptography and Coding. Lecture Notes in Computer Science, Vol. 1746. Springer, Berlin, Germany: Springer, 1999. doi: 10.1007/3-540-46665-7_1

[13] Wu YN, Li N, Zeng XY. Linear codes with few weights from cyclotomic classes and weakly regular bent fuctions. Designs Codes and Cryptography 2020; 88 (4-5): 1255-1272. doi: 10.1007/s10623-020-00744-9

[14] Xu ZF, Zhang TP, Zhang WP. On the mean value of the two-term exponential sums with Dirichlet characters. Journal of Number Theory 2007; 123 (2): 352-362. doi: 10.1016/j.jnt.2006.07.005

[15] Zhang H, Zhang WP. The fourth power mean of two-term exponential sums and its application. Mathematical Reports 2017; 19 (69): 75-81. doi: 10.1155/2014/724840

[16] Zhang WP. The fourth and sixth power mean of the classical Kloosterman sums. Journal of Number Theory 2011; 131 (2): 228-238. doi: 10.1016/j.jnt.2010.08.008

[17] Zhang WP, Han D. A new identity involving the classical Kloosterman sums and 2-dimensional Kloosterman sums. International Journal of Number Theory 2016; 12 (1): 111–119. doi: 10.1142/S179304211650007X