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Notes on multivalent Bazilević functions defined by higher order derivatives

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Abstract: In this paper we consider two subclasses $B(p,q,\alpha,\beta)$ and $B_1(p,q,\alpha,\beta)$ of p-valently Bazilević functions defined by higher order derivatives, and we defined and studied some properties of the images of the functions of these classes by the integral operators $\mathrm{I}_{n,p}$ and $\mathrm{J}_{n,p}$ for multivalent functions, defined by using higher order derivatives.

Key words: p-valent functions, p-valent starlike and convex functions, Bazilević functions, higher order derivatives, integral operator

1. Introduction

Let us denote by $\mathcal{A}(p)$, $p \in \mathbb{N} := \{1,2,\dots\}$, the class of multivalent analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,\ z \in \mathbb{U} := \{z \in \mathbb{C} : |z| < 1\},$$

and let $\mathcal{A} := \mathcal{A}(1)$.

For $0 \le \gamma < p-q$, $p > q$, $p \in \mathbb{N}$, and $q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we say that the function $f \in \mathcal{A}(p)$ is in the class $\mathbb{S}^*_{p,q}(\gamma)$ if it satisfies the inequality

$$\operatorname{Re} \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} > \gamma,\ z \in \mathbb{U},$$

and is in the class $\mathbb{K}_{p,q}(\gamma)$ if it satisfies

$$\operatorname{Re} \left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right) > \gamma,\ z \in \mathbb{U}.$$

The classes $\mathbb{S}^*_{p,q}(\gamma)$ and $\mathbb{K}_{p,q}(\gamma)$, were introduced and studied by Aouf [5, 7, 8]. Note that $\mathbb{S}^*_{p,0}(\gamma) =: \mathbb{S}^*_p(\gamma)$ and $\mathbb{K}_{p,0}(\gamma) =: \mathbb{K}_p(\gamma)$, which are, respectively, the classes of p-valent starlike and convex functions of order γ, with $0 \le \gamma < p$ (see Owa [17] and Aouf [1, 2, 10]).

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Definition 1.1 *(i) A function* $f \in \mathcal{A}(p)$ *is said to be* p-valently Bazilević functions defined by higher order derivative of type α, $(\alpha > 0)$ and order β $(0 \le \beta < p - q,\ p > q)$, *if there exists a function* $g \in \mathbb{S}^*_{p,q}(0) =: \mathbb{S}^*_{p,q}$ *such that*

$$\mathrm{Re}\left[\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\alpha}\right] > \beta,\ z \in \mathbb{U},$$

where the power is the principal one, and we denote by $B(p,q,\alpha,\beta)$ *to the class of such functions.*

(ii) Further, let $B_1(p,q,\alpha,\beta) \subset B(p,q,\alpha,\beta)$ *the subclass of functions for which* $g \in \mathcal{A}(p)$, *such that* $g^{(q)}(z) = \delta(p,q)z^{p-q}$, *and therefore* $g \in \mathbb{S}^*_{p,q}$, *where*

$$\delta(p,q) = \frac{p!}{(p-q)!},\ (p > q).$$

Remark that for special choices of the parameters we obtain the following previously studied subclasses of $B(p,q,\alpha,\beta)$ and $B_1(p,q,\alpha,\beta)$:

(i) $B(p,0,\alpha,\beta) =: B(p,\alpha,\beta)$, the class of p-valently Bazilević functions of type α $(\alpha > 0)$ and order β $(0 \le \beta < p)$ (see Irmak et al. [14], Goswami and Bansal [13], Aouf [6] and Owa [19]);

(ii) $B_1(p,0,\alpha,\beta) =: B_1(p,\alpha,\beta)$ (see Owa [19] and Aouf [6]);

(iii) $B(1,0,\alpha,\beta) =: B(\alpha,\beta)$ and $B_1(1,0,\alpha,\beta) =: B_1(\alpha,\beta)$ (see Owa and Obradović [20]);

(iv) $B(p,q,1,\beta) =: C_{p,q}(\beta) = \left\{f \in \mathcal{A}(p) : \mathrm{Re}\,\frac{zf^{(1+q)}(z)}{g^{(q)}(z)} > \beta,\ z \in \mathbb{U},\ g \in \mathbb{S}^*_{p,q}\right\}$ (see Aouf [4]), and $C_{p,0}(\beta) =: C_p(\beta)$ (see Aouf [3, 9]).

2. Integral operator $\mathrm{I}_{n,p}f^{(q)}$

Unless stated otherwise, we assume that $\alpha > 0$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, $p > q$, $0 \le \beta < p - q$, $z = re^{i\theta} \in \mathbb{U}$, and all the powers are the principal ones.

For $f \in \mathcal{A}(p)$, we define the integral operator $\mathrm{I}_{n,p}f^{(q)}$ by

$$\mathrm{I}_{0,p}f^{(q)}(z) := \left(\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right)^{\alpha},$$

and

$$\mathrm{I}_{n,p}f^{(q)}(z) := z^{-1}\int_0^1 \mathrm{I}_{n-1,p}f^{(q)}(t)dt,\ n \in \mathbb{N}.$$

Note that the integral operator $\mathrm{I}_{n,p}f^{(0)} =: \mathrm{I}_{n,p}f$ $(f \in \mathcal{A}(p))$ was studied by Owa [17, 18] and the integral operator $\mathrm{I}_{n,1}f =: \mathrm{I}_n f$ $(f \in \mathcal{A})$ was studied by Halenbeck [12], Thomas [25] and Halim and Thomas [11].

For $f \in \mathcal{A}(p)$, Owa [19] proved the following result:

Theorem A *If* $f \in B_1(p,0,\alpha,\beta) =: B_1(p,\alpha,\beta)$ $(p \in \mathbb{N},\ \alpha > 0,\ 0 \le \beta < p)$, *then*

$$\mathrm{Re}\,\mathrm{I}_{n,p}f(z) \ge \gamma_n(r) > \gamma_n(1),\ z \in \mathbb{U},\ (n \in \mathbb{N}_0) \tag{2.1}$$

where

$$\frac{\beta}{p} < \gamma_n(r) := \frac{\beta}{p} + \left(1 - \frac{\beta}{p}\right)\left(-1 + 2p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k^n(k-1+p\alpha)}\right).$$

The equality in (2.1) is attained for the function f given by

$$f(z)=\left\{\alpha\int_0^z t^{p\alpha-1}\left[\beta+(p-\beta)\frac{1-t}{1+t}\right]dt\right\}^{\frac{1}{\alpha}}.$$

Also, for $f\in\mathcal{A}(p)$, Owa [18] proved that:

Theorem B *If $f\in\mathcal{A}(p)$ satisfies*

$$\operatorname{Re}\left[\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha}\right]>0,\ z\in\mathbb{U},\ (\alpha>0),$$

then

$$\operatorname{Re}\mathrm{I}_{n,p}f(z)\geq\widetilde{\gamma}_n(r)>\widetilde{\gamma}_n(1),\ z\in\mathbb{U},\ (n\in\mathbb{N}_0) \tag{2.2}$$

and

$$0<\widetilde{\gamma}_n(r):=-1+2p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k^n(k-1+p\alpha)}<1.$$

The equality in (2.2) is attained for the function f given by

$$f(z)=\left(p\alpha\int_0^z t^{p\alpha-1}\left(\frac{1-t}{1+t}\right)dt\right)^{\frac{1}{\alpha}}.$$

The main result regarding this integral operator is the next theorem:

Theorem 2.1 *If $f\in B_1(p,q,\alpha,\beta)$, then*

$$\operatorname{Re}\mathrm{I}_{n,p}f^{(q)}(z)\geq\gamma_{p,q}^n(r)>\gamma_{p,q}^n(1),\ z\in\mathbb{U},\ r=|z|,\ (n\in\mathbb{N}_0) \tag{2.3}$$

and

$$\frac{\beta}{p-q}<\gamma_{p,q}^n(r):=\frac{\beta}{p-q}+\left(1-\frac{\beta}{p-q}\right)\left(-1+2(p-q)\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k^n[k-1+(p-q)\alpha]}\right). \tag{2.4}$$

The equality in (2.3) is attained for the function $f\in\mathcal{A}(p)$ given by

$$f^{(q)}(z)=\delta(p,q)\left\{\alpha\int_0^z t^{(p-q)\alpha-1}\left[\beta+(p-q-\beta)\frac{1-t}{1+t}\right]dt\right\}^{\frac{1}{\alpha}}.$$

Proof Since $f\in B_1(p,q,\alpha,\beta)$, then we have

$$\operatorname{Re}h(z)>\frac{\beta}{p-q},\ z\in\mathbb{U},$$

where the function h is defined by

$$h(z)=\frac{f^{(1+q)}(z)}{\delta(p,q+1)z^{p-q-1}}\left(\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right)^{\alpha-1},\ z\in\mathbb{U},$$

and $h(0)=1$. Thus, it is easy to check that

$$\left(\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right)^{\alpha}=\frac{(p-q)\alpha}{z^{(p-q)\alpha}}\int_0^z t^{(p-q)\alpha-1}h(t)dt,\ z\in\mathbb{U},$$

that is

$$\operatorname{Re}\mathrm{I}_{0,p}f^{(q)}(z)=\operatorname{Re}\left(\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right)^{\alpha}=\operatorname{Re}\left[\frac{(p-q)\alpha}{z^{(p-q)\alpha}}\int_0^z t^{(p-q)\alpha-1}h(t)dt\right],\ z=re^{i\theta}. \tag{2.5}$$

Substituting $t=\rho e^{i\theta}$ in (2.5), we have

$$\operatorname{Re}\mathrm{I}_{0,p}f^{(q)}(z)=\frac{(p-q)\alpha}{r^{(p-q)\alpha}}\int_0^r \rho^{(p-q)\alpha-1}\operatorname{Re}h\left(\rho e^{i\theta}\right)d\rho,\ z=re^{i\theta}. \tag{2.6}$$

It is well-known that for $q\in\mathcal{A}$, with $\operatorname{Re}q(z)>0$ for all $z\in\mathbb{U}$, (see [16, p. 532]) the next inequality holds:

$$\operatorname{Re}q(z)\geq\frac{1-r}{1+r},\ |z|=r<1, \tag{2.7}$$

therefore

$$\operatorname{Re}h(z)\geq\frac{\beta}{p-q}+\left(1-\frac{\beta}{p-q}\right)\frac{1-r}{1+r},\ |z|=r<1. \tag{2.8}$$

From (2.6) and (2.8) we obtain

$$\operatorname{Re}\mathrm{I}_{0,p}f^{(q)}(z)\geq\frac{(p-q)\alpha}{r^{(p-q)\alpha}}\int_0^r \rho^{(p-q)\alpha-1}\left[\frac{\beta}{p-q}+\left(1-\frac{\beta}{p-q}\right)\frac{1-\rho}{1+\rho}\right]d\rho$$

$$=\frac{\beta}{p-q}+\left(1-\frac{\beta}{p-q}\right)\left[-1+\frac{2(p-q)\alpha}{r^{(p-q)\alpha}}\int_0^r\frac{\rho^{(p-q)\alpha-1}}{1+\rho}d\rho\right],\ |z|=r<1. \tag{2.9}$$

Taking $\rho=r\varphi$ in (2.9) we deduce

$$\operatorname{Re}\mathrm{I}_{0,p}f^{(q)}(z)\geq\frac{\beta}{p-q}+\left(1-\frac{\beta}{p-q}\right)\left[-1+2(p-q)\alpha\int_0^1\frac{\varphi^{(p-q)\alpha-1}}{1+r\varphi}d\varphi\right],\ |z|=r<1,$$

and using that

$$\int_0^1\frac{\varphi^{(p-q)\alpha-1}}{1+r\varphi}d\varphi=\int_0^1\left[\varphi^{(p-q)\alpha-1}\sum_{s=0}^{\infty}(-1)^s r^s\varphi^s\right]d\varphi=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{(p-q)\alpha+k-1},$$

we have

$$\operatorname{Re} \mathrm{I}_{0,p} f^{(q)}(z) \geq \frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right) \left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k-1+(p-q)\alpha}\right] = \gamma_{p,q}^{0}(r),\ |z| = r < 1.$$

It easy to see that

$$\operatorname{Re} \mathrm{I}_{1,p} f^{(q)}(z) = \operatorname{Re} \left[\frac{1}{z} \int_0^z \mathrm{I}_{0,p} f^{(q)}(t) dt\right] = \frac{1}{r} \int_0^r \operatorname{Re} \mathrm{I}_{0,p} f^{(q)} \left(\rho e^{i\theta}\right) d\rho$$

$$\geq \frac{1}{r} \int_0^r \left\{\frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right) \left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \rho^{k-1}}{k-1+(p-q)\alpha}\right]\right\} d\rho$$

$$= \frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right) \left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k(k-1+(p-q)\alpha)}\right] = \gamma_{p,q}^{1}(r),\ |z| = r < 1,$$

and by mathematical induction, we conclude that

$$\operatorname{Re} \mathrm{I}_{n+1,p} f^{(q)}(z) = \operatorname{Re} \left[\frac{1}{z} \int_0^z \mathrm{I}_{n,p} f^{(q)}(t) dt\right] = \frac{1}{r} \int_0^r \operatorname{Re} \mathrm{I}_{n,p} f^{(q)} \left(\rho e^{i\theta}\right) d\rho$$

$$\geq \frac{1}{r} \int_0^r \left\{\frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right) \left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^n[k-1+(p-q)\alpha]}\right]\right\} d\rho$$

$$= \frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right) \left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^n(k-1+(p-q)\alpha)}\right] = \gamma_{p,q}^{n+1}(r),\ |z| = r < 1.$$

If we define the function $\Phi_{p,q}^{n,\alpha}$ by

$$\Phi_{p,q}^{n,\alpha}(r) = (p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^n[k-1+(p-q)\alpha]},\ 0 < r < 1,$$

according to the result of Thomas [25, page 20] we get $\frac{1}{2} < \Phi_{p,q}^{n,\alpha}(r) < 1$, and this inequality implies our conclusion (2.4). Moreover,

$$r\Phi_{p,q}^{n,\alpha}(r) = \int_0^r \Phi_{p,q}^{n-1,\alpha}(\rho) d\rho,\ n \in \mathbb{N},$$

thus $\left(\Phi_{p,q}^{n,\alpha}(r)\right)' < 0$ and $\gamma_{p,q}^{n}(r)$ decreases with r as $r \to 1$ for fixed n, and increases to 1 when $n \to \infty$ for fixed r, which completes our proof. □

Remark 2.2 *(i) Taking $q = 0$ in Theorem 2.1 we obtain Theorem A of Owa [19];*

(ii) Putting $\beta = q = 0$ in Theorem 2.1 we obtain Theorem B due to Owa [18];

(iii) Taking $\beta = q = 0$ and $p = 1$, in Theorem 2.1 we obtain the result of Thomas [25] and Halim and Thomas [11];

(iv) For $\beta = q = 0$ and $p = \alpha = 1$, Theorem 2.1 reduces to the result of Hallenbeck [12];

(v) Our result of Theorem 2.1 with (i) $q = 0$, (ii) $q = \beta = 0$, (iii) $q = \beta = 0$ and $\alpha = p^{-1}$ $(p \in \mathbb{N})$ improve the results of Owa [19, Lemma 4, Corollaries 3 and 4, respectively].

Putting $q = 0$ and $\alpha = 1$ in Theorem 2.1 we get the following special case:

Corollary 2.3 *If $f \in \mathcal{A}(p)$ satisfies*

$$\operatorname{Re} \frac{f'(z)}{z^{p-1}} > \beta, \; z \in \mathbb{U}, \; (0 \leq \beta < p)$$

then

$$\operatorname{Re} \mathrm{I}_{n,p} f(z) \geq \gamma_p^n(r) > \gamma_p^n(1), \; z \in \mathbb{U}, \; r = |z|, \; (n \in \mathbb{N}_0) \tag{2.10}$$

and

$$\frac{\beta}{p} < \gamma_p^n(r) = \frac{\beta}{p} + \left(1 - \frac{\beta}{p}\right) \left(-1 + 2p \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^n (k-1+p)}\right).$$

The equality in (2.10) is attained for the function

$$f(z) = z^p + 2(p-\beta) \sum_{k=1}^{\infty} (-1)^k \frac{z^{p+k}}{p+k}.$$

Remark 2.4 *Our result of Corollary 2.3 is an improvement of the result of Saitoh [24, Theorem 1, with $j = 1$ and Corollary 2], and of Aouf [6, Theorem 2, with $\alpha = n = 1$]*

For the special case $\alpha = \dfrac{1}{p-q}$, $(p > q)$ Theorem 2.1 reduces to the next special case:

Corollary 2.5 *If $f \in \mathcal{A}(p)$ satisfies*

$$\operatorname{Re} \left[\frac{f^{(1+q)}(z)}{f^{(q)}(z)} \left(\frac{f^{(q)}(z)}{\delta(p,q)} \right)^{\frac{1}{p-q}} \right] > \beta, \; z \in \mathbb{U}, \; (0 \leq \beta < p - q)$$

then

$$\operatorname{Re} \mathrm{I}_{n,p} f^{(q)}(z) \geq \gamma_{p,q}^n(r) > \gamma_{p,q}^n(1), \; z \in \mathbb{U}, \; r = |z|, \; (n \in \mathbb{N}_0) \tag{2.11}$$

and

$$\frac{\beta}{p-q} < \gamma_{p,q}^n(r) = \frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right) \left(-1 + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^{n+1}}\right) < 1.$$

The equality in (2.11) is attained for the function $f \in \mathcal{A}(p)$ given by

$$f^{(q)}(z) = \delta(p,q) \left\{ \left(\frac{2\beta}{p-q} - 1 \right) z + 2 \left(1 - \frac{\beta}{p-q} \right) \log(1+z) \right\}^{p-q}.$$

Remark 2.6 *For the special case $q = 0$, the result of Corollary 2.5 is an improvement of the result due to Owa [19, Corollary 7].*

Putting $p = 1$ and $q = 0$ in Corollary 2.5 we get:

Corollary 2.7 *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} f'(z) > \beta,\ z \in \mathbb{U}, (0 \leq \beta < 1)$$

then

$$\operatorname{Re} \mathrm{I}_n f(z) \geq \gamma_n(r) > \gamma_n(1),\ z \in \mathbb{U},\ r = |z|,\ (n \in \mathbb{N}_0) \tag{2.12}$$

and

$$\beta < \gamma_n(r) = \beta + (1 - \beta)\left(-1 + 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^{n+1}}\right).$$

The equality in (2.12) is attained for the function

$$f(z) = (2\beta - 1)z + 2\,(1 - \beta)\log(1 + z).$$

Remark 2.8 *(i) The result of Corollary 2.7 was also obtained by Owa [19, Corollary 8], Hallenbeck [12, with $n = \beta = 0$], Ling et al. [15, Corollary 3], and Patel and Rout [21, Corollary 3];*

(ii) The above corollary improve the results of Owa and Obradović [20, Theorem 4 with $\alpha = 1$ and Corollary 4], Saitoh [23, Corollary 3], Saitoh [24, Corollary 8 with $\lambda = 1$], and Ponnusamy and Karunakran [22, with $k = m = 1$].

3. Integral operator $\mathbf{J}_n f^{(q)}$

For $f \in \mathcal{A}(p)$, we define the integral operator

$$\mathrm{J}_0 f^{(q)}(z) := \frac{f^{(q)}(z)}{\delta(p, q) z^{p-q}},$$

and

$$\mathrm{J}_n f^{(q)}(z) := \frac{a + 1}{z^{a+1}} \int_0^z t^a \mathrm{J}_{n-1} f^{(q)}(t) dt,\ (a > -1,\ n \in \mathbb{N}).$$

For the operator $\mathrm{J}_n f^{(q)}$ we obtained the next result:

Theorem 3.1 *If $f \in \mathcal{A}(p)$ satisfies*

$$\operatorname{Re} \frac{f^{(q)}(z)}{\delta(p, q) z^{p-q}} > \alpha,\ z \in \mathbb{U},\ (\alpha < 1),$$

then

$$\operatorname{Re} \mathrm{J}_n f^{(q)}(z) \geq \gamma_n(r) > \gamma_n(1),\ z \in \mathbb{U},\ r = |z|,\ (n \in \mathbb{N}_0) \tag{3.1}$$

and

$$0<\gamma_n(r):=1+2(a+1)^n(1-\alpha)\sum_{k=1}^{\infty}\frac{(-r)^k}{(k+a+1)^n}<1.$$

The equality in (3.1) is attained for the function $f\in\mathcal{A}(p)$ *given by*

$$f^{(q)}(z)=\delta(p,q)z^{p-q}\left[\alpha+(1-\alpha)\frac{1-z}{1+z}\right].$$

Proof For $n=0$ the implication is trivial. For $n=1$, if we denote

$$g(z)=\frac{1}{\alpha}\left[\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}-\alpha\right],\ z\in\mathbb{U},$$

then, from our assumption we have $\operatorname{Re} g(z)>0$, $z\in\mathbb{U}$, and $g(0)=1$. Using the inequality (2.7) for the function g and letting $z=re^{i\theta}$ and $t=\rho e^{i\theta}$, for $a>-1$ we get

$$\operatorname{Re}\mathrm{J}_1 f^{(q)}(z)=\operatorname{Re}\left(\frac{a+1}{z^{a+1}}\int_0^z t^a\mathrm{J}_0 f^{(q)}(t)dt\right)\geq\frac{a+1}{r^{a+1}}\int_0^r\rho^a\left[\alpha+(1-\alpha)\frac{1-\rho}{1+\rho}\right]d\rho$$

$$=\frac{a+1}{r^{a+1}}\int_0^r\rho^a\left[1+2(1-\alpha)\sum_{k=1}^{\infty}(-\rho)^k\right]d\rho=1+\frac{2(a+1)(1-\alpha)}{r^{a+1}}\int_0^r\sum_{k=1}^{\infty}(-1)^k\rho^{k+a}d\rho$$

$$=1+2(a+1)(1-\alpha)\sum_{k=1}^{\infty}\frac{(-r)^k}{k+a+1},\ |z|=r,$$

thus (3.1) holds for $n=1$. Further, assuming that (3.1) holds for a fixed $n\in\mathbb{N}$, we have

$$\operatorname{Re}\mathrm{J}_{n+1}f^{(q)}(z)=\operatorname{Re}\left(\frac{a+1}{z^{a+1}}\int_0^z t^a\mathrm{J}_n f^{(q)}(t)dt\right)=\frac{a+1}{r^{a+1}}\int_0^r\rho^a\operatorname{Re}\mathrm{J}_n f^{(q)}\left(\rho e^{i\theta}\right)d\rho$$

$$\geq\frac{a+1}{r^{a+1}}\int_0^r\left(\rho^a+2(a+1)^n(1-\alpha)\sum_{k=1}^{\infty}\frac{(-1)^k\rho^{k+a}}{(k+a+1)^n}\right)d\rho$$

$$=1+2(a+1)^{n+1}(1-\alpha)\sum_{k=1}^{\infty}\frac{(-r)^k}{(k+a+1)^{n+1}}=\gamma_{n+1}(r),\ |z|=r.$$

Moreover, it is easy to see that $0<\gamma_n<1$, which completes our proof. □

Taking $\alpha=\dfrac{p-q}{p-q+\beta}$, $(p>q,\ \beta>0)$ in the above theorem we get the next special case:

Corollary 3.2 *If* $f\in\mathcal{A}(p)$ *satisfies*

$$\operatorname{Re}\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}>\frac{p-q}{p-q+\beta},\ z\in\mathbb{U},\ (\beta>0),$$

then

$$\operatorname{Re} \mathrm{J}_n f^{(q)}(z) \geq \gamma_n(r) > \gamma_n(1),\ z \in \mathbb{U},\ r = |z|,\ (n \in \mathbb{N}_0) \tag{3.2}$$

and

$$0 < \gamma_n(r) := 1 + \frac{2\beta(a+1)^n}{p-q+\beta} \sum_{k=1}^{\infty} \frac{(-r)^k}{(k+a+1)^n} < 1.$$

The equality in (3.2) is attained for the function $f \in \mathcal{A}(p)$ given by

$$f^{(q)}(z) = \frac{\delta(p,q)z^{p-q}}{p-q+\beta} \frac{1+\beta+(1-\beta)z}{1+z}.$$

Remark 3.3 *Putting $q = 0$ in Theorem 3.1 and in Corollary 3.2 we obtain the results of Owa [18, Theorem 2 and Corollary 4] and Owa [19, Theorem 2 and Corollary 10].*

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