On \(q\)- and \(h\)-deformations of 3d-superspaces

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Abstract: In this paper, we introduce nonstandard deformations of \((1+2)\)- and \((2+1)\)-superspaces via a contraction using standard deformations of them. This deformed superspaces are denoted by \(\mathbb{A}^{1|2}_h\) and \(\mathbb{A}^{2|1}_{h'}\), respectively. We find a two-parameter \(R\)-matrix satisfying quantum Yang–Baxter equation and thus obtain a new two-parameter nonstandard deformation of the supergroup \(\text{GL}(1|2)\). Finally, we get a new superalgebra derived from the Hopf superalgebra of functions on the quantum superspace \(\mathbb{A}^{1|2}_{p,q}\).

Key words: Quantum superspace, Hopf superalgebra, quantum supergroup, quantum Lie superalgebra, super \(*\)-algebra

1. Introduction

There are two distinct deformations for general Lie (super)groups as standard and nonstandard (or Jordanian). One of them is the well-known quantum (\(q\)-deformed) group and the other is the so-called Jordanian (\(h\)-deformed) one. Specially, quantum groups \(\text{GL}_q(2)\) \([10]\) and \(\text{GL}_h(2)\) \([9]\) have been obtained by deforming the coordinates of a plane to be noncommutative objects. In \([1]\), the authors showed that the \(h\)-deformed group can be obtained from the \(q\)-deformed Lie group through a singular limit \(q \to 1\) of a linear transformation. This method is known as the contraction procedure. Using this method, one- and two-parameter \(h\)-deformations of supergroup \(\text{GL}(1|1)\) were obtained in \([7]\) and \([2]\), respectively.

In this paper, we give some standard (as \(q\)-deformation) deformations of \((1+2)\)-superspace using the Hopf superalgebra structure of \(\mathcal{O}(\mathbb{A}^{1|2})\) and nonstandard (as \(h\)-deformation) deformations using standard deformations via a contraction. We also introduce an \((h, h')\)-deformed supergroup acting on these two-parameter \(h\)-deformed superspaces. Finally, we define involutions on \(h\)-deformed superspaces and use the generators of \((p, q)\)-deformed superalgebra \(\mathcal{O}(\mathbb{A}^{12}_{p,q})\) to get a new Lie superalgebra.

Throughout the paper, we will fix a base field \(\mathbb{K}\). The reader may consider it as the set of real numbers, \(\mathbb{R}\), or the set of complex numbers, \(\mathbb{C}\). We will denote by \(\mathbb{G}\) the Grassmann numbers and by \(\mathbb{K}'\) the set \(\mathbb{K} \cup \mathbb{G}\).

2. On \((p, q)\)-deformation of superspaces \(\mathbb{A}^{1|2}\) and \(\mathbb{A}^{2|1}\)

In order to define superalgebras and Hopf superalgebras, some minor changes are made in familiar definitions. These are briefly mentioned in the following.

A supervector space \(\mathcal{X}\) over a field \(\mathbb{K}\) is a \(\mathbb{Z}_2\)-graded vector space \(\mathcal{X}\) together with two subspaces \(\mathcal{X}_0\) and \(\mathcal{X}_1\) of \(\mathcal{X}\) such that \(\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1\). If a space \(\mathcal{X}\) is a superspace, then we denote by \(\tau(a)\) the \(\mathbb{Z}_2\)-grade of

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the element $a \in X$. If $\tau(a) = 0$, then we will call the element $a$ even and if $\tau(a) = 1$, it is called odd.

If $f : X \rightarrow Y$ is a linear map of supervector spaces and it satisfies

$$\tau(f(v)) = \tau(f) + \tau(v) \pmod{2}$$

for all $v \in X$, then $f$ is called a supervector space homomorphism.

A superalgebra (or $\mathbb{Z}_2$-graded algebra) $A$ over $K$ is a supervector space over $K$ with a map $A \times A \rightarrow A$ such that $A_i \cdot A_j \subset A_{i+j}$ for $i, j = 0, 1$. The superalgebra $A$ is called supercommutative if

$$ab = (-1)^{\tau(a)\tau(b)}ba$$

for homogeneous elements $a, b \in A$.

Let $f : A \rightarrow B$ be a map of definite degree of superalgebras. If it is a supervector space homomorphism and it obeys

$$f(ab) = (-1)^{\tau(a)\tau(f)}f(a)f(b), \quad \forall a, b \in A,$$

then $f$ is called a superalgebra homomorphism.

2.1. The algebra of polynomials on the quantum superspace $\mathbb{A}_q^{1|2}$

Let $K\langle X, \Theta_1, \Theta_2 \rangle$ be a free algebra with unit generated by $X$, $\Theta_1$, and $\Theta_2$, where the coordinate $X$ is even and the coordinates $\Theta_1$ and $\Theta_2$ are odd.

**Definition 2.1** [11] Let $I_q$ be the two-sided ideal of $K\langle X, \Theta_1, \Theta_2 \rangle$ generated by the elements $X\Theta_1 - q\Theta_1 X$, $X\Theta_2 - q\Theta_2 X$, $\Theta_1\Theta_2 + q^{-1}\Theta_2\Theta_1$, $\Theta_1^2$, and $\Theta_2^2$. The quantum superspace $\mathbb{A}_q^{1|2}$ with the function algebra

$$O(\mathbb{A}_q^{1|2}) = K\langle X, \Theta_1, \Theta_2 \rangle / I_q$$

is called $\mathbb{Z}_2$-graded quantum space (or quantum superspace).

This associative algebra over the complex number is known as the algebra of polynomials over quantum $(1+2)$-superspace. In accordance with the above definition, we have

$$X\Theta_i = q\Theta_i X, \quad \Theta_i\Theta_j = -q^{i-j}\Theta_j\Theta_i, \quad (i, j = 1, 2) \quad (2.1)$$

where $q \in K - \{0\}$.

**Example 2.2** If we consider the generators of the algebra $O(\mathbb{A}_q^{1|2})$ as linear maps, then we can find the matrix representations of them. In fact, it can be seen that there exists a representation $\rho : O(\mathbb{A}_q^{1|2}) \rightarrow M(3, K')$ such that matrices

$$\rho(X) = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^2 \end{pmatrix}, \quad \rho(\Theta_1) = \begin{pmatrix} 0 & 0 & \varepsilon_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(\Theta_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.2)$$

representing the coordinate functions satisfy relations (2.1) for all $\varepsilon_1, \varepsilon_2$.

**Remark 2.3** In the next section, we will assume that $\varepsilon_1$ and $\varepsilon_2$ are two Grassmann numbers.
The following definition gives the product rule for tensor product of \( \mathbb{Z}_2 \)-graded algebras.

**Definition 2.4** The product rule is defined by

\[
(a_1 \otimes a_2)(a_3 \otimes a_4) = (-1)^{r(a_2)r(a_3)}(a_1a_3 \otimes a_2a_4)
\]

in the \( \mathbb{Z}_2 \)-graded algebra \( A \otimes A \), where \( A \) is the \( \mathbb{Z}_2 \)-graded algebra and \( a_i \)'s are homogeneous elements in \( A \).

A Hopf superalgebra is a supervector space \( A \) over \( \mathbb{K} \) with two algebra homomorphisms \( \Delta : A \rightarrow A \otimes A \), called the coproduct, \( \epsilon : A \rightarrow \mathbb{K} \), called the counit, and an algebra antihomomorphism \( S : A \rightarrow A \), called the antipode, such that

\[
\begin{align*}
(\Delta \otimes \text{id}) \circ \Delta &= (\text{id} \otimes \Delta) \circ \Delta, \\
m \circ (\epsilon \otimes \text{id}) \circ \Delta &= \text{id} = m \circ (\text{id} \otimes \epsilon) \circ \Delta, \\
m \circ (S \otimes \text{id}) \circ \Delta &= \eta \circ \epsilon = m \circ (\text{id} \otimes S) \circ \Delta,
\end{align*}
\]

and \( \Delta(1) = 1 \otimes 1 \), \( \epsilon(1) = 1 \), \( S(1) = 1 \), where \( m \) is the multiplication map, id is the identity map and \( \eta : \mathbb{K} \rightarrow A \).

*Note.* An element of a Hopf superalgebra \( A \) is expressed as a product on the generators and its antipode \( S \) is calculated with the property

\[
S(ab) = (-1)^{r(a)r(b)}S(b)S(a), \quad \forall a, b \in A.
\]

We denote the unital extension of \( \mathcal{O}(A_q^{1/2}) \) by \( \mathcal{F}(A_q^{1/2}) \) adding the unit and \( x^{-1} \), the inverse of \( x \), which obeys \( xx^{-1} = 1 = x^{-1}x \). The following theorem says that the superalgebra \( \mathcal{F}(A_q^{1/2}) \) has a Hopf algebra structure [4]:

**Theorem 2.5** [4] The superalgebra \( \mathcal{F}(A_q^{1/2}) \) is a Hopf superalgebra with the defining coproduct, counit, and antipode on the algebra \( \mathcal{F}(A_q^{1/2}) \) as follows:

1. **The coproduct** \( \Delta : \mathcal{F}(A_q^{1/2}) \rightarrow \mathcal{F}(A_q^{1/2}) \otimes \mathcal{F}(A_q^{1/2}) \) is defined by

\[
\Delta(X) = X \otimes X, \quad \Delta(\Theta_1) = \Theta_1 \otimes X + X \otimes \Theta_1, \quad \Delta(\Theta_2) = \Theta_2 \otimes X^2 + X^2 \otimes \Theta_2.
\]

2. **The counit** \( \epsilon : \mathcal{F}(A_q^{1/2}) \rightarrow \mathbb{K} \) is given by

\[
\epsilon(X) = 1, \quad \epsilon(\Theta_i) = 0, \quad (i = 1, 2).
\]

3. **The algebra** \( \mathcal{F}(A_q^{1/2}) \) admits a \( \mathbb{K} \)-algebra antihomomorphism (antipode) \( S : \mathcal{F}(A_q^{1/2}) \rightarrow \mathcal{F}(A_q^{-1/2}) \) defined by

\[
S(X) = X^{-1}, \quad S(\Theta_1) = -X^{-1}\Theta_1X^{-1}, \quad S(\Theta_2) = -X^{-2}\Theta_2X^{-2}.
\]

### 2.2. The algebra of polynomials on the quantum superspace \( \mathbb{A}_{p,q}^{2,1} \)

Let \( \mathbb{K} \langle \Phi, Y_1, Y_2 \rangle \) be a free algebra with unit generated by \( \Phi \), \( Y_1 \) and \( Y_2 \), where \( \tau(\Phi) = 1 \) and \( \tau(Y_1) = 0 = \tau(Y_2) \).
Definition 2.6 [5] Let $\Lambda(\mathbb{A}_q^{1|2})$ be the algebra with the generators $\Phi$, $Y_1$, and $Y_2$ satisfying the relations
\begin{equation}
\Phi^2 = 0, \quad \Phi Y_1 = q p^{-1} Y_1 \Phi, \quad \Phi Y_2 = p q Y_2 \Phi, \quad Y_1 Y_2 = p q^{-1} Y_2 Y_1.
\end{equation}
We call $\Lambda(\mathbb{A}_q^{1|2})$ exterior algebra of the $\mathbb{Z}_2$-graded space $\mathbb{A}_q^{1|2}$.

Remark 2.7 The exterior algebra $\Lambda(\mathbb{A}_q^{1|2})$ of the superspace $\mathbb{A}_q^{1|2}$ can be thought of as a two-parameter deformation of the $(2+1)$-superspace $\mathbb{A}^{2|1}$. Thus, we denote this algebra by $\mathcal{O}(\mathbb{A}_{p,q}^{2|1})$.

Example 2.8 If we consider the generators of the algebra $\mathcal{O}(\mathbb{A}_{p,q}^{2|1})$ as linear maps, then we can find the matrix representations of them. In fact, it can be seen that there exists a representation $\rho: \mathcal{O}(\mathbb{A}_{p,q}^{2|1}) \rightarrow M(3, \mathbb{K}')$ such that matrices
\begin{align*}
\rho(\Phi) &= \begin{pmatrix} 0 & 0 & \epsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\rho(Y_1) &= \begin{pmatrix} q & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}, \\
\rho(Y_2) &= \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{align*}
representing the coordinate functions satisfy relations (2.4) for all $c, \epsilon$.

3. Two-parameter $h$-deformation of the superspaces
In this section, we introduce a two-parameter $h$-deformation of the superspace $\mathbb{A}_h^{1|2}$ (and its dual) from the $(p, q)$-deformation via a contraction similar to the method of [1].

We consider the $q$-deformed algebra of functions on the quantum superspace $\mathbb{A}_q^{1|2}$ generated by $X$, $\Theta_1$, and $\Theta_2$ with the relations (2.1) and we introduce new even coordinate $x$ and odd coordinates $\theta_1$, $\theta_2$ with the change of basis in the coordinates of the $q$-superspace using the following $g$ matrix:
\begin{equation}
X = \begin{pmatrix} X \\ \Theta_1 \\ \Theta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tilde{h}' \\ 0 & 1 & 0 \\ \tilde{h} & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \theta_1 \\ \theta_2 \end{pmatrix} = g x, \quad \tilde{h} = \frac{h}{q-1}, \quad \tilde{h}' = \frac{h'}{pq-1}
\end{equation}
where $h$ and $h'$ ($h \neq 0 \neq h'$) are two new deformation parameters that will be replaced with $q$ and $p$ ($q \neq 1 \neq pq$) in the limits $q \rightarrow 1$ and $p \rightarrow 1$.

We now assume that the parameters $h$ and $h'$ are both Grassmann numbers ($h^2 = 0 = h'^2$, $hh' = -h'h$) and anticommute with $\theta_i$ for $i = 1, 2$. When the relations (2.1) are used, one gets
\begin{equation}
x \theta_1 = q \theta_1 x, \quad x \theta_2 = q \theta_2 x + hx, \quad \theta_2 \theta_1 = -q \theta_1 \theta_2, \quad \theta_1^2 = 0, \quad \theta_2^2 = -h \theta_2 x.
\end{equation}
Note that the parameter $h'$ does not enter the above relations. By taking the limit $q \rightarrow 1$, we obtain the following exchange relations, which define the $h$-superspace $\mathbb{A}_h^{1|2}$:

Definition 3.1 [4] Let $\mathcal{O}(\mathbb{A}_h^{1|2})$ be the algebra with the generators $x$, $\theta_1$, and $\theta_2$ satisfying the relations
\begin{equation}
x \theta_1 = \theta_1 x, \quad x \theta_2 = \theta_2 x + hx, \quad \theta_2 \theta_1 = -\theta_2 \theta_1, \quad \theta_1^2 = 0, \quad \theta_2^2 = -h \theta_2 x.
\end{equation}
We call $\mathcal{O}(\mathbb{A}_h^{1|2})$ the algebra of functions on the $\mathbb{Z}_2$-graded quantum space $\mathbb{A}_h^{1|2}$.
Example 3.2 Let us assume that $\varepsilon_1$ and $\varepsilon_2$ are two Grassmann numbers. If the $g$ matrix in (3.1) is used, the matrix representation in (2.2) takes the following form:

$$\rho(x) = q \left( \begin{array}{cccccc}
1 - \tilde{h}h' & 0 & 0 \\
0 & 1 - \tilde{h}h' & -q^{-1}\tilde{h}'\varepsilon_2 \\
0 & 0 & q(1 - \tilde{h}h')
\end{array} \right), \quad \rho(\theta_1) = \left( \begin{array}{ccc}
0 & 0 & \varepsilon_1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right), \quad \rho(\theta_2) = -\left( \begin{array}{ccc}
q\tilde{h} & 0 & 0 \\
0 & q\tilde{h} & -(1 + \tilde{h}h')\varepsilon_2 \\
0 & 0 & q^2\tilde{h}
\end{array} \right).$$

(3.4)

These matrices satisfy the relations (3.2), for all $\varepsilon_1$ and $\varepsilon_2$.

Proof Existing claims come from the fact that $\rho$ is an algebra homomorphism. $\square$

In the case of dual (exterior) $h'$-superspace, we use the transformation

$$\hat{X} = g\hat{x}$$

(3.5)

with the components $\varphi$, $y_1$, and $y_2$ of $\hat{x}$. The definition is given below.

Definition 3.3 Let $\mathcal{O}(\mathbb{A}_{h'}^{2|1}) := \Lambda(\mathbb{A}_h^{1|2})$ be the algebra with the generators $\varphi$, $y_1$, and $y_2$ satisfying the relations

$$\varphi y_1 = y_1\varphi, \quad \varphi y_2 = y_2\varphi + h' y_2^2, \quad y_1 y_2 = y_2 y_1, \quad \varphi^2 = h' y_2\varphi$$

(3.6)

where $\tau(\varphi) = 1$ and $\tau(y_1) = 0 = \tau(y_2)$. We call $\Lambda(\mathbb{A}_h^{1|2})$ the quantum exterior algebra of the $\mathbb{Z}_2$-graded quantum space $\mathbb{A}_h^{1|2}$.

Remark 3.4 The parameter $h$ does not enter the relations (3.6). The exterior algebra $\Lambda(\mathbb{A}_h^{1|2})$ of the superspace $\mathbb{A}_h^{1|2}$ can be thought of as an $h'$-deformation of the $(2+1)$-superspace $A^{2|1}$.

4. An $R$-matrix and its properties

The relations in (2.1) can be written in a compact form as follows:

$$p \mathbf{X} \otimes \mathbf{X} = \hat{R}_{p,q} \mathbf{X} \otimes \mathbf{X}$$

(4.1)

with an $R$-matrix given by [6]

$$\hat{R}_{p,q} = \left( \begin{array}{cccccccccc}
p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p - 1 & 0 & q & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & pq & 0 \\
0 & pq^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -pq^{-1} & 0 \\
0 & q^{-1} & 0 & 0 & 0 & p - 1 & 0 & 0 \\
0 & 0 & 0 & -q & 0 & p - 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array} \right)$$

where $p, q \in \mathbb{K} - \{0\}$. This matrix satisfies the graded braid equation and the matrix $R_{p,q} = P \hat{R}_{p,q}$ satisfies the graded Yang–Baxter equation where $P$ is the super permutation matrix.
It can be considered that a change of basis in the quantum superspaces leads to a two-parameter $R$-matrix. The corresponding $R$-matrix can be obtained as

$$
\hat{R}_{h,h'} = \lim_{(p,q)\to (1,1)} \left[ (g \otimes g)^{-1} \hat{R}_{p,q}(g \otimes g) \right]
$$

where it is assumed that $\otimes$ is graded. As a result, we obtain the following $R$-matrix

$$\hat{R}_{h,h'} = \begin{pmatrix}
1 + hh' & 0 & h' & 0 & 0 & 0 & -h' & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
h & 0 & hh' & 0 & 0 & 0 & 1 & 0 & -h' \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-h & 0 & 1 & 0 & 0 & 0 & hh' & 0 & -h' \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -h & 0 & 0 & 0 & -h & 0 & hh' - 1
\end{pmatrix}.
$$

The equation in (4.1) with the new $R$-matrix $\hat{R}_{h,h'}$ takes the form

$$x \otimes x = \hat{R}_{h,h'} x \otimes x,$$

that is, the relations (3.3) are equivalent to this equation.

The $R$-matrix $\hat{R}_{h,h'}$ has some interesting properties. Some of them are listed below, where sometimes we write $\hat{R} = \hat{R}_{h,h'}$ for simplicity.

1. The matrix $\hat{R}_{h,h'}$ satisfies the graded braid equation $\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}$, where $\hat{R}_{12} = \hat{R} \otimes I_3$ and $\hat{R}_{12} = I_3 \otimes \hat{R}$.

2. The matrix $R_{h,h'} = P \hat{R}_{h,h'}$ satisfies the graded Yang–Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12},$$

where $R_{13}$ acts both on the first and third spaces.

3. The matrix $\hat{R}_{h,h'}$ holds $\hat{R}_{h,h'}^2 = I_9$; thus, it has two eigenvalues $\pm 1$.

4. If we set $hh' = 0$, then the matrix $R_{h,h'}$ can be decomposed in the form

$$R_{h,h'} = R(h) R(h')$$

where

$$R(h) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-h & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
h & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & h & 0 & 0 & 0 & h & 0 & 1
\end{pmatrix}, \quad R(h') = R^t(h)|_{h=h'}.
$$

It can be checked that these matrices satisfy the graded Yang–Baxter equation.
5. If $P_{\pm}$ are the projections onto the eigenspaces $\pm 1$ of $\hat{R}_{h,h'}$, then we have

$$\hat{R}_{h,h'} = P_+ - P_-.$$ 

Let $O(A_{1,2}^{h} \mid 2)$ and $O(A_{2,1}^{h} \mid 1)$ be the quotients of algebras generated by $x, \theta_1, \theta_2$ and $\varphi, y_1, y_2$ modulo the two-sided ideals generated by $\text{Ker}P_-$ and $\text{Ker}P_+$, respectively. Then $O(A_{1,2}^{h})$ and $O(A_{2,1}^{h})$ are isomorphic to $O(\mathbb{A}_h^{1\mid 2})$ with defining relations (3.3) and $O(\mathbb{A}_{h'}^{2\mid 1})$ with defining relations (3.6), respectively. That is, we can write

$$P_- x \otimes x = 0 \quad \text{and} \quad (-1)^{\tau(x)} P_+ \hat{x} \otimes \hat{x} = 0.$$

5. The quantum superbialgebra $O(M_{h,h'}(1\mid 2))$

Let $T$ be a 3x3 matrix in $\mathbb{Z}_2$-graded space given by

$$T = \begin{pmatrix} a & \alpha & \beta \\ \gamma & b & c \\ \delta & d & e \end{pmatrix} = (t_{ij})$$

where $a, b, c, d, e$ are even and $\alpha, \beta, \gamma, \delta$ are odd. The coordinate ring of such matrices over a field $\mathbb{K}$ is simply the polynomial ring in nine variables, that is $O(M(1\mid 2)) = \mathbb{K}[a, b, c, d, e, \alpha, \beta, \gamma, \delta]$.

In this section, we will assume that the matrix entries of $T$ belong to a free superalgebra and define a two-parameter $h$-analogue of $O(M(1\mid 2))$. To do so, let $x, \theta_1, \theta_2$ be elements of the superalgebra $O(\mathbb{A}_h^{1\mid 2})$ subject to the relations (3.3) and $\varphi, y_1, y_2$ be elements of $O(\mathbb{A}_{h'}^{2\mid 1})$ subject to the relations (3.6), and $t_{ij}$ be nine generators which supercommute with the elements of $O(\mathbb{A}_h^{1\mid 2})$ and $O(\mathbb{A}_{h'}^{2\mid 1})$. It is well known that the supermatrix $T$ defines the linear transformations $T : \mathbb{A}_h^{1\mid 2} \rightarrow \mathbb{A}_h^{1\mid 2}$ and $T : \mathbb{A}_{h'}^{2\mid 1} \rightarrow \mathbb{A}_{h'}^{2\mid 1}$. Let $\mathbf{x} = (x, \theta_1, \theta_2)^t$ and $\hat{\mathbf{x}} = (\varphi, y_1, y_2)^t$. Thus, we can give the following theorem.

**Theorem 5.1** Under the above hypotheses, the following conditions are equivalent:

(i) $T\mathbf{x} = \mathbf{x}' \in \mathbb{A}_h^{1\mid 2}$ and $T\hat{\mathbf{x}} = \hat{\mathbf{x}}' \in \mathbb{A}_{h'}^{2\mid 1}$,
(ii) the relations are satisfied

\[ aα = (1 + hh')αα - h'(αδ + da), \quad aβ = βα + h'(a^2 - ea - βδ) - hβ^2, \]
\[ aγ = (1 + hh')γa + h(γβ - ca), \quad ac = ca - hcβ - h'γa + hh'γβ, \]
\[ aδ = δa + h(a^2 - ea + δβ) + h'δ^2, \quad ad = da + hao + h'dδ - hh'aδ, \]
\[ ae = ea + hβ(a - e) + h'(e - a)δ, \quad αβ = -(1 + hh')βα + h'(βd + ea), \]
\[ αγ = -γα, \quad ac = ca, \quad αδ = -δα - hao + h'δd - hh'ad, \]
\[ αd = da + h'd^2, \quad ae = ea + hβα + h'ed - hh'dβ, \]
\[ βγ = -γβ + hcβ - h'γa - hh'ca, \quad βc = (1 - hh')cβ - h'(γβ + ca), \]
\[ βδ = -δβ + (hβ + h'δ)(e - a), \quad βd = dβ + haβ + h'de - hh'ea, \]
\[ βe = eβ + h'(e^2 - ea - δβ) - hβ^2, \quad γc = cγ + he^2, \]
\[ γd = dγ, \quad γe = cγ + hec - h'dγ - hh'cδ, \quad cd = dc, \quad ee = (1 - hh')ec + h'(eγ - δc), \]
\[ δe = eδ + h(e^2 - ea + βδ) + h'd^2, \quad de = (1 - hh')ed + h(βd - ea), \]
\[ δc = cδ - hδc, \quad cδ = hδc, \quad δc = hδc, \quad δe = hδ(e - a), \]
\[ b_{ij} = t_{ij}b, \quad a(hβ + h'δ) = (hβ + h'δ)a, \quad e(hβ + h'δ) = (hβ + h'δ)e. \]

**Proof** A direct verification shows that the relations (5.1) respect the ideals defining \( A_h^{12} \) and \( A_{h'}^{21} \).

Standard FRT construction [8], namely, the relations (5.1), is obtained via the matrix \( \hat{R}_{h,h'} \) given in Section 4:

**Theorem 5.2** A 3x3-matrix \( T \) is a \( \mathbb{Z}_2 \)-graded quantum supermatrix if and only if

\[ \hat{R}_{h,h'}T_1T_2 = T_1T_2\hat{R}_{h,h'} \]

where \( T_1 = T \otimes I_3 \) and \( T_2 = PT_1P \).

**Definition 5.3** The superalgebra \( \mathcal{O}(M_{h,h'}(1|2)) \) is the quotient of the free algebra \( \mathbb{K}\{a,b,c,d,e,α,β,γ,δ\} \) by the two-sided ideal \( J_{h,h'} \) generated by the relations (5.1) of Theorem 5.1.

**Remark 5.4** The quantum matrix space \( M_{p,q}(1|2) \) is obtained in [6]. It is clear that a change of basis in the quantum superspace leads to the similarity transformation \( T = g^{-1}T'g \), where \( T' \in M_{p,q}(1|2) \). Therefore, the entries of the transformed quantum matrix \( T \) fulfill the commutation relations (5.1) of the matrix elements of the matrix \( T \) in \( M(1|2) \).

**Theorem 5.5** The superalgebra \( \mathcal{O}(M_{h,h'}(1|2)) \) with the following two algebra homomorphisms of superalgebras

1. the coproduct \( Δ : \mathcal{O}(M_{h,h'}(1|2)) \to \mathcal{O}(M_{h,h'}(1|2)) \otimes \mathcal{O}(M_{h,h'}(1|2)) \) determined by \( Δ(t_{ij}) = \sum_{k=1}^{3} t_{ik} \otimes t_{kj} \),
(2) the counit \( \epsilon : O(M_{h,h'}(1|2)) \to \mathbb{K} \) determined by \( \epsilon(t_{ij}) = \delta_{ij} \) becomes a super bialgebra.

**Proof** It can be easily checked the properties of the costructures hold:

(i) The coproduct \( \Delta \) is coassociative in the sense of

\[
(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta
\]

where \( \text{id} \) denotes the identity map on \( M_{h,h'}(1|2) \) and \( \Delta(ab) = \Delta(a)\Delta(b), \Delta(1) = 1 \otimes 1 \).

(ii) The counit \( \epsilon \) has the property

\[
m \circ (\epsilon \otimes \text{id}) \circ \Delta = \text{id} = m \circ (\text{id} \otimes \epsilon) \circ \Delta
\]

where \( m \) stands for the algebra product and \( \epsilon(ab) = \epsilon(a)\epsilon(b), \epsilon(1) = 1 \).

It is well known that \( O(A_{1|2}) \) is comodule algebra over the bialgebra \( O(M(1|2)) \). The following theorem gives a quantum version of this fact.

**Theorem 5.6** There exist algebra homomorphisms

\[
\delta_L : O(A_{h}^{1|2}) \to O(M_{h,h'}(1|2)) \otimes O(A_{h}^{1|2}), \quad \delta_L(x_i) = \sum_{k=1}^{3} t_{ik} \otimes x_k,
\]

\[
\tilde{\delta}_L : O(A_{h'}^{2|1}) \to O(M_{h,h'}(1|2)) \otimes O(A_{h'}^{2|1}), \quad \tilde{\delta}_L(\hat{x}_i) = \sum_{k=1}^{3} t_{ik} \otimes \hat{x}_k
\]

where \( x_i \in \{x, \theta_1, \theta_2\} \) and \( \hat{x}_i \in \{\varphi, y_1, y_2\} \).

**Proof** Using the relations (3.3) and (3.6) together with (5.1), it is enough to check that

\[
\delta_L(x_1 \theta_1 - \theta_1 x) = \delta_L(x)\delta_L(\theta_1) - \delta_L(\theta_1)\delta_L(x) = 0,
\]

etc., in \( O(M_{h,h'}(1|2)) \otimes O(A_{h}^{1|2}) \). To see that \( \delta_L \) defines a comodule structure we check that

\[
(\Delta \otimes \text{id}) \circ \delta_L = (\text{id} \otimes \delta_L) \circ \delta_L, \quad m \circ (\epsilon \otimes \text{id}) \circ \delta_L = \text{id}.
\]

A quantum supergroup (Hopf superalgebra) can be regarded as a generalization of the notion of a supergroup. It is defined by

\[
O(GL_{h,h'}(1|2)) = O(M_{h,h'}(1|2))[t]/(ts\text{det}_{h,h'} - 1).
\]

This case is also inviting to generalize the corresponding notions of differential geometry [12]. A differential calculus on \( O(GL_{h,h'}(1|2)) \) will be discussed in the next work.

6. A Lie superalgebra derived from \( F(A_{h}^{1|2}) \)

It is known that an element of a Lie group can be represented by exponential of an element of its Lie algebra. In [3], by virtue of this fact, using the generators of the superalgebra \( F(A_{h}^{1|2}) \), a new superalgebra is obtained
from this algebra. In this section, we will obtain a new superalgebra from \( \mathcal{F}(A_{p,q}^{1/2}) \). Thus, let us begin with the definition of \( \mathcal{F}(A_{p,q}^{1/2}) \) which is an extension to two parameters of \( \mathcal{F}(A_{q}^{1/2}) \).

**Definition 6.1** Let \( I_{p,q} \) be the two-sided ideal of \( \mathbb{K}(X, \Theta_1, \Theta_2) \) generated by the elements \( X\Theta_1 - q\Theta_1X, X\Theta_2 - p\Theta_2X, \Theta_1\Theta_2 + pq^{-2}\Theta_2\Theta_1, \Theta_1^2, \) and \( \Theta_2^2 \). The quantum superspace \( A_{p,q}^{1/2} \) with the function algebra

\[ \mathcal{O}(A_{p,q}^{1/2}) = \mathbb{K}(X, \Theta_1, \Theta_2)/I_{p,q} \]

is called quantum superspace.

In accordance with this definition, we have

\[ X\Theta_1 = q\Theta_1X, \quad X\Theta_2 = p\Theta_2X, \quad \Theta_1\Theta_2 = -pq^{-2}\Theta_2\Theta_1, \quad \Theta_1^2 = 0 \quad (6.1) \]

where \( p, q \in \mathbb{K} \setminus \{0\} \).

**Example 6.2** If we consider the generators of the algebra \( \mathcal{O}(A_{p,q}^{1/2}) \) as linear maps, then we can find the matrix representations of them. In fact, it can be seen that there exists a representation \( \rho : \mathcal{O}(A_{p,q}^{1/2}) \rightarrow M(3, \mathbb{K}') \) such that matrices

\[
\rho(X) = \begin{pmatrix} q & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & pq \end{pmatrix}, \quad \rho(\Theta_1) = \begin{pmatrix} 0 & 0 & \varepsilon_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(\Theta_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 \\ 0 & 0 & 0 \end{pmatrix}
\]

representing the coordinate functions satisfy relations (6.1) for all \( \varepsilon_1, \varepsilon_2 \).

Let \( \mathbb{K}(u, \xi_1, \xi_2) \) be a free algebra generated by \( u, \xi_1, \xi_2 \), where \( \tau(u) = 0, \tau(\xi_1) = 1 = \tau(\xi_2) \). Let \( \mathcal{L} \) be the quotient of the free algebra \( \mathbb{K}(u, \xi_1, \xi_2) \) by the two-sided ideal \( J_0 \) generated by the elements \( u\xi_k - \xi_ku, \xi_1\xi_2 + \xi_2\xi_1, \xi_k^2 \) for \( k = 1, 2 \).

Now, we will show that the Hopf superalgebra of Theorem 2.5 can be embedded into the enveloping superalgebra of a Lie superalgebra, with Lie structure and a deformed coproduct. Thus, let us define the generators of the algebra \( \mathcal{F}(A_{p,q}^{1/2}) \) as

\[ X := e^u, \quad \Theta_k := e^{\xi_k^u} \]

for \( k = 1, 2 \). The first equality implies that the generator \( X \) is invertible. Then, by direct calculations we can prove the following lemma.

**Lemma 6.3** The generators \( u, \xi_1, \xi_2 \) have the following commutation relations (Lie (anti-)brackets), for \( j, k = 1, 2 \)

\[ [u, \xi_k] = \imath h_k \xi_k, \quad [\xi_j, \xi_k]_+ = 0, \quad (6.2) \]

where \( q = e^{\imath h_1}, \quad p = e^{\imath h_2} \) with \( \imath = \sqrt{-1} \) and \( h_1, h_2 \in \mathbb{R} \).

We denote the algebra for which the generators obey the relations (6.2) by \( \mathcal{L}_{h_1,h_2} := \mathcal{L}(A_{p,q}^{1/2}) \). Let \( U(\mathcal{L}_{h_1,h_2}) \) be the algebra defined by (6.2). The Hopf superalgebra structure of \( U(\mathcal{L}_{h_1,h_2}) \) can be read off from Theorem 2.5.
The superalgebra $U(L_{h_1, h_2})$ is a Hopf superalgebra with coproduct, counit, and antipode on the algebra $L_{h_1, h_2}$ defined by

$$\Delta(u_i) = u_i \otimes 1 + 1 \otimes u_i, \quad \epsilon(u_i) = 0, \quad S(u_i) = -u_i,$$

for $u_i \in \{u, \xi_1, \xi_2\}$.

Example 6.5 There exists a Lie algebra homomorphism $\mu$ from $L_{h_1, h_2}$ into $M(3, \mathbb{K}')$.

Proof We see that there exists an algebra homomorphism $\rho$ from $F(A_{1|2}^{q_1})$ into $M(3, \mathbb{K}')$ such that the relations (6.1) hold. As a consequence of this fact, there exists a Lie algebra homomorphism $\mu$ from $L_{h_1, h_2}$ into $M(3, \mathbb{K}')$. The action of $\mu$ on the generators of $L_{h_1, h_2}$ is of the form

$$\mu(u) = \begin{pmatrix} \imath h_1 & 0 & 0 \\ 0 & \imath h_2 & 0 \\ 0 & 0 & \imath (h_1 + h_2) \end{pmatrix}, \quad \mu(\xi_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^{-i(h_1 + h_2)\varepsilon_1} & 0 & 0 \end{pmatrix}, \quad \mu(\xi_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e^{-2(h_1 + h_2)\varepsilon_2} & 0 \end{pmatrix}$$

(6.3)

where $\varepsilon_1$ and $\varepsilon_2$ are two Grassmann numbers. To see that the relations (6.2) are preserved under the action of $\mu$, we use the fact that

$$[\mu(a), \mu(b)] = \mu([a, b]),$$

for all $a, b \in L_{h_1, h_2}$.

7. *-Structures on the algebras $O(A_{1|2}^{1|2})$ and $O(A_{h}^{2|1})$

It is possible to define the star operation (or involution) on the Grassmann generators. However, there are two possibilities to do so*. If $\alpha$ and $\beta$ are two Grassmann generators and $\lambda$ is a complex number and $\bar{\lambda}$ its complex conjugate, the star operation, denoted by $\star$, is defined by

$$(\lambda\alpha)^\star = \bar{\lambda}\alpha^\star, \quad (\alpha\beta)^\star = \beta^\star\alpha^\star, \quad (\alpha^\star)^\star = \alpha$$

and the superstar operation, denoted by $\#$, is defined by

$$(\lambda\alpha)^\# = \bar{\lambda}\alpha^\#, \quad (\alpha\beta)^\# = \alpha^\#\beta^\#, \quad (\alpha^\#)^\# = -\alpha.$$

It is easily shown that there exists a star operation on the algebra $O(A_{h}^{1|2})$ if $q$ is a complex number of modulus one:

**Proposition 7.1** (i) If $\bar{q} = q^{-1}$ then the algebra $O(A_{h}^{1|2})$ equipped with the involution determined by

$$X^\star = X, \quad \Theta_i^\star = \Theta_i \quad (i = 1, 2)$$

(7.1)

becomes a $\star$-algebra.

(ii) If $\bar{p} = p^{-1}$ and $\bar{q} = q^{-1}$ then the algebra $O(A_{p,q}^{2|1})$ equipped with the involution determined by

$$\Phi^\star = \Phi, \quad Y_i^\star = -Y_i \quad (i = 1, 2)$$

(7.2)

becomes a $\star$-algebra.


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7.1. \(\ast\)-Structures on the algebra \(O(A_{h}^{1|2})\)

As noted in Section 3, the relations in (3.3) do not include the parameter \(h'\). Thus, we can rearrange the change of basis in the coordinates (see, equation (3.1)) as

\[
\begin{pmatrix}
x \\
\Theta_1 \\
\Theta_2
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\frac{h}{q-1} & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix}.
\]

(7.3)

This case can help us to define a star operation on the algebra \(O(A_{h}^{1|2})\) by a coordinate transformation using the generators of the algebra \(O(A_{q}^{1|2})\) and to prove the following lemma.

**Lemma 7.2** For a certain special choice of \(h\), there exists an involution on the algebra \(O(A_{h}^{1|2})\).

**Proof** Using the equation (7.3), we introduce the coordinates \(x, \theta_1,\) and \(\theta_2\) with the change of basis in the coordinates of the superspace \(A_{q}^{1|2}\) as follows:

\[
x = X, \quad \theta_1 = \Theta_1, \quad \theta_2 = \Theta_2 - \frac{h}{q-1} X.
\]

Then, with \(|q| = 1\) and (7.1)

\[
\theta_2^* = \Theta_2^* - \frac{q\bar{h}}{1-q}X^* = \theta_2 + \frac{h + q\bar{h}}{q-1}x
\]

so that, if we demand that \(\bar{h} = -h\), we obtain \(\theta_2^* = \theta_2 - hx\). Note that

\[
(x^*)^* = x, \quad (\theta_1^*)^* = \theta_1, \quad (\theta_2^*)^* = \theta_2,
\]

for all \(h\). \(\square\)

**Proposition 7.3** If \(\bar{h} = -h\), then the algebra \(O(A_{h}^{1|2})\) supplied with the involution determined by

\[
x^* = x, \quad \theta_1^* = \theta_1, \quad \theta_2^* = \theta_2 - hx
\]

(7.4)

becomes a \(\ast\)-algebra.

**Proof** Since \(\bar{h} = -h\), we have

\[
(x\theta_1 - \theta_1 x)^* = \theta_1 x - x\theta_1,
\]

\[
(x\theta_2 - \theta_2 x - hx^2)^* = (\theta_2 - hx) x - x(\theta_2 - hx) + hx^2 = (\theta_2 x - x\theta_2 + hx^2),
\]

\[
(\theta_1 \theta_2 + \theta_2 \theta_1)^* = (\theta_2 - hx) \theta_1 + \theta_1 (\theta_2 - hx) = \theta_2 \theta_1 + \theta_1 \theta_2,
\]

\[
(\theta_2^* + h\theta_2 x)^* = (\theta_2 - hx)(\theta_2 - hx) + x(\theta_2 - hx)(-h) = \theta_2^* + h\theta_2 x.
\]

Hence, the ideal \((x\theta_1 - \theta_1 x, x\theta_2 - \theta_2 x - hx^2, \theta_1 \theta_2 + \theta_2 \theta_1, \theta_1^2, \theta_2^* + h\theta_2 x)\) is \(\ast\)-invariant and the quotient algebra

\[
\mathbb{K}(x, \theta_1, \theta_2)/(x\theta_1 - \theta_1 x, x\theta_2 - \theta_2 x - hx^2, \theta_1 \theta_2 + \theta_2 \theta_1, \theta_1^2, \theta_2^* + h\theta_2 x)
\]

becomes a \(\ast\)-algebra. \(\square\)
Remark 7.4 Of course, we can consider the change of basis in the coordinates of the superspace $A^1_{q|2}$ in (3.1). In this case, since

$$x^* = (1 + \tilde{h} - \overline{h'})x + (\tilde{h}' - \overline{h'})\theta_2,$$

$$\theta_1^* = \theta_1,$$

$$\theta_2^* = (1 - \overline{h} (\tilde{h}' - \overline{h'}))\theta_2 + (\tilde{h} - \overline{h})x,$$

we have again (7.4) with the choices $\tilde{h} = -h$ and $\overline{h}' = h'$.

7.2. $\star$-Structure on the algebra $O(A^2_{h'} | 1)$

Since the relations in (3.6) do not include the parameter $h$, we can rearrange the change of basis in the coordinates (see, equation (3.1)) as

$$
\begin{pmatrix}
\Phi \\
Y_1 \\
Y_2
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & \frac{h'}{pq-1} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\varphi \\
y_1 \\
y_2
\end{pmatrix}.
$$

(7.5)

There exists a special case, where the algebra $O(A^2_{h'} | 1)$ admits an involution. The proofs of the following lemma and proposition can be done in a similar way to Lemma 7.2 and Proposition 7.3.

Lemma 7.5 If $\overline{h}' = h'$, there exists an involution on the algebra $O(A^2_{h'} | 1)$.

Proposition 7.6 If $\overline{h}' = h'$, then the algebra $O(A^2_{h'} | 1)$ supplied with the involution determined by

$$
\varphi^* = \varphi - h'y_2, \quad y_i^* = -y_i, \quad (i = 1, 2)
$$

(7.6)

becomes a $\star$-algebra.

References


