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Pre-Markov operators

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Abstract: In operator theory characterizing extreme points has been systematically studied in a convex set of linear operators from an algebra to another. This paper presents some new characterizations. We define pre-Markov operators and identify when the second adjoint of a linear positive operator being an extreme point in the collection of all Markov operators between the unital second order duals of two unital f-algebras. Moreover a characterization of extreme points is given in the collection of all contractive operators between unital f-algebras. In addition, we give a condition that makes an order bounded algebra homomorphism is a lattice homomorphism.

Key words: Markov operator, f-algebra, algebra homomorphism, lattice homomorphism, contractive operator, Arens multiplication

1. Introduction

A positive linear operator T between two unital f-algebras, with point separating order duals, A and B is called a Markov operator for which $T(e_1) = e_2$ where e_1, e_2 are unit elements of A and B respectively. Let A and B be semiprime f-algebras with point separating order duals such that their second order duals $A^{\sim\sim}$ and $B^{\sim\sim}$ are unital f-algebras. In this case, we will call a positive linear operator $T : A \to B$ to be a pre-Markov operator, if the second adjoint operator of T is a Markov operator. Recall that a semiprime f-algebra A can be embedded as a Riesz subspace and a ring ideal in the f-algebra $Orth(A)$ of all orthomorphisms on A, by identifying $a \in A$ with $\pi_a \in Orth(A)$ where $\pi_a(b) = a.b$ for all $b \in A$. The identity operator I_A on A is a unit element in $Orth(A)$ and $A = Orth(A)$ if and only if A has a unit element. Hence we identify A with $\pi(A)$. One can easily see that

$$A \cap [0, I_A] = \left\{a \in A : a^2 \leq a\right\} = \{a \in A : 0 \leq ab \leq b \text{ for all } 0 \leq b \in A\}.$$

A positive linear operator T between two semiprime f-algebras, with point separating order duals, A and B is said to be contractive if $Ta \in B \cap [0, I_B]$ whenever $a \in A \cap [0, I_A]$, where I_A and I_B are the identity operators on A and B respectively.

The collection of all pre-Markov operators is a convex set. In this paper, first of all, we characterize pre-Markov algebra homomorphisms. In this regard, we show that a pre-Markov operator is an algebra homomorphism if and only if its second adjoint operator is an extreme point in the collection of all Markov operators from $A^{\sim\sim}$ to $B^{\sim\sim}$ (Theorem 3.1). In addition, we characterize the extreme points of all contractive

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operators $T : A \to B$ whenever A and B are Archimedean semiprime f-algebras provided B is relatively uniformly complete (Proposition 3.5). For the second aim, let A and B be Archimedean semiprime f-algebras and $T : A \to B$ a linear operator. Huijsman and De Pagter proved in [8] the following:

(i) If T is a positive algebra homomorphism then it is a lattice homomorphism;

(ii) In addition, if the domain A is relatively uniformly complete and T is an algebra homomorphism then it is a lattice homomorphism and the assumption that the domain A of T is relatively uniformly complete is not reduntant (Theorem 5.1 and Example 5.2.);

(iii) In addition, if the domain A has a unit element and T is an order bounded algebra homomorphism then it is a lattice homomorphism (Theorem 5.3).

We prove that any order bounded algebra homomorphism $T : A \to B$ is a lattice homomorphism, if B is relatively uniformly complete (Corollary 3.7). In this regard, first we give an alternate proof of Lemma 6 in [10] for order bounded operators with the relatively uniformly complete region instead of positive operators with Dedekind complete region (Propositions 3.6 and 3.8). In the last part, we give a necessary and sufficient condition for a positive operator to be a lattice homomorphism (Proposition 3.11).

2. Preliminaries

For unexplained terminology and the basic results on vector lattices and semiprime f-algebras we refer to [1, 11, 13, 15]. The real algebra A is called a Riesz algebra or lattice-ordered algebra if A is a Riesz space such that $ab \in A$ whenever a, b are positive elements in A. The Riesz algebra is called an f-algebra if A satisfies the condition that

$$a \wedge b = 0 \text{ implies } ac \wedge b = ca \wedge b = 0 \text{ for all } 0 \leq c \in A.$$

In an Archimedean f-algebra A, all nilpotent elements have index 2. Indeed, assume that $a^3 = 0$ for some $0 \leq a \in A$. Since the equality $\left(a^2 - na\right) \wedge \left(a - na^2\right) = 0$ implies $\left(a^2 - na\right) \wedge a^2 = \left(a^2 - na\right) = 0$ we get $a^2 = 0$ as A is Archimedean. The same argument is true for all $n \geq 3$. Throughout this paper A is assumed to be an Archimedean semiprime f-algebra with point separating order dual $A^{\sim}$ [15]. By definition, if zero is the unique nilpotent element of A, that is, $a^2 = 0$ implies $a = 0$, A is called semiprime f-algebra. It is well known that every f-algebra with unit element is semiprime.

Let A be a lattice ordered algebra. If A is a lattice ordered space, then the first order dual space $A^{\sim}$ of A is defined as the collection of all order bounded linear functionals on A and $A^{\sim}$ is a Dedekind complete Riesz space. The second order dual space of A is denoted by $A^{\sim\sim}$. Let $a \in A$, $f \in A^{\sim}$ and $F, G \in A^{\sim\sim}$. Define $f \cdot a \in A^{\sim}$, by

$$(f \cdot a)(b) = f(ab)$$

and $F \cdot f \in A^{\sim}$, by

$$(F \cdot f)(a) = F(f \cdot a)$$

and $F \cdot G \in A^{\sim\sim}$, by

$$(F \cdot G)(f) = F(G \cdot f)$$

The last equality is called the Arens multiplication in $A^{\sim\sim}$ [2].

The second order dual space $A^{\sim\sim}$ of a semiprime f-algebra A is again an f-algebra with respect to the Arens multiplication [4]. In the literature, there are several studies, for example [5–7, 9], that respond the question "Under what conditions does the f-algebra $A^{\sim\sim}$ have a unit element?".

Let A and B be semiprime f-algebras with point separating order duals such that their second order duals $A^{\sim\sim}$ and $B^{\sim\sim}$ have unit elements E_1 and E_2 respectively. Let $T : A \to B$ be an order bounded operator. We denote the second adjoint operator of T by T^{**}. Since A and B have point separating order duals, the linear operator $J_1 : A \to A^{\sim\sim}$, which assigns to $a \in A$ the linear functional $\widehat{a}$ defined on $A^{\sim}$ by $\widehat{a}(f) = f(a)$ for all $a \in A$, is an injective algebra homomorphism. Therefore we will identify A with $J_1(A)$, and B with $J_2(B)$ in the similar sense.

Definition 2.1 *Let A and B be semiprime f-algebras with point separating order duals such that their second order duals $A^{\sim\sim}$ and $B^{\sim\sim}$ are unital f-algebras. In this case, we call a positive linear operator $T : A \to B$ to be a pre-Markov operator, if the second adjoint operator of T is a Markov operator. That is, the second adjoint operator $T^{**} : A^{\sim\sim} \to B^{\sim\sim}$ of T is a positive linear and $T^{**}(E_1) = E_2$, where E_1 and E_2 are the unitals of $A^{\sim\sim}$ and $B^{\sim\sim}$ respectively.*

Recall that a positive operator $T : A \to B$ satisfying $0 \le T(a) \le E_2$ whenever $0 \le a \le E_1$ is called a contractive operator.

In this point we remark that , if A and B are semiprime f-algebras with point separating order duals and $T : A \to B$ is a positive linear operator, then T^{**} is positive. Indeed, let $0 \le F \in A^{\sim\sim}$ and $0 \le g \in B^{\sim}$. Then $0 \le g \circ T \in A^{\sim}$ and therefore $F(g \circ T) = T^{**}(F) \ge 0$.

Proposition 2.2 *Let A and B be semiprime f-algebras with point separating order duals such that their second order duals $A^{\sim\sim}$ and $B^{\sim\sim}$ have unit elements E_1 and E_2 respectively. $T : A \to B$ is contractive if and only if T^{**} is contractive.*

Proof Suppose that T is contractive. Then T^{**} is positive. Let $F \in [0, E_1] \cap A^{\sim\sim}$. In order to prove that T^{**} is contractive we shall show that $T^{**}(E_1) \le E_2$. Due to [9],

$$\begin{aligned} E_1(f) &= \sup f(A \cap [0, E_1]) \\ E_2(g) &= \sup g(B \cap [0, E_2]) \end{aligned}$$

for all $f \in A^{\sim}$ and $g \in B^{\sim}$. Let $a \in A \cap [0, E_1]$ and $0 \le g \in B^{\sim}$. Since T is contractive, $T(a) \in B \cap [0, E_2]$ so $g(T(a)) \le E_2(g)$ which implies that $T^{**}E_1(g) = E_1(g \circ T) \le E_2(g)$. Thus $T^{**}(E_1) \le E_2$. Conversely, assume that T^{**} is contractive. Let $a \in A \cap [0, E_1]$ and $0 \le g \in B^{\sim}$. Then $\widehat{Ta}(g) = g(Ta) \le T^{**}E_1(g) \le E_2(g)$ Thus $0 \le Ta = \widehat{Ta} \le E_2$. □

Corollary 2.3 *Let A and B be semiprime f-algebras with point separating order duals such that their second order duals $A^{\sim\sim}$ and $B^{\sim\sim}$ have unit elements E_1 and E_2 respectively. If $T : A \to B$ is a pre-Markov operator then T is contractive.*

Proof Since $T^{**}(E_1) = E_2$ and T^{**} is positive, T^{**} is contractive. By Proposition 2.2 we have the conclusion. □

3. Main results

Theorem 3.1 *Let A and B be semiprime f-algebras with point separating order duals such that their second order duals $A^{\sim\sim}$ and $B^{\sim\sim}$ have unit elements E_1 and E_2 respectively. A pre-Markov operator $T : A \to B$ is an algebra homomorphism if and only if its second adjoint operator T^{**} is an algebra homomorphism.*

Proof Suppose that the pre-Markov operator T is an algebra homomorphism. Since T^{**} is a Markov operator, due to [8], it is enough to show that it is a lattice homomorphism. Let $F, G \in A^{\sim\sim}$ such that $F \wedge G = 0$. Since $A^{\sim\sim}$ and $B^{\sim\sim}$ are semiprime f- algebras, $F \cdot G = 0$. We shall show that $T^{**}(F) \cdot T^{**}(G) = 0$. Let $a, b \in A$ and $f \in B^{\sim}$. Then it follows from the following equations

$$\begin{aligned}((f \cdot Ta) \circ T)(b) &= (f \cdot Ta)(Tb) = f(TaTb) = f(T(ab)) \\ &= (f \circ T)(ab) = ((f \circ T) \cdot a)(b)\end{aligned}$$

that

$$(f \cdot Ta) \circ T = (f \circ T) \cdot a. \tag{3.1}$$

On the other hand, the following equations

$$((G \circ T^*) \cdot f) \circ T)(a) = ((G \circ T^*) \cdot f)(Ta) = (G \circ T^*)(f \cdot Ta) = G((f \cdot Ta) \circ T)$$

hold. Thus $((G \circ T^*) \cdot f) \circ T)(a) = G((f \cdot Ta) \circ T)$. From here, by setting Equation (3.1), we conclude that

$$((G \circ T^*) \cdot f) \circ T)(a) = G((f \circ T) \cdot a) = (G \cdot (f \circ T))(a)$$

which implies

$$((G \circ T^*) \cdot f) \circ T) = (G \cdot (f \circ T)). \tag{3.2}$$

Taking into account Equation (3.2), we get

$$\begin{aligned}(T^{**}(F) \cdot T^{**}(G))(f) &= T^{**}(F)((T^{**}(G) \cdot f)) = (F \circ T^*)((G \circ T^*) \cdot f) \\ &= F((G \circ T^*) \cdot f) \circ T) = F(G \cdot (f \circ T))\end{aligned}$$

thus we have

$$(T^{**}(F) \cdot T^{**}(G))(f) = (F \cdot G)(f \circ T) = 0$$

as desired. Conversely suppose that T^{**} is an algebra homomorphism. Let $a, b \in A$. It follows from

$$T(ab) = \widehat{T(ab)} = T^{**}\left(\widehat{ab}\right) = T^{**}\left(\widehat{a} \cdot \widehat{b}\right) = T^{**}(\widehat{a}) \cdot T^{**}\left(\widehat{b}\right) = \widehat{Ta} \cdot \widehat{Tb} = Ta.Tb$$

that T is an algebra homomorphism. □

In the proof of Theorem 3.1 we proved the following corollary as well.

Corollary 3.2 *Let A, B and their second order duals $A^{\sim\sim}$ and $B^{\sim\sim}$ be semiprime f-algebras and $T : A \to B$ a positive algebra homomorphism. Then T^{**} is a lattice homomorphism.*

Theorem 3.3 *Let A and B be semiprime f-algebras with point separating order duals and $T : A \to B$ a positive linear operator. If the second order duals $A^{\sim\sim}$ and $B^{\sim\sim}$ have unit elements and T is an algebra homomorphism, then T is an extreme point of the contractive operators from A to B.*

Proof Suppose that T is a positive algebra homomorphism. Then due to [14, Theorem 4.3], T is a contractive operator. Let $2T = T_1 + T_2$ for some contractive operators T_1, T_2 from A to B. In this case, $2T^{**} = T_1^{**} + T_2^{**}$. By Proposition 2.2, T^{**}, T_1^{**} and T_2^{**} are contractive and by Corollary 3.2, T^{**} is a lattice homomorphism. Taking into account [3, Theorem 3.3], we derive that T^{**} is an extreme point in the collection of all contractive operators from $A^{\sim\sim}$ to $B^{\sim\sim}$. Thus $T^{**} = T_1^{**} = T_2^{**}$ and therefore $T = T_1 = T_2$. □

At this point, we recall the definition of uniform completion of an Archimedean Riesz space. If A is an Archimedean Riesz space and $\widehat{A}$ is the Dedekind completion of A, then $\overline{A}$, the closure of A in $\widehat{A}$ with respect to the relatively uniform topology [11], is so called that relatively uniformly completion of A [12]. If A is an semiprime f-algebra then the multiplication in A can be extended in a unique way into a lattice ordered algebra multiplication on $\overline{A}$ such that A becomes a subalgebra of $\overline{A}$ and $\overline{A}$ is an relatively uniformly complete semiprime f-algebra. In [14, Theorem 3.4] it is shown that a positive operator T from a Riesz space A to a uniformly complete space B, has a unique positive linear extension $\overline{T} : \overline{A} \to B$ to the relatively uniformly completion $\overline{A}$ of A, defined by,

$$\overline{T}\,(x) = \sup\{T(a) : 0 \leq a \leq x\}$$

for $0 \leq x \in \overline{A}$. We also recall that $\overline{A}$ satisfies the Stone condition (that is, $x \wedge nI^* \in \overline{A}$, for all $x \in \overline{A}$, where I denotes the identity on A of $OrthA$) due to Theorem 2.5 in [7]. For the completeness we give the easy proof of the following proposition.

Proposition 3.4 *Let A and B be Archimedean semiprime f-algebras such that B is relatively uniformly complete. In this case, $T : A \to B$ is contractive if and only if $\overline{T}$ is contractive.*

Proof Suppose that T is contractive. Let $x \in \overline{A} \cap \left[0, \overline{I}\right]$, here $\overline{I}$ is the unique extension to $\overline{A}$ of the identity mapping $I : A \to A$. Since T is contractive, $a \in A \cap [0, x]$ implies that I is an upper bound for the set $\{T\,(a) : a \leq x, a \in A\}$, so $\overline{T}\,(x) \leq I$. Therefore $\overline{T}$ is contractive. The converse implication is trivial, since $\overline{T}$ is the extension of T, we get $0 \leq \overline{T}\,(a) = T\,(a) \leq I$ whenever $a \in A \cap [0, I]$. □

Proposition 3.5 *Let A and B be Archimedean semiprime f-algebras such that B is relatively uniformly complete and let $T : A \to B$ be a contractive operator. Then T is an extreme point in the collection of all contractive operators from A to B if and only if $\overline{T}$ is an extreme point of all contractive operators from $\overline{A}$ to B.*

Proof Suppose that $\overline{T}$ is an extreme point in the set of all contractive operators from $\overline{A}$ to B. We shall show that for arbitrary $\varepsilon > 0$ and contractive operator S from A to B satisfying $\varepsilon T - S \geq 0$ implies that $T = S$. Let $0 \leq x \in \overline{A}$. Then there exists a positive sequence $(a_n)_n$ in A converging relatively uniformly to x. Since $\overline{T}$ and $\overline{S}$ are relatively uniformly continuos, the sequence $\varepsilon\overline{T}\,(a_n)\; - \overline{S}\,(a_n) = \varepsilon T\,(a_n) - S\,(a_n)$ converges to $\varepsilon\overline{T}\,(x)\; - \overline{S}\,(x)$. Therefore, since $(a_n)_n$ is positive sequence and $\varepsilon T - S \geq 0$, we get $\varepsilon\overline{T}\; - \overline{S} \geq 0$. Since $\overline{T}$ is an extreme point, we have $\overline{T} = \overline{S}$, so that $T = S$. Conversely assume that T is an extreme point in the set of all contractive operators from A to B. Let $\varepsilon > 0$ be any number and let S be any contractive operator from $\overline{A}$ to B satisfying $\varepsilon\overline{T}\; - S \geq 0$. Let U be the restriction of S to A. Since S is contractive, by

Proposition 3.4, $S\mid_A = U$ is contractive and by the uniquness of the extension, we infer that $S = \overline{U}$. Hence $\left(\varepsilon\overline{T} - S\right)\mid_A = \varepsilon T - U \geq 0$. Thus $\overline{T} = S$, which shows that $\overline{T}$ is an extreme point. □

After proving the following Propositions 3.6 and 3.8 for order bounded operators with the relatively uniformly complete region, we remarked that both were proved in [10] for the positive operators with Dedekind complete region. They might be regarded as the alternate proofs.

Proposition 3.6 *Let $T : A \to B$ be an order bounded operator where A and B are Archimedean f-algebras and B is, in addition, relatively uniformly complete. Then T is an algebra homomorphism iff $\overline{T}$ is an algebra homomorphism.*

Proof Suppose that $T : A \to B$ is an algebra homomorphism and x, y be positive elements in $\overline{A}$. By [14], since

$$xy = \sup\left\{R_y(a) : 0 \leq a \leq x, a \in A\right\}$$

and

$$R_y(a) = \sup\left\{ab : 0 \leq b \leq y, b \in A\right\}.$$

Now as $\overline{T}$ is relatively uniformly continuous, we get,

$$\begin{aligned}\overline{T}\left(R_y(a)\right) &= \sup\left\{\overline{T}(ab) = T(ab) = T(a)\,T(b) : 0 \leq b \leq y, b \in A\right\} \\ &= T(a)\sup\left\{T(b) : 0 \leq b \leq y, b \in A\right\} \\ &= T(a)\,\overline{T}(y)\end{aligned}$$

and then

$$\begin{aligned}\overline{T}(xy) &= \sup\left\{\overline{T}\left(R_y(a)\right) : 0 \leq a \leq x, a \in A\right\} \\ &= \sup\left\{T(a)\,\overline{T}(y) : 0 \leq a \leq x, a \in A\right\} \\ &= \overline{T}(y)\sup\left\{T(a) : 0 \leq a \leq x, a \in A\right\} \\ &= \overline{T}(x)\,\overline{T}(y)\end{aligned}$$

Hence $\overline{T}$ is an algebra homomorphism. The converse is trivial. □

In [8], both were proved that an algebra homomorphism $T : A \to B$ need not be a lattice homomorphism if the domain A is not relatively uniformly complete (Example 5.2) and an order bounded algebra homomorphism $T : A \to B$ is a lattice homomorphism whenever the domain A has a unit element. We remarked that Proposition 3.6 yields that the second result also holds for an order bounded algebra homomorphism without unitary domain but the region is relatively uniformly complete.

Corollary 3.7 *Let A be an Archimedean semiprime f-algebra and B a relatively uniformly complete Archimedean f-algebra. Then any order bounded algebra homomorphism $T : A \to B$ is a lattice homomorphism.*

Proof By Proposition 3.6, $\overline{T}$ is an algebra homomorphism and since $\overline{A}$ is relatively uniformly complete, $\overline{T}$ is a lattice homomorphism [8]. Thus T is a lattice homomorphism. □

Proposition 3.8 *Let A be an Archimedean f-algebra and let B be a relatively uniformly complete semiprime f- algebra. Then the operator $T : A \to B$ is a lattice homomorphism iff $\overline{T}$ is a lattice homomorphism.*

Proof Suppose that T is a lattice homomorphism. Let $x \in \overline{A}$. Let $a \in [0, x^+] \cap A$ and $b \in [0, x^-] \cap A$. Since T is a lattice homomorphism, we have

$$T(a \wedge b) = T(a) \wedge T(b) = 0.$$

On the other hand, it follows from the equality

$$T(a) \wedge \overline{T}\left(x^-\right) = \sup\left\{T(a) \wedge T(b) : 0 \le b \le x^-, b \in A\right\}$$

that

$$\overline{T}\left(x^+\right)\overline{T}\left(x^-\right) = \sup\left\{T(a) \wedge \overline{T}\left(x^-\right) : 0 \le a \le x^+, a \in A\right\} = 0$$

which its turn is equivalent to $\overline{T}$ is a lattice homomorphism, as B is semiprime. Converse is trivial. □

In this point, we remark that Lemma 3.1 and Theorem 3.3 in [3] are also true for Archimedean semiprime f-algebras without the Stone condition on the domain A whenever B is relatively uniformly complete.

Proposition 3.9 *Let A and B be Archimedean semiprime f-algebras, B relatively uniformly complete and $T : A \to B$ a contractive operator. Assume that $\overline{A}$ has unit element. For $y \in \overline{A}$, define $H_x(y) = \overline{T}(xy) - \overline{T}(x)\overline{T}(y)$. Then $\overline{T} \mp H_x$ are contractive mappings for all $x \in \overline{A} \cap [0, I]$.*

Proof By Proposition 3.4, $\overline{T}$ is contractive. Since $\overline{A}$ satisfies the Stone condition, due to [3, Lemma 3.1], we have the conclusion. □

Corollary 3.10 *Let A and B be Archimedean semiprime f-algebras such that B is relatively uniformly complete and let $T : A \to B$ be a contractive operator. If $\overline{A}$ has unit element, then $T \mp T_a$ are contractive for all $a \in A \cap [0, I]$, here $T_a(b) = T(ab) - T(a)T(b)$.*

Proof By Proposition 3.9, $\overline{T} \mp H_x$ are contractive mappings for all $x \in \overline{A} \cap [0, I]$. Let $a \in A \cap [0, I]$ and $0 \le b \in A$. Then $0 \le \left(\overline{T} \mp H_a\right)(b) = T(b) \mp T_a(b)$ holds. Thus $T \mp T_a$ is positive. Let $b \in A \cap [0, I]$. It follows from

$$0 \le \left(\overline{T} \mp H_a\right)(b) = T(b) \mp T_a(b) \le I$$

that $T \mp T_a$ are contractive. □

Proposition 3.11 *Let A and B be Archimedean semiprime f-algebras such that B is relatively uniformly complete and let $T : A \to B$ be a positive linear operator. T is contractive and it is an extreme point in the collection of all contractive operators from A to B if and only if T is an algebra homomorphism.*

Proof Let T be an extreme point in the collection of all contractive operators from A to B. Then by Proposition 3.5, $\overline{T}$ is an extreme point of all contractive operators from $\overline{A}$ to B. It follows from [3, Theorem 3.3] that $\overline{T}$ is an algebra homomorphism. By Proposition 3.6, T is an algebra homomorphism. Conversely, if T is an algebra homomorphism, then due to [14, Theorem 4.3], T is a contractive operator. By Proposition 3.4, $\overline{T}$ is contractive and by Proposition 3.6, $\overline{T}$ is an algebra homomorphism. Thus $\overline{T}$ is an extreme point in the set of all contractions from $\overline{A}$ to B due to [3, Theorem 3.3]. By using Proposition 3.5, we have the conclusion. □

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