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On symplectic 8-manifolds admitting $Spin(7)$-structure

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Abstract: In this paper we study symplectic 8-manifolds admitting $Spin(7)$- structure. We give examples and show that many of the symplectic 8-manifolds constructed by Pasquotto satisfy the Chern number relations required to admit a $Spin(7)$-structure.

Key words: Calibrations, special holonomy, symplectic manifolds

1. Introduction

The Lie groups G_2 and $Spin(7)$ are the exceptional holonomy groups in Berger's classification of holonomy groups of Riemannian manifolds. They can only appear as the holonomy groups of 7 and 8 dimensional Riemannian manifolds, respectively. Manifolds with these holonomy groups play an important role in M-theory and gauge theories in higher dimensions. Hence, understanding the structure of these special geometries forms a very active research area for many geometers and physicists (cf. [4], [9], [7]). In [1] Arikan, Cho and Salur initiate the study of 7-dimensional manifolds with both contact and G_2 structures. They try to understand G_2 manifolds by using the techniques coming from contact geometry. Using the associative calibration they find a contact 1-form on a 7-manifold with G_2-structure which satisfies certain compatibility conditions. They also construct explicit almost contact metric structures on manifolds with G_2-structures which are also studied by Özdemir, Solgun and Aktay in [12].

There is a similar interdisciplinary research area for 8-dimensional case; namely 8-manifolds with both symplectic and $Spin(7)$-structures which are studied in this paper. However, this case is harder than the 7-dimensional case because every spin 7-manifold M has a G_2 structure and Arikan et al. prove that M has an almost contact structure, too. For an 8-dimensional manifold M, being spin is not enough to have a $Spin(7)$-structure. Additionally, the Pontryagin classes (or Chern classes in symplectic case) of M must satisfy a certain equation.

In this article, we prove certain results on the existence of symplectic 8-manifolds with $Spin(7)$-structures and give some examples. Some of our results heavily depend on the work of Pasquotto [13] where she constructs 8-dimensional symplectic manifolds for every Chern number system which satisfies certain modular equations. We use Pasquotto's constructions to find a large family of symplectic 8-manifolds which may have a $Spin(7)$-structure.

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The paper is organized as follows. In Section 2 we give the preliminaries about symplectic and $Spin(7)$-structures. In Section 3 we start with some examples and then we continue with summarizing Pasquotto's results. Finally in the same section, we show that many of her constructions satisfy the Chern number relations satisfied by manifolds with $Spin(7)$-structure.

2. Preliminaries

In this section, we give some background materials.

2.1. Spin structures

Let E be an oriented n-dimensional Riemannian vector bundle over a manifold M, and let $P_{SO(n)}(E)$ be its bundle of oriented orthonormal frames. There is a naturally defined universal covering homomorphism $\xi_0 : Spin(n) \to SO(n)$ with kernel $\{-1, 1\} \simeq Z_2$ for $n \geq 3$. If we can lift the structure group of E from $SO(n)$ to $Spin(n)$, then oriented vector bundle E is spinnable.

Definition 2.1 *For $n \geq 3$, a spin structure on E is a principal $Spin(n)$-bundle $P_{Spin(n)}(E)$ together with a 2-sheeted covering*

$$\xi : P_{Spin(n)}(E) \to P_{SO(n)}(E)$$

such that $\xi(p.g) = \xi(p).\xi(g)$ for all $p \in P_{Spin(n)}(E)$ and all $g \in Spin(n)$.

An oriented Riemannian manifold M is called a spin manifold if its tangent bundle TM carries a spin structure.

For an oriented Riemannian manifold M, the obstruction to having a spin structure is the Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}_2)$. Hence, M has a spin structure if and only if $w_2(M) = 0$.

Proposition 2.2 *Let M and N be two spin manifolds. Then, $M \times N$ and $M \# N$ are spin manifolds, too.*

In dimension 8, there is a necessary condition for an oriented Riemannian manifold M to be spin. It is given by the $\hat{A}$-genus, which can be computed using the following formula:

$$\hat{A}[M] = \frac{1}{5760}(7p_1^2 - 4p_2)[M] \tag{2.1}$$

, where p_i is the i^{th} Pontryagin class of M. Borel and Hirzebruch [2] prove that the $\hat{A}$-genus must be an integer for a spin manifold.

2.2. Symplectic structure

A *symplectic form* on a smooth manifold M is a closed non-degenerate differential 2-form ω where nondegenerate means the skew-symmetric pairing $\omega_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ is nondegenerate for all $p \in M$. Since skew-symmetric pairings are always singular in odd dimensions, M must be even dimensional. A smooth manifold M^{2n} with a symplectic form ω is called a *symplectic manifold*. Furthermore, the assigned symplectic form ω is also called *symplectic structure*.

Remark 2.3 *i) If (M^{2n}, ω) is a symplectic manifold, any even dimensional submanifold $S \subset M$ with $\omega_{|_S}$ is a symplectic manifold.*
ii) If (M_1, ω_1) and (M_2, ω_2) are two symplectic manifolds, then $M_1 \times M_2$ is a symplectic manifold with symplectic form $\omega = \pi_1^* \omega_1 + \pi_2^* \omega_2$. Furthermore, there is a symplectic sum operation $\#_S$ which makes $M_1 \#_S M_2$ symplectic.

Definition 2.4 *An almost complex structure on a smooth oriented manifold M is an endomorphism of the tangent bundle $J : TM \to TM$ such that $J^2 = -id_M$.*

Every symplectic manifold (M, ω) with a Riemannian metric g has an almost complex structure J on M which is compatible with ω, i.e.

$$g(v, w) = \omega(v, Jw)$$

.

2.3. Spin(7)-structure on 8-dimensional manifolds

Let

$$\begin{aligned} \Phi_0 = dx^{1234} + dx^{1256} + dx^{1278} + dx^{1357} - dx^{1368} - dx^{1458} - dx^{1467} \\ - dx^{2358} - dx^{2367} - dx^{2457} + dx^{2468} + dx^{3456} + dx^{3478} + dx^{5678} \end{aligned} \tag{2.2}$$

where $dx^{ijkl} = dx^i \wedge dx^j \wedge dx^k \wedge dx^l$ is the standard Cayley 4-form on $\mathbb{R}^8$. The subgroup of $SO(8)$ that preserves Φ_0 is $Spin(7)$ (hence the action of $Spin(7)$ on $\mathbb{R}^8$ preserves the Euclidean metric g_0 and the orientation).

Definition 2.5 *A 4-form Φ on an 8-dimensional vector space V is called admissible if there exists a basis of V in which it can be identified with the 4-form Φ_0.*

The space of admissible 4-forms on V is denoted by $\mathcal{A}(V) \subset \bigwedge^4 V$.

Definition 2.6 *A $Spin(7)$-structure on an 8-dimensional manifold M is an admissible 4-form $\Phi \in \Gamma(\mathcal{A}(TM)) \subset \Omega^4(M)$.*

Each 8-manifold with a $Spin(7)$-structure Φ is canonically equipped with a metric g. Hence, we can think of a $Spin(7)$-structure on M as a pair (Φ, g) such that for all $p \in M$ there is an isomorphism between T_pM and $\mathbb{R}^8$ which identifies (Φ_p, g_p) with (Φ_0, g_0). In [8] (M, Φ, g) is also called an almost $Spin(7)$-manifold. Furthermore, if the holonomy group of M with the Levi-Civita connection ∇_g of the metric g, $\mathrm{Hol}(M, g) \subseteq Spin(7)$, then M is called a $Spin(7)$-manifold. $\mathrm{Hol}(M, g) \subseteq Spin(7)$ if and only if $\nabla_g \Phi = 0$ (or equivalently $d\Phi = 0$ since Φ is self-dual) [7].

Existence of a $Spin(7)$-structure on an 8-dimensional manifold M is equivalent to reducing the structure group from $SO(8)$ to its subgroup $Spin(7)$. Every 8-dimensional manifold with a $Spin(7)$-structure is a spin manifold, but the converse is not true. Existence of a $Spin(7)$-structure is guaranteed by more topological conditions. The following result gives the necessary and sufficient conditions for an 8-dimensional manifold M to admit a $Spin(7)$-structure (cf. [3], [10]).

Theorem 2.7 *Let M be a differentiable 8-manifold. M admits a Spin(7)-structure if and only if $w_1(M) = w_2(M) = 0$ and for appropriate choice of orientation on M we have that*

$$p_1(M)^2 - 4p_2(M) \pm 8\chi(M) = 0. \tag{2.3}$$

First, we consider some simple specific cases. In this case, the above equation can be defined by a combination of Chern classes as in the following corollary.

Corollary 2.8 *[10] Let M be a complex manifold of dimension 4. Then M admits a topological $Spin(7)^+$-structure if and only if*

$$c_1[c_1^3 - 4c_1c_2 + 8c_3] = 0 \tag{2.4}$$

Proof The Pontryagin classes of M are even Chern classes of the complexified tangent bundle of M. Moreover, the Euler class of M is the top Chern class of M. We have the following relations between the Pontryagin classes and Chern classes of a complex vector bundle ξ, where $\xi_{\mathbb{R}}$ is the underlying real vector bundle of ξ.

$$\begin{aligned} p_k(\xi_{\mathbb{R}}) &= (-1)^k \Sigma_{i+j=2k} c_i(\xi)(-1)^j c_j(\xi) \\ p_k(\xi_{\mathbb{R}}) &= c_k(\xi)^2 - 2c_{k-1}(\xi)c_{k+1}(\xi) + \ldots \pm 2c_{2k}(\xi) \\ p_1 &= c_1^2 - 2c_2, \\ p_2 &= c_2^2 - 2c_1c_3 + 2c_4 \text{ and } \chi = c_4. \end{aligned} \tag{2.5}$$

Hence,

$$c_1^4 - 4c_1^2c_2 + 8c_1c_3 = 0. \tag{2.6}$$

□

Therefore, the following holds for the product of two compact, spin 4-manifolds.

Corollary 2.9 *[10] Let X and Y be compact spin 4-manifolds. Then the product $M = X \times Y$ admits a topological Spin(7)-structure if and only if*

$$9\sigma(X)\sigma(Y) = 4\chi(X)\chi(Y). \tag{2.7}$$

In particular, $X \times X$ has a such structure if and only if

$$3\sigma(X) = \pm 2\chi(X), \tag{2.8}$$

where σ is the signature of M and χ is the Euler characteristic of M.

3. Symplectic 8-manifolds with $Spin(7)$-structure

Now we give examples of symplectic 8-manifolds with $Spin(7)$-structure. First we construct examples using some well-known manifolds in literature. Then we use Pasquotto's results to show the possible existence of many more.

Example 3.1 Let $(N^6, \omega_N, \Omega_N)$ be a Calabi-Yau 3-fold where ω_N is the Kähler (symplectic) 2-form and Ω_N is a holomorphic volume form. When $M^8 = N \times \mathbb{R}^2$, then the holonomy group $\mathrm{Hol}(M) \subseteq \mathrm{SU}(3) \subset Spin(7)$. Thus, M^8 is a symplectic $Spin(7)$-manifold with

$$\omega = \pi_1^* \omega_N + \pi_2^*(ds \wedge dt)$$

,

$$\Phi = (\mathrm{Re}(\Omega_N) - \omega_N \wedge ds) \wedge dt - \mathrm{Im}(\Omega_N) \wedge ds - \omega_N^2/2$$

, where (s,t) are the coordinates on $\mathbb{R}^2$.

Example 3.2 Let (N^7, φ) be a G_2-manifold with associative 3-form φ. Then $M^8 = N \times \mathbb{R}$ has holonomy $\mathrm{Hol}(M) \subseteq G_2 \subset Spin(7)$ and so is a $Spin(7)$-manifold with

$$\Phi = \varphi \wedge dt - (*\varphi)$$

, where t is the coordinate on $\mathbb{R}$. In [1] Arikan et al. showed that when N^7 is open, then there exists a contact structure on N^7 with contact form α. Hence, M^8 will also be a symplectic manifold with symplectic 2-form $\omega = d(e^t \alpha)$.

Example 3.3 Let $(M^8, \omega_M, \Omega_M)$ be a Calabi-Yau 4-fold with holonomy group $\mathrm{Hol}(M) = \mathrm{SU}(4)$. Since $\mathrm{SU}(4) = Spin(6) \subset Spin(7)$, M is a $Spin(7)$-manifold with

$$\Phi = -\frac{\omega_M^2}{2} + \mathrm{Re}(\Omega_M).$$

Clearly, ω_M is the symplectic 2-form on M.

Now we construct 8-dimensional manifolds by using symplectic 4-manifolds given by Gompf [5] and Halic [6].

Example 3.4 Let $\mathbb{E}_1 = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ be an elliptic surface and let $\mathbb{E}_n$ denote the simply connected, relatively minimal elliptic surface with topological Euler characteristic $\chi(\mathbb{E}_n) = 12n > 0$ and no multiple fibers. The diffeomorphism type of E_n is unique, $\sigma(\mathbb{E}_n) = -8n$ and $\mathbb{E}_\ltimes$ may be obtained symplectically by taking the fiber sum of n copies of a rational elliptic surface $\mathbb{E}_1$. In particular, for $n = 2k$ and $\sigma(\mathbb{E}_{2k}) = -16k$, $\mathbb{E}_{2k}$ is spin since $\sigma(\mathbb{E}_{2k}) = 0$ $(mod 16)$ and $\chi(\mathbb{E}_2) = 24k$ and $3\sigma = 2\chi$ holds. $\mathbb{E}_{2k} \times \mathbb{E}_{2k}$ admits a topological $Spin(7)$ structure since $\mathbb{E}_{2k}$ is spin and satisfies the condition given by Gompf [5] .

Example 3.5 We will construct a symplectic manifold Q_2 that is a torus bundle over a genus 2 surface and has a symplectic section with square 0. First, we consider the manifold Z described by Thurston in [T]. This manifold is a quotient of $\mathbb{R}^4$ by the action of a discrete group G of symplectomorphisms. The group G is generated by unit translations parallel to the x_1-,x_2- and x_3-axes, together with the map $(x_1, ..., x_4) \to (x_1 + x_2, x_2, x_3, x_4 + 1)$. The standard symplectic form $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ descends to a symplectic form on Z. Projection onto the last two coordinates induces a bundle structure $\pi \ : Z \to \mathbb{T}^2$ with torus fibers that are symplectic. We have a section $\varphi : \mathbb{T} \to Z$ given by $\varphi(x_3, x_4) = (0, 0, x_3, x_4)$, which is a symplectic embedding, and the image of φ has a canonical normal framing via the vector field $\frac{\partial}{\partial x_1}$. The manifold Z is parallelizable by the frame field $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4})$. Thus $\sigma(Z) = \chi(Z) = 0$ and Z is spin. Then $Z \times Z$ admits $Spin(7)$-structure.

Example 3.6 [K3 surface] [10] Let $V^n(d) \subset \mathbb{CP}^{n+1}$ be the nonsingular complex hypersurface with degree d. Then,

$$c_1(V^n(d)) = (n+2-d)g$$

, where g is the canonical generator of $H^2(V^n(d);\mathbb{Z})$. The following holds;

$$V^n(d) \text{ is spin } \iff n+d \text{ is even.}$$

Specifically, $V^2(4) = \{(w_0, w_1, w_2, w_3) | w_0^4 + w_1^4 + w_2^4 + w_3^4 = 0\} \subset \mathbb{CP}^3$ is a spin manifold with signature 16, which is called a Kummer (or K3) surface. Thus, $K3 \times K3$ admits a $Spin(7)$-structure.

3.1. Pasquotto's results

In this section we summarize Pasquotto's results. Details can be found in [13]. The Riemann-Roch theorem gives relations which a system of integers must satisfy in order to appear as the system of Chern numbers of an almost complex manifold. In dimension 8, these relations are given below:

$$\begin{array}{lcl} -c_4 + c_1c_3 + 3c_2^2 + 4c_2c_1^2 & \equiv & 0 \ (mod\ 720) \\ 2c_1^4 + c_1^2c_2 & \equiv & 0 \ (mod\ 12) \\ -2c_4 + c_1c_3 & \equiv & 0 \ (mod\ 4) \end{array}$$

Let a given quintuple of integer numbers $(c_4, c_1c_3, c_2^2, c_1^2c_2, c_1^4)$ satisfy the system of congruence relations given above. Then there exist integers (a, j, k, m, b);

$$\begin{array}{lcl} a & = & c_4 \\ 720j & = & -c_4 + c_1c_3 + 3c_2^2 + 4c_2c_1^2 - c_1^4 \\ 12k & = & 2c_1^4 + c_1^2c_2 \\ 4m & = & -2c_4 + c_1c_3 \\ b & = & c_1^4. \end{array}$$

This system is equivalent to the following:

$$\begin{array}{lcl} c_4 & = & a \\ c_1c_3 & = & 4m + 2a \\ c_1^4 & = & b \\ c_1^2c_2 & = & 12k - 2b \\ 3c_2^2 & = & 720j - a - 4m - 48k + 9b. \end{array}$$

Pasquotto proved that if $a + m = 0\ (mod\ 3)$, one can construct a symplectic 8-manifold with these Chern numbers. She applied blow-ups (either at a point or along a submanifold) or fiber sum operations to certain manifolds, which were used as building blocks, until the given modular equations were satisfied by the resulting manifold.

In the following subsections we summarize how the Chern numbers change under the blow-ups.

3.1.1. Blowing-up at a point

Let M be an 8-dimensional symplectic (hence almost complex) manifold and $\hat{M}$ be a blow up of M at a point. Pasquotto proves the following relations in [13].

$$\begin{array}{lcl} c_4[\hat{M}] & = & c_4[M]+3 \\ c_1c_3[\hat{M}] & = & c_1c_3[M]+6 \\ c_2^2[\hat{M}] & = & c_2^2[M]-4 \\ c_1^2c_2[\hat{M}] & = & c_1^2c_2[M]-18 \\ c_1^4[\hat{M}] & = & c_1^4[M]-81. \end{array}$$

Hence, (a', b', j', k', m') change as following;

$$\begin{array}{lcl} a' & = & a+3 \\ 4m' & = & 4m \\ 720j' & = & 720j \\ 12k' & = & 12k-180 \\ b' & = & b-81. \end{array}$$

3.1.2. Blowing up along a submanifold

Let (a', b', j', k', m') be the new quintuple after blowing up along a symplectic surface C, with genus g and normal bundle ν_C. Then,

$$\begin{array}{lcl} a' & = & a+4(1-g) \\ 4m' & = & 4m-4(1-g) \\ 720j' & = & 720j \\ 12k' & = & 12k-144(1-g)-36 < c_1(\nu_C),[C] > \\ b' & = & b-64(1-g)-16 < c_1(\nu_C),[C] > . \end{array}$$

Moreover, (a', b', j', k', m') changes in the following way after blowing up along symplectic 4-dimensional submanifold N with normal bundle ν_N:

$$\begin{array}{lcl} a' & = & a+c_2[N] \\ 4m' & = & 4m+c_1^2[M]-3c_2[N] \\ 720j' & = & 720j \\ 12k' & = & 12k-13c_1^2[N]-c_2[N]-18 < c_1(N)c_1(\nu_N) \\ & & -6 < c_1^2(\nu_N),[N] > \\ b' & = & b-6c_1^2[N]-8 < c_1(N)c_1(\nu_N),[N] > -3 < c_1^2(\nu_N),[N] > \\ & & + < c_2(\nu_N),[N] > . \end{array}$$

3.2. Symplectic spin manifolds

Let (a, j, k, m, b) be the quintuple which are related to Chern numbers as in [13].

Lemma 3.7 *Suppose M^8 is a connected, symplectic manifold which satisfies the quintuple (a, j, k, m, b). If M is a spin manifold, then*

$$135b - 720k = \ 0 \ (mod \ 5760). \tag{3.1}$$

Proof $\hat{A}$-genus, which is more generally defined, is given by the following formula for an 8-manifold.

$$\hat{A}[M] = \frac{1}{5760}(7p_1^2 - 4p_2)[M]. \tag{3.2}$$

It is an integer when M is a spin manifold.

Since M admits an almost complex structure, the p_i's can be written as a combination Chern numbers in the following way $p_1 = c_1^2 - 2c_2$ and $p_2 = c_2^2 - 2c_1c_3 + 2c_4$. We have

$$\begin{array}{lcl} 7c_1^4 - 28c_1^2c_2 + 24c_2^2 + 8c_1c_3 - 8c_4 & \equiv & 0 \ (mod\ 5760) \\ 135b - 720k & \equiv & 0 \ (mod\ 5760). \end{array}$$

□

Lemma 3.8 *Suppose M^8 is a connected, symplectic manifold which satisfies the quintuple (a, j, k, m, b). If M carries a $Spin(7)$-structure, then*

$$9b - 48k + 32m + 16a = 0. \tag{3.3}$$

Proof

If M has a $Spin(7)$-structure, then $p_1(M)^2 - 4p_2(M) + 8\chi(M) = 0$. Since M is symplectic, one can write the Pontryagin classes in terms of the Chern classes; $p_1 = c_1^2 - 2c_2$ and $p_2 = c_2^2 - 2c_1c_3 + 2c_4$. Hence,

$$\begin{array}{lcl} c_1^4 - 4c_1^2 + 8c_1c_3 & = & 0 \\ b - 4(12k - 2b) + 8(4m + 2a) & = & 0 \\ 9b - 48k + 32m + 16a & = & 0. \end{array}$$

□

3.3. Examples

In this subsection we give a concrete example of an 8-dimensional symplectic manifold which satisfies the Chern number relations (3.1) and (3.3). These relations are satisfied by an 8-manifold with a $Spin(7)$-structure. Furthermore, by using a computer software we find a huge subfamily of symplectic 8-manifolds constructed by Pasquotto which also satisfy (3.1) and (3.3).

Example 3.9 Let $\mathbb{E}_1 = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ be the symplectic 4-dimensional elliptic curve given in [5]. Then $\mathbb{E}_2 = \mathbb{E}_1 \#_{\mathbb{E}_1} \mathbb{E}_1$ and $\mathbb{E}_n$ can be constructed inductively. We have

$$\chi(\mathbb{E}_n) = 12n \text{ and } \sigma(\mathbb{E}_n) = -8n \text{ implies } c_1^2(\mathbb{E}_n) = 0.$$

Let (N, ω) be a closed symplectic 4-manifold (e.g. the elliptic curve given above) and $E \to N$ be a complex line bundle over N. Let $\rho : S \to N$ be the S^2-bundle over N, obtained from projectifying the bundle $E \oplus \mathbb{C} \longrightarrow N$. S must be symplectic by Thurston's theorem [11]. The Chern classes of the 6-dimensional manifold S on $\mathbb{E}_n$ with the fibers S^2 can be calculated as follows:

$$\begin{array}{lcl} c_1^3[S] & = & 0 \\ c_1c_2[S] & = & 24n \\ c_3[S] & = & 24n. \end{array}$$

Secondly, let $M = S \times F$ with F a compact Riemann surface of genus g and use product formula to calculate;

$$\begin{array}{lcl} c_4[M] & = & 4 \cdot 12n \\ c_1c_3[M] & = & 4 \cdot 12n \\ c_2^2[M] & = & 8 \cdot 12n \\ c_1^2c_2[M] & = & 4 \cdot 2 \cdot 12n \\ c_1^4[M] & = & 0. \end{array}$$

Therefore, M satisfies the equation (3.3).

Example 3.10 (**8-manifolds constructed by Pasquotto**)
Let M be the symplectic 8-dimensional manifold constructed by Pasquotto in [13]. We consider the constructions in two cases.

Case I: ($j \neq 0$) In this case, quintuples (a, m, k, b, j) of M can be expressed as follows:

$$\begin{aligned} a &= 48n + 12 \\ 4m &= -12 \\ 12k &= -192n - 468 \\ b &= -128n - 208. \end{aligned}$$

The quintuple satisfies equation (3.1). Now, let $\hat{M}$ be the 8-manifold obtained by blowing up M at x points, at y copies of E_-, at z copies of F_-, and at u copies of X_{n-}. The submanifolds E_-, F_- and X_{n-} are defined in [13].

If $\hat{M}$ admits a $Spin(7)$-structure, then $\hat{M}$ must satisfy the equation

$$384n + 96 + 39x + 32y - 32z + (-24n - 18)u = 0.$$

By using the software MAGMA we see that there are many solutions, some of which are given below. Actually, the number of positive solutions is greater than one thousand in the interval [0,50]. Furthermore, the number of solutions gets drastically larger when the interval is larger than $[0, 50]$.

Some of positive integer solutions (x, y, z, u, n) in the interval $[0, 50]$ are the following quintuples:

(2, 1, 1, 15, 4)	(4, 1, 7, 14, 4)	(2, 5, 8, 15, 8)	(8, 1, 13, 12, 2)
(2, 1, 4, 11, 1)	(6, 2, 14, 13, 4)	(4, 1, 13, 6, 1)	(2, 6, 6, 15, 4)
(2, 1, 4, 15, 8)	(2, 5, 8, 11, 1)	(8, 1, 10, 12, 1)	(4, 2, 5, 14, 2)
(2, 1, 7, 15, 12)	(4, 1, 10, 14, 6)	(2, 5, 11, 15, 12)	(8, 2, 11, 12, 1)
(2, 5, 5, 15, 4)	(6, 3, 15, 13, 4)	(4, 1, 13, 14, 8)	(2, 6, 9, 11, 1)

Case II: ($j = 0$)

In this case the constructed 8-dimensional symplectic manifold M cannot be spin as it does not satisfy (3.1) and even after any of the blow-ups done in the $j \neq 0$ case. Hence, we won't find any family of examples with this method.

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