Benedicks and Donoho-Stark type theorems

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Abstract: In this paper, we prove a Benedicks type theorem and a Donoho-Stark type theorem, for the generalized Fourier transform $\mathcal{F}_\alpha$ associated to some differential operators that we call Flensted-Jensen operators, in various spaces such $L^1_\alpha(K)$, $L^2_\alpha(K)$ and $L^1_\alpha(K) \cap L^2_\alpha(K)$, where $K = \mathbb{R}_+ \times \mathbb{R}$.

Key words: Generalized Fourier transform, Benedicks theorem, Donoho-Stark theorem, uncertainty principle

1. Introduction

The uncertainty principle is a characterization of a quantum mechanical system. This principle says that one cannot measure, simultaneously and as accurately as one wants, the position and momentum of a quantum particle. In harmonic analysis, the uncertainty principle can be summarized by the following sentence:

a nonzero function and its Fourier transform cannot be localized as precisely as one wishes.

We can distinguish two formulations of this principle, quantitative and qualitative. In 1927, W. Heisenberg [13] gave a physical interpretation of the quantitative uncertainty principle that he wrote in the form of the following formula called Heisenberg inequality:

$$\forall f \in L^2(\mathbb{R}), \quad \int_\mathbb{R} x^2 |f(x)|^2 dx \cdot \int_\mathbb{R} y^2 |\hat{f}(y)|^2 dy \geq \frac{1}{4} \left( \int_\mathbb{R} |f(x)|^2 dx \right)^2,$$

where $\hat{f}$ is the Fourier transform of $f$, defined for all $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, by

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} f(x) e^{-ixy} dx.$$

Equality cases are realized only by Gaussians of the form

$$f(x) = Ce^{-ax^2}, \quad x \in \mathbb{R},$$

where $C$ and $a$ are constants with $a > 0$.

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By a qualitative uncertainty principle one means a result that, without giving quantitative estimates for a function $f$ and its Fourier transform $\hat{f}$, says that $f$ and $\hat{f}$ cannot both be sharply localized unless $f = 0$. Several authors have published works in the context of the qualitative uncertainty principle. We can cite for example, [1–7, 12, 14–16]. For further references about uncertainty principle, we refer the reader to the book [11] and the survey [10].

In [7], Donoho and Stark studied a new version of qualitative uncertainty principle. This uncertainty principle relies on the notion of $\varepsilon$–concentrated, where a function $f$ belongs to $L^2(\mathbb{R})$ called $\varepsilon$–concentrated on a measurable set $E$ if

$$\|f - f|_E\|_2 \leq \varepsilon \|f\|_2.$$ 

Both others in [7] established that if $f$ is $\varepsilon_1$–concentrated on $E$ and $\hat{f}$ is $\varepsilon_2$-concentrated on $F$, then

$$m(E)m(F) \geq (1 - \varepsilon_1 - \varepsilon_2)^2$$

The aim of this paper is to establish Benedicks and Donoho-Stark type theorems associated to the following operators, that we call Flensted-Jensen operators,

$$\left\{ \begin{array}{ll}
D &= \frac{\partial}{\partial \theta}, \\
D_\alpha &= \frac{\partial^2}{\partial y^2} + [(2\alpha + 1) \coth y + \tanh y] \frac{\partial}{\partial y} - \frac{1}{\cosh^2 y} \frac{\partial^2}{\partial \theta^2} + (\alpha + 1)^2,
\end{array} \right.$$ 

where $\alpha > 0$ and $(y, \theta) \in \mathbb{K} = [0, +\infty) \times \mathbb{R}$. This system was first considered by Flensted-Jensen in [9] for $\alpha = n - 1$, where $n$ is a positive integer, in the frame work of simply connected semisimple Lie group. The operators $D$ and $|D_{n-1} - n^2|$ with the identity generate the algebra $\mathbf{D}(\tilde{G}/K)$ of left invariant differential operators on $\tilde{G}/K$, where $\tilde{G}$ is the universal covering group of $G = U(n, 1)$ and $K$ is the subgroup $U(n)$. A several works on the theory of uncertainty principle, related to the operators $D$ and $D_\alpha$ were studiedin [15–17].

The outline of this paper is given as follows: Section 2 is devoted to recall some results concerning the harmonic analysis associated to the operators $D$ and $D_\alpha$. In section 3, we prove a Benedicks type theorem. In the last section we obtain a various versions of Donoho-Stark theorem.

2. Preliminaries

For $(y, \theta) \in \mathbb{K}$, the following system

$$\left\{ \begin{array}{ll}
Du(y, \theta) &= i\lambda u(y, \theta), \\
D_\alpha u(y, \theta) &= -\mu^2 u(y, \theta), \quad \lambda, \mu \in \mathbb{R} \\
u(0, 0) &= 1, \quad \frac{\partial u}{\partial y}(0, \theta) = 0, \quad \theta \in \mathbb{R}
\end{array} \right.$$ 

has a unique solution given by

$$\varphi_{\lambda, \mu}(y, \theta) = e^{i\lambda \theta} (\cosh y)^\lambda \varphi_{\lambda}^{\alpha, \lambda}(y),$$

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where $\varphi_{\lambda, \mu}^{\alpha}(y)$ is the Jacobi function defined by

$$
\varphi_{\lambda, \mu}^{\alpha}(y) = \,_{2}F_{1}\left(\frac{\alpha + \lambda + 1 + i\mu}{2}, \frac{\alpha + \lambda + 1 - i\mu}{2}; \alpha + 1; - \sinh^{2} y \right).
$$

Recall that $\,_{2}F_{1}$ is the Gaussian hypergeometric function (see [8]).

From [18], we have

$$
\sup_{(y, \theta) \in K} |\varphi_{\lambda, \mu}(y, \theta)| = 1, \quad (\lambda, \mu) \in \mathbb{R}_{+}
$$

Let $L = \mathbb{R} \times [0, +\infty]$ and $\mathbb{R}_{+} = L \cup \Omega$, where

$$
\Omega = \bigcup_{m \in \mathbb{N}} D_{m}^{+} \cup D_{m}^{-},
$$

with

$$
D_{m}^{+} = \{ (\alpha + 2m + 1 + \eta, i\eta) \mid \eta > 0 \} \quad \text{and} \quad D_{m}^{-} = \{ (-\alpha - 2m - 1 - \eta, i\eta) \mid \eta > 0 \}.
$$

Let $1 \leq p < +\infty$. Consider $L_{p}^{1}(K)$, the space of measurable functions $f$ on $K$ verifying

$$
\|f\|_{p, m_{\alpha}} = \left( \int_{K} |f(y, \theta)|^{p} \, dm_{\alpha}(y, \theta) \right)^{\frac{1}{p}} < +\infty,
$$

where

$$
dm_{\alpha}(y, \theta) = 2^{2(\alpha+1)}(\sinh y)^{2\alpha+1} \cosh y \, dy \, d\theta.
$$

For $p = \infty$, we put

$$
\|f\|_{\infty, m_{\alpha}} = \text{ess sup}_{(y, \theta) \in K} |f(y, \theta)|.
$$

The generalized Fourier transform of $f$ associated to Flensted-Jensen operators is given by

$$
\forall (\lambda, \mu) \in \mathbb{R}_{+}, \quad \mathcal{F}_{\alpha} f(\lambda, \mu) = \int_{K} f(y, \theta) \varphi_{\lambda, \mu}^{\alpha}(y, \theta), dm_{\alpha}(y, \theta).
$$

where $f \in L_{p}^{1}(K)$.

Denote $\gamma_{\alpha}$ the positive measure, defined on $\mathbb{R}_{+}$ by

$$
\int_{\mathbb{R}} g \, d\gamma_{\alpha}(\lambda, \mu) = \frac{1}{(2\pi)^{2}} \int_{\mathbb{R} \times [0, +\infty]} g(\lambda, \mu) \frac{d\lambda d\mu}{|C_{1}(\lambda, \mu)|^{2}}
$$

$$
+ \frac{1}{(2\pi)^{2}} \sum_{m=0}^{+\infty} \left\{ \int_{0}^{+\infty} g(\alpha + 2m + 1 + \eta, i\eta) C_{2}(\alpha + 2m + 1 + \eta, i\eta) d\eta
$$

$$
+ \int_{0}^{+\infty} g(-\alpha - 2m - 1 - \eta, i\eta) C_{2}(-\alpha - 2m - 1 - \eta, i\eta) d\eta \right\},
$$

where

$$
C_{1}(\lambda, \mu) = \frac{2^{\alpha+1-i\mu} \Gamma(\alpha + 1) \Gamma(i\mu)}{\Gamma\left(\frac{\alpha + \lambda + 1 + i\mu}{2}\right) \Gamma\left(\frac{\alpha - \lambda + 1 + i\mu}{2}\right)}, \quad (\lambda, \mu) \in \mathbb{R} \times [0, +\infty[.
$$
and

$$C_2(\lambda, \mu) = -i \text{Res}_{z=\mu} \left[ C_1(\lambda, z)C_1(\lambda, -z) \right]^{-1}, \quad (\lambda, \mu) \in \Omega.$$ 

We have from [18] the following inversion formula

$$\mathcal{F}^{-1}_\alpha g(y, \theta) = \int_{\mathbb{R}} g(\lambda, \mu) \varphi_{\lambda, \mu}(y, \theta) d\gamma_\alpha(\lambda, \mu). \quad (2.2)$$

For $1 \leq p < +\infty$, denote $L^p_\alpha(\hat{K})$ the space of measurable functions $g : \hat{K} \rightarrow \mathbb{C}$ verifying

$$\|g\|_{p, \gamma_\alpha} = \left( \int_{\mathbb{R}} |g(\lambda, \mu)|^p d\gamma_\alpha(\lambda, \mu) \right)^{\frac{1}{p}} < +\infty.$$ 

For $p = \infty$, we denote

$$\|g\|_{\infty, \gamma_\alpha} = \text{ess sup}_{(\lambda, \mu) \in \mathbb{R}} |g(\lambda, \mu)|.$$

The generalized Fourier transform $\mathcal{F}_\alpha$ extended to an isometry between $L^2_\alpha(\mathbb{K})$ and $L^2_\alpha(\hat{K})$. In particular, for $f \in L^2_\alpha(\hat{K})$, we have the Plancherel formula

$$\|\mathcal{F}_\alpha f\|_{2, \gamma_\alpha} = \|f\|_{2, m_\alpha}. \quad (2.3)$$

For $f \in L^1_\alpha(\mathbb{K})$, we have

$$\|\mathcal{F}_\alpha f\|_{\infty, \gamma_\alpha} \leq \|f\|_{1, m_\alpha}. \quad (2.4)$$

In the following sections, we consider $E \subset \mathbb{K}$ and $F \subset \hat{K}$ tow measurable subsets. For a function $f \in L^2_\alpha(\mathbb{K})$, we denote by

$T_E$ the time-limiting operator

$$T_E f = \chi_E f,$$

$P_F$ the frequency-limiting operator

$$\mathcal{F}_\alpha(P_F f) = \chi_F \mathcal{F}_\alpha(f),$$

where $\chi_A$ is the characteristic function of the set $A$.

If $0 < \gamma_\alpha(F) < \infty$, then for $f \in L^2_\alpha(\mathbb{K})$ we have

$$P_F f(y, \theta) = \int_{F} \mathcal{F}_\alpha f(\lambda, \mu) \varphi_{\lambda, \mu}(y, \theta) d\gamma_\alpha(\lambda, \mu). \quad (2.5)$$

The operators $P_F$ is bounded from $L^2_\alpha(\mathbb{K})$ into itself and

$$\|P_F f\|_{2, m_\alpha} \leq \|f\|_{2, m_\alpha}. \quad (2.6)$$
3. Benedicks type theorem

In order to prove the main theorem of this section, we start by proving the following lemmas.

**Lemma 3.1** If $0 < m_\alpha(E) < \infty$ and $\gamma_\alpha(F) < \infty$, then the Hilbert-Schmidt norm of $P_T E$ is finite and we have

$$\|P_T E\|_{HS} \leq \sqrt{m_\alpha(E) \gamma_\alpha(F)}.$$

**Proof** Let $f \in L^2_\alpha(K)$, from relation (2.5)

$$P_T E f(y, \theta) = \int_K F_\alpha T_E f(\lambda, \mu) \varphi_{\lambda, \mu}(y, \theta) d\gamma_\alpha(\lambda, \mu)$$

$$= \int_K \chi_f(\lambda, \mu) \left\{ \int_K \chi_E(s, t) f(s, t) \varphi_{-\lambda, \mu}(s, t) dm_\alpha(s, t) \right\} \varphi_{\lambda, \mu}(y, \theta) d\gamma_\alpha(\lambda, \mu).$$

Denote

$$g_{s, t}(\lambda, \mu) = \chi_f(\lambda, \mu) \varphi_{-\lambda, \mu}(s, t)$$

and

$$N(s, t, y, \theta) = \chi_E(s, t) F^{-1}_\alpha(g_{s, t})(y, \theta).$$

Using Fubini’s theorem, we obtain

$$P_T E f(y, \theta) = \int_K f(s, t) N(s, t, y, \theta) dm_\alpha(s, t).$$

$N$ is called the kernel of integral operator $P_T E$ and the Hilbert-Schmidt norm of this operator is given by

$$\|P_T E\|_{HS} = \|N\|_{L^1_\alpha(K) \otimes L^1_\alpha(K)}.$$

Therefore,

$$\|N\|_{L^1_\alpha(K) \otimes L^1_\alpha(K)} = \left( \int_K |\chi_E(s, t)|^2 \left( \int_K |F^{-1}_\alpha(g_{s, t})(y, \theta)|^2 dm_\alpha(y, \theta) \right) dm_\alpha(s, t) \right)^{1/2}.$$

By applying Plancherel formula (2.3), we get

$$\|N\|_{L^1_\alpha(K) \otimes L^1_\alpha(K)} = \left( \int_K \chi_E(s, t) \left( \int_K \chi_f(\lambda, \mu) \varphi_{-\lambda, \mu}(s, t) d\gamma_\alpha(\lambda, \mu) \right) dm_\alpha(s, t) \right)^{1/2}.$$

We deduce from relation (2.1) that

$$\|P_T E\|_{HS} \leq \sqrt{m_\alpha(E) \gamma_\alpha(F)}.$$ 

**Lemma 3.2** Let $f \in L^2_\alpha(K)$. Then

$$(1 - \|P_T E\|) \|f\|_{2, m_\alpha} \leq \left( \|T_E f\|_{2, m_\alpha}^2 + \|P f\|_{2, m_\alpha}^2 \right)^{1/2}.$$
Proof Let $I$ be the identity operator, we have

$$I = P_E T_E + P_{E^c} + P_{F^c}.$$  

For $f \in L_\alpha^2(\mathbb{K})$, we get

$$\|f - P_E T_E f\|_{2,m_\alpha}^2 = \|P_E T_{E^c} f + P_{F^c} f\|_{2,m_\alpha}^2 = \|P_E T_{E^c} f\|_{2,m_\alpha}^2 + \|P_{F^c} f\|_{2,m_\alpha}^2.$$  

It follows by using (2.6) that

$$\|f - P_E T_E f\|_{2,m_\alpha}^2 \leq \|P_E T_{E^c} f\|_{2,m_\alpha}^2 + \|P_{F^c} f\|_{2,m_\alpha}^2. \quad (3.1)$$  

On the other hand, we have

$$\|f - P_E T_E f\|_{2,m_\alpha}^2 \geq \|f\|_{2,m_\alpha}^2 - \|P_E T_{E^c} f\|_{2,m_\alpha}^2.$$  

Since

$$\|P_E T_{E^c} f\|_{2,m_\alpha} \leq \|P_E T_{E}\| \|f\|_{2,m_\alpha},$$  

therefore

$$\|f - P_E T_E f\|_{2,m_\alpha}^2 \geq (1 - \|P_E T_{E}\|)\|f\|_{2,m_\alpha}^2. \quad (3.2)$$  

Combining relations (3.1) and (3.2) we obtain the wanted result.

\[ \square \]

**Theorem 3.3** Let $f \in L_\alpha^2(\mathbb{K})$. If $supp(f) \subset E$, $supp(F_\alpha f) \subset F$ and $0 < m_\alpha(E)\gamma_\alpha(F) < 1$ then $f = 0$.

**Proof** Let $f \in L_\alpha^2(\mathbb{K})$, from lemma 3.1, we obtain

$$\|P_E T_E\| \leq \|P_E T_{E}\|_{HS} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)} < 1.$$  

Applying lemma 3.2, we get

$$\|f\|_{2,m_\alpha}^2 \leq (1 - \|P_E T_{E}\|)^{-2} (\|T_{E^c} f\|_{2,m_\alpha}^2 + \|P_{F^c} f\|_{2,m_\alpha}^2) \leq (1 - \sqrt{m_\alpha(E)\gamma_\alpha(F)})^{-2} (\|T_{E^c} f\|_{2,m_\alpha}^2 + \|P_{F^c} f\|_{2,m_\alpha}^2).$$

Hence $supp f \subset E$ and $supp F_\alpha f \subset F$ then

$$T_{E^c} f = 0 \quad \text{and} \quad P_{F^c} f = 0.$$  

Therefore $f = 0$.  

\[ \square \]
4. Donoho-Stark uncertainty principle

4.1. $L^2$ version of Donoho-Stark theorem

We start by giving the definition of $\varepsilon$-concentrated functions.

**Definition 4.1** Let $f \in L^2_\alpha(K)$, $E$ and $F$ be measurable subsets, respectively, of $K$ and $\hat{K}$. We call

1. $f$ is an $\varepsilon_E$-concentrated on $E$ if there exists a vanishing function $g$ on $K \setminus E$, such that

$$\|f - g\|_{2,\alpha} \leq \varepsilon_E \|f\|_{2,\alpha}. \quad (4.1)$$

2. $\mathcal{F}_\alpha(f)$ is an $\varepsilon_F$-concentrated on $F$ if there exists a vanishing function $h$ on $\hat{K} \setminus F$, such that

$$\|\mathcal{F}_\alpha(f) - h\|_{2,\gamma} \leq \varepsilon_F \|\mathcal{F}_\alpha f\|_{2,\gamma}. \quad (4.2)$$

**Lemma 4.2** Let $f \in L^2_\alpha(K)$, $E$ and $F$ be measurable subsets, respectively, of $K$ and $\hat{K}$. We have

1. $f$ is $\varepsilon_E$-concentrated on $E$ if and only if

$$\|f - T_E f\|_{2,\alpha} \leq \varepsilon_E \|f\|_{2,\alpha}. \quad (4.3)$$

2. $\mathcal{F}_\alpha f$ is $\varepsilon_F$-concentrated on $F$ if and only if

$$\|f - P_F f\|_{2,\alpha} \leq \varepsilon_F \|f\|_{2,\alpha}. \quad (4.4)$$

**Proof**

1. Let $f$ be a $\varepsilon_E$-concentrated on $E$. There exits a vanishing function $g$ on $E^c$, such that

$$\|f - g\|_{2,\alpha} \leq \varepsilon_E \|f\|_{2,\alpha}. \quad (4.3)$$

On the other hand, we have

$$f(y, \theta) - T_E f = \chi_{E^c} f.$$

Then

$$\|f - T_E f\|_{2,\alpha} = \int_K |f(y, \theta) - T_E f(y, \theta)|^2 d\alpha(y, \theta)$$

$$= \int_{E^c} |f(y, \theta) - g(y, \theta)|^2 d\alpha(y, \theta)$$

$$\leq \|f - g\|_{2,\alpha}. \quad (4.3)$$

Then from relation (4.3), we get

$$\|f - T_E f\|_{2,\alpha} \leq \varepsilon_E \|f\|_{2,\alpha}. \quad (4.4)$$
2. Let $\mathcal{F}_\alpha f$ be a $\varepsilon_F$-concentrated to $F$, then there exists a vanishing function $h$ on $F^c$, such that
\[
\|\mathcal{F}_\alpha (f) - h\|_{2,\gamma_\alpha} \leq \varepsilon_F \|\mathcal{F}_\alpha f\|_{2,\gamma_\alpha}.
\] (4.4)
Moreover
\[
\mathcal{F}_\alpha f - \mathcal{F}_\alpha (P_F f) = \mathcal{F}_\alpha f - \chi_F \mathcal{F}_\alpha f = \chi_{F^c}\mathcal{F}_\alpha f.
\]
Then
\[
\|\mathcal{F}_\alpha f - \mathcal{F}_\alpha (P_F f)\|_{2,\gamma_\alpha}^2 = \int_K |\mathcal{F}_\alpha f(\lambda, \mu) - \mathcal{F}_\alpha (P_F f)(\lambda, \mu)|^2 d\gamma_\alpha(\lambda, \mu)
\]
\[
= \int_{F^c} |\mathcal{F}_\alpha f(\lambda, \mu) - h(\lambda, \mu)|^2 d\gamma_\alpha(\lambda, \mu)
\]
\[
\leq \|\mathcal{F}_\alpha f - h\|_{2,\gamma_\alpha}^2.
\]
By relation (4.4), we obtain the following result
\[
\|\mathcal{F}_\alpha f - \mathcal{F}_\alpha (P_F f)\|_{2,\gamma_\alpha} \leq \varepsilon_F \|\mathcal{F}_\alpha f\|_{2,\gamma_\alpha}.
\]
Applying Plancherel’s formula (2.3) on both terms of the above inequality we get
\[
\|f - P_F f\|_{2,\gamma_\alpha} \leq \varepsilon_F \|f\|_{2,\gamma_\alpha}.
\]

\[\square\]

**Lemma 4.3** For $f \in L^2_{\alpha}(\mathbb{K})$ we have
\[
\|P_F T_E f\|_{2,\gamma_\alpha} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)} \|f\|_{2,\gamma_\alpha}.
\]

**Proof** Assume that $m_\alpha(E)$ and $\gamma_\alpha(F)$ are finite. Applying Lemma 3.1 we get
\[
\|P_F T_E\|_{HS} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)}
\]
considering
\[
\|P_F T_E\| = \sup_{f \in L^2_{\alpha}(\mathbb{K}) \setminus \{0\}} \frac{\|P_F T_E f\|_{2,\gamma_\alpha}}{\|f\|_{2,\gamma_\alpha}} \leq \|P_F T_E\|_{HS}
\]
then for $f \in L^2_{\alpha}(\mathbb{K}) \setminus \{0\}$ we have
\[
\frac{\|P_F T_E f\|_{2,\gamma_\alpha}}{\|f\|_{2,\gamma_\alpha}} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)}
\]
which allows us to deduce the wanted result.

\[\square\]

**Theorem 4.4** Consider a nonzero function $f \in L^2_{\alpha}(\mathbb{K})$. If $f$ is an $\varepsilon_E$-concentrated on $E$, $\mathcal{F}_\alpha f$ is an $\varepsilon_F$-concentrated on $F$ and $\varepsilon_E + \varepsilon_F < 1$, then
\[
\sqrt{m_\alpha(E)\gamma_\alpha(F)} \geq 1 - \varepsilon_E - \varepsilon_F.
\]
Proof  Let \( f \in L^2_\alpha(\mathbb{K}) \setminus \{0\} \), we have
\[
\| f - P_f T_E f\|_{2,m_\alpha} \leq \| f - P_f f\|_{2,m_\alpha} + \| P_f f - P_f T_E f\|_{2,m_\alpha}.
\]
From relations (4.2), (2.6) and (4.1), we obtain
\[
\| f - P_f T_E f\|_{2,m_\alpha} \leq \varepsilon_f \| f\|_{2,m_\alpha} + \| f - T_E f\|_{2,m_\alpha}
\leq (\varepsilon_E + \varepsilon_f) \| f\|_{2,m_\alpha}.
\]
which allows us to get the following inequality
\[
\| P_f T_E f\|_{2,m_\alpha} \geq \| f\|_{2,m_\alpha} - \| f - P_f T_E f\|_{2,m_\alpha}
\geq (1 - \varepsilon_E - \varepsilon_f) \| f\|_{2,m_\alpha}.
\]
Applying lemma 4.3 we conclude that
\[
\sqrt{m_\alpha(E)\gamma_\alpha(F)} \geq (1 - \varepsilon_E - \varepsilon_f).
\]

4.2. \( L^1 \) version of Donoho-Stark theorem
In this section, we study the case of a function \( f \in L^1_\alpha(\mathbb{K}) \).

The operator \( T_E \) verifies the following inequality on \( L^1_\alpha(\mathbb{K}) \).
\[
\| T_E f\|_{1,m_\alpha} \leq \| f\|_{1,m_\alpha} \tag{4.5}
\]
We say that \( f \) is an \( \varepsilon_E \)–concentrated on \( E \) in \( L^1_\alpha(\mathbb{K}) \) if
\[
\| f - T_E f\|_{1,m_\alpha} \leq \varepsilon_E \| f\|_{1,m_\alpha}.
\]
We denote by \( B^1_\alpha(F) \) the following subset
\[
B^1_\alpha(F) = \{ g \in L^1_\alpha(\mathbb{K}) \mid P_f g = g \}.
\]
We say that \( f \) is an \( \varepsilon_E \)–bandlimited on \( F \) if there is a function \( g \in B^1_\alpha(F) \) such that
\[
\| f - g\|_{1,m_\alpha} \leq \varepsilon_E \| f\|_{1,m_\alpha}.
\]
We begin with the following lemma in order to prove the Donoho-Stark type theorem on \( L^1_\alpha(\mathbb{K}) \).

Lemma 4.5 Consider a nonzero function \( f \in B^1_\alpha(F) \), we have
\[
\frac{\| T_E f\|_{1,m_\alpha}}{\| f\|_{1,m_\alpha}} \leq m_\alpha(E)\gamma_\alpha(F).
\]
Proof Let $f \in B^1_\alpha(F) \setminus \{0\}$, according to relation (2.5) we get

$$f(y, \theta) = \int_K \chi_F(\lambda, \mu) F_{\alpha} f(\lambda, \mu) \varphi_{\lambda, \mu}(y, \theta) d\gamma_\alpha(\lambda, \mu).$$

Therefore by Fubini’s theorem, we obtain

$$f(y, \theta) = \int_K f(s, t) \left( \int_F \varphi_{\lambda, \mu}(s, t) \varphi_{\lambda, \mu}(y, \theta) d\gamma_\alpha(\lambda, \mu) \right) dm_\alpha(s, t).$$

From relation (2.1), we get

$$\|f\|_{\infty, m_\alpha} \leq \gamma_\alpha(F) \|f\|_{1, m_\alpha}. \quad (4.6)$$

Furthermore,

$$\|T_E f\|_{1, m_\alpha} = \int_K \chi_E(y, \theta) |f(y, \theta)| dm_\alpha(y, \theta) \leq m_\alpha(E) \|f\|_{\infty, m_\alpha}$$

by using the relation (4.6), we get

$$\|T_E f\|_{1, m_\alpha} \leq m_\alpha(E) \gamma_\alpha(F) \|f\|_{1, m_\alpha}.$$

Then, we gain the wanted result.

□

Theorem 4.6 Consider a nonzero function $f \in L^1_\alpha(K)$ and $\varepsilon_E$, $\varepsilon_F$ two real numbers such that $\varepsilon_E + \varepsilon_F < 1$. If $f$ is $\varepsilon_E$-concentrated on $E$ and $\varepsilon_F$-bandlimited on $F$ in $L^1_\alpha(K)$ then

$$m_\alpha(E) \gamma_\alpha(F) \geq \frac{1 - \varepsilon_E - \varepsilon_F}{1 + \varepsilon_F}.$$

Proof We consider $f \in L^1_\alpha(K) \setminus \{0\}$, we have

$$\|T_E f\|_{1, m_\alpha} = \|f + T_E f - f\|_{1, m_\alpha}.$$

By applying the triangular inequality, we obtain

$$\|T_E f\|_{1, m_\alpha} \geq \|f\|_{1, m_\alpha} - \|f - T_E f\|_{1, m_\alpha}.$$

Since $f$ is $\varepsilon_E$-concentrated on $E$, then

$$\|T_E f\|_{1, m_\alpha} \geq (1 - \varepsilon_E) \|f\|_{1, m_\alpha}. \quad (4.7)$$

On the other hand, $f$ is $\varepsilon_F$-bandlimited so there exists a function $g \in B^1_\alpha(F)$ such that

$$\|f - g\|_{1, m_\alpha} \leq \varepsilon_F \|f\|_{1, m_\alpha}. \quad (4.8)$$

Furthermore, from relation (4.5) we get

$$\|T_E g\|_{1, m_\alpha} \geq \|T_E f\|_{1, m_\alpha} - \|T_E f - T_E g\|_{1, m_\alpha} \geq \|T_E f\|_{1, m_\alpha} - \|f - g\|_{1, m_\alpha}. $$

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Using both relations (4.7) and (4.8), we get
\[ \| T_E g \|_{1,m_\alpha} \geq (1 - \varepsilon_E - \varepsilon_g) \| f \|_{1,m_\alpha}. \]
On the other hand, we have
\[ \| g \|_{1,m_\alpha} \leq (1 + \varepsilon_g) \| f \|_{1,m_\alpha}. \]
Therefore,
\[ \frac{\| T_E g \|_{1,m_\alpha}}{\| g \|_{1,m_\alpha}} \geq \frac{1 - \varepsilon_E - \varepsilon_g}{1 + \varepsilon_g}. \]
Then, by lemma 4.5 we obtain the wanted result. □

In the sequel, we give an \( L_1^\alpha \cap L_2^\alpha \) version of Donoho-Stark theorem for the generalized Fourier transform \( \mathcal{F}_\alpha \).

**Theorem 4.7** Consider a nonzero function \( f \in L_1^\alpha(\mathbb{K}) \cap L_2^\alpha(\mathbb{K}) \). If \( f \) is \( \varepsilon_E \)-concentrated on \( E \) in \( L_1^\alpha(\mathbb{K}) \) and \( \mathcal{F}_\alpha f \) is \( \varepsilon_g \)-cocentered on \( F \) in \( L_2^\alpha(\mathbb{K}) \) then
\[ m_\alpha(E) \gamma_\alpha(F) \geq (1 - \varepsilon_E)^2(1 - \varepsilon_g)^2. \]

**Proof** Assume that \( m_\alpha(E) \) and \( \gamma_\alpha(F) \) are finite. For a nonzero function \( f \in L_1^\alpha(\mathbb{K}) \cap L_2^\alpha(\mathbb{K}) \), we have
\[ \| f \|_{2,m_\alpha} \leq \| f - P_E f \|_{2,m_\alpha} + \| P_E f \|_{2,m_\alpha}. \]
Plancherel’s formula (2.3) gives us the following inequality
\[ \| f \|_{2,m_\alpha} \leq \| \mathcal{F}_\alpha f - \mathcal{F}_\alpha (P_E f) \|_{2,\gamma_\alpha} + \| \chi_E \mathcal{F}_\alpha f \|_{2,\gamma_\alpha}. \]
Since \( \mathcal{F}_\alpha f \) is \( \varepsilon_g \)-cocentered on \( F \) in \( L_2^\alpha(\mathbb{K}) \), we obtain by using relation (4.2)
\[ \| f \|_{2,m_\alpha} \leq \varepsilon_g \| \mathcal{F}_\alpha f \|_{2,\gamma_\alpha} + \left( \int_F |\mathcal{F}_\alpha f(\lambda, \mu)|^2 d\gamma_\alpha(\lambda, \mu) \right)^{\frac{1}{2}} \]
\[ \leq \varepsilon_g \| f \|_{2,m_\alpha} + \sqrt{\gamma_\alpha(F)} \| \mathcal{F}_\alpha f \|_{\infty, \gamma_\alpha}. \]
Furthermore from relation (2.4), we obtain
\[ (1 - \varepsilon_g) \| f \|_{2,m_\alpha} \leq \sqrt{\gamma_\alpha(F)} \| f \|_{1,m_\alpha}. \]  
(4.9)

On the other hand, we have
\[ \| f \|_{1,m_\alpha} \leq \| f - T_E f \|_{1,m_\alpha} + \| T_E f \|_{1,m_\alpha}. \]
Seeing that \( f \) is \( \varepsilon_E \)-concentrated on \( E \) in \( L_1^\alpha(\mathbb{K}) \), we conclude from relation (4.1) that
\[ \| f \|_{1,m_\alpha} \leq \varepsilon_E \| f \|_{1,m_\alpha} + \int_E |f(y, \theta)| dm_\alpha(y, \theta) \]
\[ \leq \varepsilon_E \| f \|_{1,m_\alpha} + \sqrt{m_\alpha(E)} \| f \|_{2,m_\alpha}. \]
Therefore,
\[ (1 - \varepsilon_E) \| f \|_{1,m_\alpha} \leq \sqrt{m_\alpha(E)} \| f \|_{2,m_\alpha}. \]  
(4.10)
Combining (4.9) and (4.10) we reach the needed result. □

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References


