An algorithm to check the equality of total domination number and double of domination number in graphs

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Received: 17.01.2020 • Accepted/Published Online: 18.06.2020 • Final Version: 21.09.2020

Abstract: In graph theory, domination number and its variants such as total domination number are studied by many authors. Let the domination number and the total domination number of a graph \( G \) without isolated vertices be \( \gamma(G) \) and \( \gamma_t(G) \), respectively. Based on the inequality \( \gamma_t(G) \leq 2\gamma(G) \), we investigate the graphs satisfying the upper bound, that is, graphs \( G \) with \( \gamma_t(G) = 2\gamma(G) \). In this paper, we present some new properties of such graphs and provide an algorithm which can determine whether \( \gamma_t(G) = 2\gamma(G) \) or not for a family of graphs not covered by the previous results in the literature.

Key words: Domination number, total domination number

1. Introduction

A graph \( G \) is a pair of sets \( V(G) \) and \( E(G) \). \( V(G) \) is a nonempty set and an element of \( V(G) \) is called a vertex of \( G \). \( E(G) \) is a set of unordered pairs of vertices of \( G \) and an element of \( E(G) \) is called an edge of \( G \). An edge \( \{u, v\} \) is denoted as \( uv \) for convenience in the text. Whenever \( uv \) is an edge in a graph, we say \( u \) and \( v \) are adjacent or neighbors and also say \( u \) and \( v \) are endpoints of \( uv \).

The neighborhood of a vertex \( v \in V(G) \), denoted by \( N_G(v) \), is the set of all neighbors of \( v \) in \( G \). For any subset \( S \subseteq V(G) \), the neighborhood of \( S \) is the union \( \bigcup_{v \in S} N_G(v) \) and is denoted by \( N_G(S) \). The closed neighborhood of a subset \( S \subseteq V(G) \), denoted by \( N_G[S] \), is \( N_G(S) \cup S \). In particular, the closed neighborhood of a vertex \( v \) of \( G \) is denoted by \( N_G[v] \). A vertex \( v \) is called isolated if \( v \) is adjacent to no vertex in the graph \( G \), i.e., \( N_G(v) = \emptyset \). Along this paper, we use \( N(S) \) and \( N[S] \) instead of \( N_G(S) \) and \( N_G[S] \), respectively, as long as the graph \( G \) is implicit in the text.

A set \( S \subseteq V(G) \) of vertices is called a dominating set of \( G \) if every vertex not in \( S \) is adjacent to at least one member of \( S \), in short, when closed neighborhood of \( S \) is \( V(G) \). The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set of \( G \). A subset \( S \) of \( V(G) \) is called a \( \gamma \)-set of \( G \) whenever \( S \) is a minimum dominating set, that is, if \( S \) is a dominating set of \( G \) satisfying \( |S| = \gamma(G) \).

If \( G \) has no isolated vertices, a subset \( S \subseteq V(G) \) is called a total dominating set of \( G \) if every member of \( V(G) \) is adjacent to a vertex in \( S \), i.e., neighborhood of \( S \) is \( V(G) \). The total domination number \( \gamma_t(G) \) of \( G \) without isolated vertices, denoted by \( \gamma_t(G) \), is the minimum size of a total dominating set of \( G \). Note that

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2010 AMS Mathematics Subject Classification: 05C69, 05C85

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there is no total dominating set for graphs having an isolated vertex and therefore, total domination number is defined only for graphs with no isolated vertex. A minimum total dominating set of \( G \) (a total dominating set of \( G \) with cardinality \( \gamma_t(G) \)) is called a \( \gamma_t \)-set of \( G \). See Figure 1 for \( \gamma \)-sets and \( \gamma_t \)-sets of a graph.

![Figure 1](image)

**Figure 1.** In the given graph \( G \), \( \gamma(G) = 2 \) and \( \gamma_t(G) = 3 \). Note that \( \gamma \)-sets of \( G \) are \{1, 6\}, \{2, 6\} and \{4, 6\}, and \( \gamma_t \)-sets of \( G \) are \{2, 3, 6\} and \{2, 5, 6\}.

Since every \( \gamma_t \)-set is also a dominating set, we obtain the inequality \( \gamma(G) \leq \gamma_t(G) \). For any \( \gamma \)-set \( S \), one can expand \( S \) to a total dominating set of cardinality at most \( 2\gamma(G) \) by inserting a neighbor of each vertex in \( S \) and consequently, we get the inequality \( \gamma_t(G) \leq 2\gamma(G) \). Therefore, we have the inequality chain \( \gamma(G) \leq \gamma_t(G) \leq 2\gamma(G) \) involving domination and total domination numbers which is first beheld by [2]. We say a graph \( G \) is a \((\gamma_t, 2\gamma)\)-graph whenever \( \gamma_t(G) = 2\gamma(G) \). Characterization of all \((\gamma_t, 2\gamma)\)-graphs still appears to be an unsolved problem, however, some results on specific families are provided by several authors. [5] characterized \((\gamma_t, 2\gamma)\)-trees in a constructive way, whereas [6] solved the problem for block graphs and gave necessary and sufficient conditions for a graph to be a \((\gamma_t, 2\gamma)\)-block graph. [1] extended these two results for a larger family of graphs (including the class of chordal graphs) and showed that the problem of determining whether a given graph from that family is a \((\gamma_t, 2\gamma)\)-graph or not is polynomial time solvable. In addition to these three works, [3] characterized all \((\gamma_t, 2\gamma)\)-cubic graphs.

In this paper, we study determining whether a given graph is a \((\gamma_t, 2\gamma)\)-graph or not. In general, both of finding the domination number and calculating the total domination number are NP-complete problems (see, [4] and [7], respectively.) Therefore, there does not exist an algorithm with polynomial time complexity which computes both \( \gamma \) and \( \gamma_t \). By using some structural properties of \((\gamma_t, 2\gamma)\)-graphs we establish an algorithm which runs in polynomial time to check whether a given graph is a \((\gamma_t, 2\gamma)\)-graph or not. Even though the algorithm we provide does not work for some graphs, it works for a family of graphs which belongs to none of the classes above.

The rest of this paper is organized as follows: Preliminaries are provided in Section 2, main results of this paper are presented in Section 3, and concluding remarks together with discussion are given in Section 4.

2. Preliminaries

We first provide some definitions required for the statements of the results in this paper. Two vertices \( u \) and \( v \) in \( G \) are called **true twins** if \( N[u] = N[v] \) holds, i.e. \( u \) and \( v \) are true twins if and only if any vertex other than \( u \) and \( v \) is adjacent to either both or none of \( u \) and \( v \).

We borrow some notation and definition from [1] required for the building blocks of this paper. For every vertex \( v \) in \( G \), split \( N[v] \) into three sets \( T(v), D(v) \) and \( M(v) \). The set \( T(v) \) consists of \( v \) itself and its true twins. Let \( D(v) = \{ u \in N[v] : N[u] \subset N[v] \} \). That is, \( u \in D(v) \) if and only if every neighbor of \( u \) other than
v is also a neighbor of v but u is not a true twin of v. All other vertices adjacent to v belong to M(v), i.e. a neighbor of v is in M(v) if and only if it is adjacent to a vertex (other than v) which is not a neighbor of v.

Figure 2. In the given graph, N(7) = {1, 2, 5, 6, 7, 8}, T(7) = {7, 8}, D(7) = {1, 6} and M(7) = {2, 5}. Since none of N(2) = {1, 3, 7, 8} and N(5) = {4, 6, 7, 8} includes D(7) = {1, 6}, we see that 7 is a special vertex. Note also that 7 and 8 are true twins and they are the only special vertices in this graph.

In [1], the results are based on a set of vertices (named special vertices) satisfying some structural properties. Special vertices are also the keystones of this paper. A vertex v is called special if none of the vertices in M(v) has a neighborhood containing D(v). In other words, a vertex v is special if and only if the set of vertices adjacent to every vertex in D(v) is T(v). Note that if D(v) is an empty set and M(v) is not, then v is not special. Moreover, note that if a vertex is special, then any true twin of it is special as well. See Figure 2 for an example of a special vertex.

Notice that being true twins induces an equivalence relation on V(G) and we consider associated equivalence classes of the set of special vertices. In other words, we partition the set of special vertices of G in such a way that two vertices are in the same part if and only if they are true twins. A set obtained by selecting exactly one vertex from each part is called an S(G)-set. Hence, for any special vertex v, every S(G)-set contains exactly one member in T(v).

A graph H is a subgraph of G if V(H) ⊆ V(G) and E(H) ⊆ E(G). For a subset A ⊆ V(G), G − A is the graph obtained by removing the vertices in A together with the edges incident to them from the graph G. A subgraph H of G is an induced subgraph of G whenever H = G − A for some A. The subgraph of G induced by the set A ⊆ V(G) is the graph G − B where B = V(G)\A. In other words, the subgraph of G induced by A is the graph whose vertex set is A and whose edge set consists of all the edges in E(G) that have both endpoints in A.

A graph is called (G1, . . . , Gk)-free if none of G1, . . . , Gk is an induced subgraph of G. A set S ⊆ V(G) is called a packing in G if N[u] and N[v] share no element for every pair of vertices u and v in S. A set is called an efficient dominating set of G if it is both a packing and a dominating set in G. Let H1, H2 and C6 be the graphs shown in Figure 3.

Figure 3. The graphs H1, H2 and C6.
[1] provided the following three results.

**Lemma 2.1** Let $G$ be a $(\gamma_t, 2\gamma)$-graph and $A$ be a subset of $V(G)$. Then, $A$ is a $\gamma$-set of $G$ if and only if $A$ is an efficient dominating set of $G$.

**Lemma 2.2** Let $S$ be an $S(G)$-set. If $S$ is an efficient dominating set of $G$, then $G$ is a $(\gamma_t, 2\gamma)$-graph.

**Theorem 2.3** Let $S$ be an $S(G)$-set of a $(H_1, H_2, C_6)$-free graph $G$. Then, $G$ is a $(\gamma_t, 2\gamma)$-graph if and only if $S$ is an efficient dominating set of $G$.

Lemma 2.2 gives a sufficient condition for $G$ to be a $(\gamma_t, 2\gamma)$-graph. However, this sufficient condition is not always necessary. For example, $C_6$ is a $(\gamma_t, 2\gamma)$-graph but it has no special vertices. On the other hand, Theorem 2.3 shows that this sufficient condition is also necessary whenever the graph has no $H_1, H_2$ or $C_6$ as an induced subgraph. Therefore, if the given graph $G$ has an induced $H_1, H_2$ or $C_6$, then Theorem 2.3 has no conclusion. In this paper, we fill some part of the gap on determining whether a given graph is a $(\gamma_t, 2\gamma)$-graph or not.

### 3. Main results

In this section, we present an algorithm which might detect $(\gamma_t, 2\gamma)$-graphs and non-$(\gamma_t, 2\gamma)$-graphs. We first provide some results on $(\gamma_t, 2\gamma)$-graphs and special vertices which justify why the algorithm works.

**Lemma 3.1** Let $G$ be a $(\gamma_t, 2\gamma)$-graph and $D$ be a $\gamma$-set of $G$. If $v$ is a special vertex, then $|T(v) \cap D| = 1$, that is, exactly one true twin (or itself) of $v$ is in $D$.

**Proof** Let $A = \{v_1, \ldots, v_k\}$. By Lemma 2.1 we see that $A$ is a packing and a dominating set. Therefore, $N[v_1], \ldots, N[v_k]$ is a partition of $V(G)$ and hence $v$ belongs to exactly one of them, say $N[v_1]$. Note that it suffices to show that $v \in T(v_1)$. We will show that $v$ belongs to neither $D(v_1)$ nor $M(v_1)$.

Suppose that $v \in D(v_1)$. Then, since $v$ and $v_1$ are not true twins, $v_1$ has a neighbor which is not adjacent to $v$. Thus, $v_1$ must be in $M(v)$. On the other hand, as $v \in D(v_1)$ we have $D(v) \subseteq N[v] \subset N[v_1]$ which contradicts with the fact that $v$ is special. Hence, we get $v \notin D(v_1)$.

Now assume that $v \in M(v_1)$. Clearly $v$ and $v_1$ are not adjacent for every $i \geq 2$. If $w$ is a neighbor of $v$ in $N(v_i)$ for some $i \geq 2$, then $w \in M(v)$, because $w$ is adjacent to $v_i$ which is not a neighbor of $v$. Therefore, the set $D(v)$ is contained in $N[v_1]$. Then as $v$ is special, $v_1$ cannot be in $M(v)$. Moreover, $v$ and $v_1$ are not true twins and hence, $v_1$ must be in $D(v)$. Then we see that $N[v_1] \subseteq N[v]$, that is, every neighbor of $v_1$ other than $v$ is also a neighbor of $v$. However, in this case, the set $\{v, v_2, w_2, \ldots, v_k, w_k\}$ (where $w_i \in N(v_i)$ for every $i \geq 2$ and one of $w_2, \ldots, w_k$ is a neighbor of $v$ which exists since $v \in M(v_1)$) is a total dominating set of size $2k - 1$, a contradiction.

Notice that in a dominating set, if a vertex is replaced by one of its true twins then the resulting set is a dominating set as well. Consequently, we obtain the following corollaries.

**Corollary 3.2** Let $G$ be a $(\gamma_t, 2\gamma)$-graph and $S$ be an $S(G)$-set. Then, $S$ is a packing and there exists a $\gamma$-set of $G$ containing $S$.

**Corollary 3.3** If an $S(G)$-set is not a packing, then $G$ is not a $(\gamma_t, 2\gamma)$-graph.
Considering the empty graph as a \((\gamma_t, 2\gamma)\)-graph, we have the following lemma.

**Lemma 3.4** Let \(G\) be a \((\gamma_t, 2\gamma)\)-graph, \(X\) be a subset of a \(\gamma\)-set of \(G\) and \(G' = G - N[X]\). Then, \(\gamma(G') = \gamma(G) - |X|\) and \(\gamma_t(G') = 2(\gamma(G) - |X|)\) and hence, \(G'\) is a \((\gamma_t, 2\gamma)\)-graph as well.

**Proof** First note that if \(G'\) is empty, then the claim is trivial. Let \(X = \{x_1, \ldots, x_k\}\) and \(\{x_1, \ldots, x_k, y_1, \ldots, y_m\}\) be a \(\gamma\)-set of \(G\) (so, \(\gamma(G) = k + m\) and \(\gamma_t(G) = 2k + 2m\)). Recall that \(N[x_1], \ldots, N[x_k], N[y_1], \ldots, N[y_m]\) is a partition of \(V(G)\).

Now let \(D\) be a \(\gamma\)-set of \(G'\). Note that \(D \cup X\) is a dominating set of \(G\) and hence, we get \(\gamma(G') + k \geq \gamma(G) = k + m\) i.e., \(\gamma(G') \geq m\). On the other hand, \(\{y_1, \ldots, y_m\}\) is a dominating set of \(G'\) and therefore, we obtain \(\gamma(G') = m\).

Next, let \(T\) be a \(\gamma_t\)-set of \(G'\) and \(w_i\) be a neighbor of \(x_i\) in \(G\) for \(i = 1, \ldots, k\). Then, \(T \cup \{x_1, w_1, \ldots, x_k, w_k\}\) is a \(\gamma_t\)-set of \(G\) and hence, we get \(\gamma_t(G') + 2k \geq \gamma_t(G) = 2k + 2m\), that is, \(\gamma_t(G') \geq 2m\). Moreover, we have \(\gamma_t(G') \leq 2\gamma(G') = 2m\) and thus, we get \(\gamma_t(G') = 2m = 2\gamma(G')\).

Combining Corollary 3.2 and Lemma 3.4 yields the following result.

**Proposition 3.5** If \(G\) is a \((\gamma_t, 2\gamma)\)-graph, then \(G - N[S]\) is a \((\gamma_t, 2\gamma)\)-graph and \(\gamma(G - N[S]) = \gamma(G) - |S|\) for any \(S(G)\)-set \(S\).

Proposition 3.5 implies that removing the vertices of an \(S(G)\)-set together with their neighbors from a \((\gamma_t, 2\gamma)\)-graph \(G\) reveals another \((\gamma_t, 2\gamma)\)-graph which is an induced subgraph of \(G\). However, the converse of this implication is not always true. If \(G - N[S]\) is a \((\gamma_t, 2\gamma)\)-graph for an \(S(G)\)-set, then \(G\) itself does not have to be a \((\gamma_t, 2\gamma)\)-graph. See Figure 4 for an example.

\[\text{Figure 4. In the given graph } G, 2 \text{ is the unique special vertex. Note that } \gamma(G - N[2]) = 1 \text{ and } \gamma_t(G - N[2]) = 2 \text{ and hence, } G - N[2] \text{ is a } (\gamma_t, 2\gamma)\text{-graph. However, } G \text{ is not a } (\gamma_t, 2\gamma)\text{-graph since } \gamma(G) = 2 \text{ and } \gamma_t(G) = 3.\]

**Proposition 3.6** Let \(G\) be a \((\gamma_t, 2\gamma)\)-graph. Let \(G_0 = G\) and \(G_{i+1} = G_i - N[S_i]\) for \(i = 0, \ldots, m\) where each \(S_i\) is a nonempty \(S(G_i)\)-set. If \(v\) is a special vertex of \(G_{m+1}\), then \(N_G[v] \cap N_{G_i}[v] = \emptyset\) for \(i = 0, \ldots, m\).

**Proof** By Proposition 3.5 we see that \(G_1, \ldots, G_{m+1}\) are \((\gamma_t, 2\gamma)\)-graphs. By Corollary 3.2, there exists a \(\gamma\)-set of \(G_{m+1}\) containing \(v\), say \(D\). Proposition 3.5 yields that \(\gamma(G_m) = |D| + |S_m|\). On the other hand, \(D \cup S_m\) is a dominating set of \(G_m\) and therefore, \(D \cup S_m\) is a \(\gamma\)-set of \(G_m\). Applying Proposition 3.5 recursively gives that \(D \cup (\cup_{i=0}^m S_m)\) is a \(\gamma\)-set of \(G_0 = G\). As \(G\) is a \((\gamma_t, 2\gamma)\)-graph, \(D \cup (\cup_{i=0}^m S_m)\) is a packing in \(G\) and the result follows. \(\square\)
By using Proposition 3.6 we can construct a procedure which concludes $\gamma_t(G) = 2\gamma(G)$, $\gamma_t(G) \neq 2\gamma(G)$ or no comment, see Algorithm 1.

Data: A graph $G$
Result: $\gamma_t(G) = 2\gamma(G)$, $\gamma_t(G) \neq 2\gamma(G)$ or “test fails”

Let $A$ be an $S(G)$-set;
if $A$ is an efficient dominating set of $G$ then
| $\gamma_t(G) = 2\gamma(G)$
else
  initialization;
  $H = G$;
  $T = \emptyset$;
  while $H \neq \emptyset$ do
    Let $S_H$ be the set of all special vertices of $H$;
    if $S_H = \emptyset$ then
      Test fails
    end
    if $T \cap N_G(s) \neq \emptyset$ for some $s \in S_H$ then
      $\gamma_t(G) \neq 2\gamma(G)$
    end
    Let $X_H$ be an $S(H)$-set;
    if $X_H$ is not a packing in $H$ then
      $\gamma_t(G) \neq 2\gamma(G)$
    else
      $H \leftarrow H - N_G[X_H]$;
      $T \leftarrow T \cup N_G[X_H]$;
    end
  end
  if $H = \emptyset$ then
    Test fails
  end
end

Algorithm 1: A test that might detect $(\gamma_t, 2\gamma)$-graphs and non-$(\gamma_t, 2\gamma)$-graphs.

4. Conclusion and discussion

In this paper, some new properties of $(\gamma_t, 2\gamma)$-graphs and based on these results we provide an algorithm to designate whether a given graph $G$ satisfies $\gamma_t(G) = 2\gamma(G)$ or not. Pseudo code of the procedure is given in Algorithm 1. It is easy to see that in Algorithm 1 the number of operations is bounded above by a polynomial in terms of the number of vertices in the given graph. Therefore, Algorithm 1 has a polynomial time complexity.

Moreover, Algorithm 1 works for some graphs which belong to none of the classifications in previous papers on $(\gamma_t, 2\gamma)$-graphs. As an example, consider the graph $G$ shown in Figure 5. Since $G$ is not a cubic graph, [3] has no conclusion on $G$. In addition, $G$ has both induced $C_6$ and $H_1$ and hence, [1] provides no result for $G$. However, Algorithm 1 implies that $G$ is not a $(\gamma_t, 2\gamma)$-graph. On the other hand, there are some graphs, such as $C_6$, whose output is “test fails” in Algorithm 1 although it is a $(\gamma_t, 2\gamma)$-graph. Improving the existing algorithm to make it work for a larger family of graphs and investigating more properties of $(\gamma_t, 2\gamma)$-graphs are topics of ongoing research.
Figure 5. In graph $G$, special vertices are 2 and 14, and hence $\{2,14\}$ is the only $S(G)$-set. Special vertices of $G - N[\{2,14\}]$ are 7 and 9, but they share a common neighbor. Therefore, $G$ is not a $(\gamma_t,2\gamma)$-graph by Algorithm 1. Indeed, $\gamma(G) = 4$ ($\{2,7,9,14\}$ is a $\gamma$-set) and $\gamma_t(G) = 7$ ($\{2,3,7,8,9,13,14\}$ is a $\gamma_t$-set).

Acknowledgments

This work is supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) under grant no. 118E799.

References


