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Some results on top generalized local cohomology modules with respect to a system of ideals

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Abstract: Let R be a commutative Noetherian ring and Φ be a system of ideals of R. In this paper, we study the annihilators and the set of attached prime ideals of top generalized local cohomology modules with respect to a system of ideals.

Key words: Annihilators, attached prime ideals, cohomological dimension, generalized local cohomology

1. Introduction

Throughout this paper, R is a commutative Noetherian ring and M is a finitely generated R-module. Let Φ be a nonempty set of ideals of R. We call Φ a system of ideals of R if whenever $\mathfrak{a}, \mathfrak{b} \in \Phi$, then there is an ideal $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$ (see [5]). For every R-module N, one can define

$$\Gamma_\Phi(N) = \{x \in N \mid \mathfrak{a}x = 0 \text{ for some } \mathfrak{a} \in \Phi\}.$$

Then $\Gamma_\Phi(-)$ is an additive, covariant, R-linear and left exact functor from the category of R-modules to itself. The functor $\Gamma_\Phi(-)$ is called the general local cohomology functor with respect to Φ. For each $i \geq 0$, the ith right derived functor of $\Gamma_\Phi(-)$ is denoted by $H^i_\Phi(-)$. For an ideal $\mathfrak{a}$ of R, if $\Phi = \{\mathfrak{a}^n \mid n \in \mathbb{N}_0\}$, then the functor $H^i_\Phi(-)$ coincides with the ordinary local cohomology functor $H^i_\mathfrak{a}(-)$. Some basic properties of the local cohomology modules with respect to Φ were shown in [1, 5, 7].

In [6], the author introduced the bifunctor $H^i_\Phi(-,-)$ as follows: Let M, N be two R-modules. Then the module $H^i_\Phi(M,N)$ is defined as

$$H^i_\Phi(M,N) \cong \varinjlim_{\mathfrak{a}\in\Phi} \operatorname{Ext}^i_R(M/\mathfrak{a}M, N).$$

This bifunctor is contravariant in the first variable and covariant in the second variable. If $\Phi = \{\mathfrak{a}^n \mid n \in \mathbb{N}_0\}$, then the bifunctor $H^i_\Phi(-,-)$ is naturally equivalent to the bifunctor $H^i_\mathfrak{a}(-,-)$ of Herzog in [17].

In [19], a nonzero R-module N is said to be secondary precisely when $N \neq 0$ and, for each $r \in R$, either $rN = N$ or there exists $n \in \mathbb{N}$ such that $r^n N = 0$. Then the ideal $\mathfrak{p} := \sqrt{\operatorname{Ann}_R N}$ is a prime ideal and N is called $\mathfrak{p}$-secondary. A secondary representation of an R-module N is an expression of N as a sum of

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finitely many secondary submodules of N. If an R-module N has a secondary representation, then it is said to be representable. A secondary representation of $N = N_1 + N_2 + \cdots + N_n$ is called minimal if the prime ideals $\mathfrak{p}_i = \sqrt{\mathrm{Ann}_R N_i}, i = 1, 2, \ldots, n$ are all distinct and none of N_i is redundant. The set $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ is independent of the choice of the minimal secondary representation of N. It is denoted by $\mathrm{Att}_R N$ and called the set of attached prime ideals of N.

In [24], a prime ideal $\mathfrak{p}$ of R is called an attached prime of an R-module N (not necessarily admitting a secondary representation) if $\mathfrak{p} = \mathrm{Ann}_R(N/T)$ for some submodule T of N. If N admits a secondary representation, then this definition agrees with the preceding one of attached prime ideals.

The set of attached prime ideals of local cohomology modules, which have been studied by many authors, is one of the most interesting properties. In the case of generalized local cohomology modules with respect to an ideal, we see that $\mathrm{Att}_R(H^d_{\mathfrak{m}}(M, N)) = \mathrm{Ass}_R\, M \cap \mathrm{Supp}_R N$ whenever $(R, \mathfrak{m})$ is a d-dimensional Gorenstein local ring ([15, Lemma 2.4]) or $(R, \mathfrak{m})$ is a d-dimensional Cohen-Macaulay local ring ([16, Corollary 3.4]).

In this paper, we study the cohomological dimension of M, N with respect to Φ, the annihilators and the set of attached prime ideals of top generalized local cohomology modules with respect to Φ. In Section 2, we show some properties of cohomological dimension of M, N with respect to a Φ, which is donoted by

$$\mathrm{cd}_\Phi(M, N) := \sup\{i \mid H^i_\Phi(M, N) \neq 0\}.$$

This direction of research is motivated by Amjadi and Naghipour's work [2] and the one of Divaani et al. [14]. In particular, $\mathrm{cd}_\Phi(R, N)$ is denoted by $\mathrm{cd}_\Phi(N)$. The next section shows some results on the annihilators of top generalized local cohomology modules with respect to Φ.

Theorem 1.1 (See Theorem 3.3) *Let M, N be two nonzero finitely generated R-modules with cohomological dimension $c := \mathrm{cd}_\Phi(M, N)$. Then*

$$\mathrm{Ann}_R H^c_\Phi(M, N) = \mathrm{Ann}_R(N/T_\Phi(M, N)),$$

where $T_\Phi(M, N)$ is the largest submodule of N such that $\mathrm{cd}_\Phi(M, T_\Phi(M, N)) < c$.

A consequence of Theorem 3.3 is Corollary 3.4 which shows that

$$\mathrm{Ann}_R H^c_\Phi(M, R) = \bigcap_{\mathrm{cd}_\Phi(M, R/\mathfrak{p}_i) = c} \mathfrak{q}_i$$

where $c = \mathrm{cd}_\Phi(M, R)$ and $0 = \bigcap_{\mathfrak{p}_i \in \mathrm{Ass}_R R} \mathfrak{q}_i$ is a reduced primary decomposition of the zero ideal of R, $\mathfrak{q}_i$ is a $\mathfrak{p}_i$-primary ideal of R.

The set of attached prime ideals of top generalized local cohomology modules with respect to Φ are presented in Section 4. The first main result of Section 4 is the following Theorem.

Theorem 1.2 (See Theorem 4.4) *Let M, N be two nonzero finitely generated R-modules such that $p := \mathrm{pd} M < \infty$ and $d := \dim N = \dim R$. Assume that $\mathrm{cd}_\Phi(M, N) = p + d$. Then*

$$\mathrm{Att}_R(H^{p+d}_\Phi(M, N)) = \{\mathfrak{p} \in \mathrm{Ass}_R\, N \mid \mathrm{cd}_\Phi(M, R/\mathfrak{p}) = p + d\}.$$

Moreover, $\mathrm{Att}_R H^{p+d}_\Phi(M, N) \subseteq \mathrm{Att}_R H^d_\Phi(N)$.

Let R be a Cohen-Macaulay local ring with $\dim R > 0$. Assume that M is a nonzero Cohen-Macaulay finitely generated R-module with $\mathrm{pd} M < \infty$ and $\mathrm{cd}_\Phi(M,R) = \dim R$. Theorem 4.10 says that

$$\mathrm{Att}_R(H_\Phi^{\dim R}(M,R)) = \mathrm{Att}_R(H_\Phi^{\dim M}(M)).$$

The last section is devoted to the study of the set of attached prime ideals of generalized local cohomology modules with respect to a pair of ideals which were introduced in [21]. The results in Section 5 are special cases of Section 4, when we consider Φ as the set $\tilde{W} = \{\mathfrak{a} \text{ is an ideal of } R \mid I^n \subseteq \mathfrak{a} + J \text{ for some integer } n\}$.

Throughout this paper, M is always a finitely generated R-module with finite projective dimension. We denote the projective dimension of M by $\mathrm{pd} M$. For each ideal $\mathfrak{a}$ of R, the supremum of i's such that $H_\mathfrak{a}^i(M,N) \neq 0$ is denoted by $\mathrm{cd}_\mathfrak{a}(M,N)$, and we abbreviate $\mathrm{cd}_\mathfrak{a}(R,N)$ by $\mathrm{cd}_\mathfrak{a}(N)$. We denote by $\widehat{R}$ and $\widehat{N}$ the $\mathfrak{m}$-adic completion of R and R-module N, respectively.

2. Cohomological dimension with respect to a system of ideals

Firstly, we investigate the cohomological dimension with respect to a system of ideals. Results of this section will be used in the following sections. We also extend some facts of [2].

Definition 2.1 *Let M be a finitely generated R-module of finite projective dimension and N be an R-module. The cohomological dimension of M,N with respect to Φ is defined as*

$$\mathrm{cd}_\Phi(M,N) := \sup\{i \mid H_\Phi^i(M,N) \neq 0\},$$

if this supremum exists, otherwise, we define it $-\infty$.

Lemma 2.2 *[6, Lemma 2.1] Let M be a finitely generated R-module. Then*

$$H_\Phi^i(M,N) \cong \varinjlim_{\mathfrak{a}\in\Phi} H_\mathfrak{a}^i(M,N)$$

for all R-module N and $i \geq 0$.

If M is a nonzero finitely generated R-module of finite projective dimension and N is an R-module of finite Krull dimension, then it follows from Lemma 2.2, [12, Corollary 3.2] and [6, Proposition 5.2] that

$$\mathrm{cd}_\Phi(M,N) \leq \min\{\dim R, \mathrm{pd} M + \dim N\}$$

and

$$\mathrm{cd}_\Phi(M,N) \leq \sup\{\mathrm{cd}_\mathfrak{a}(M,N) \mid \mathfrak{a} \in \Phi\}.$$

Proposition 2.3 *Let M be a finitely generated R-module and N be an R-module. Then*

$$\mathrm{cd}_\Phi(M,N) \leq \sup\{\mathrm{cd}_\Phi(M,K) \mid K \textit{ is a finitely generated submodule of } N\}.$$

Proof Since $H_\Phi^i(M,-)$ commutes with direct limits and N is a direct limit of all finitely generated submodules of N, the assertion follows. □

Proposition 2.4 *Let M, N be two finitely generated R-modules and K be an R-module such that* $\mathrm{Supp}_R K \subseteq \mathrm{Supp}_R N$. *Then* $\mathrm{cd}_\Phi(M, K) \leq \mathrm{cd}_\Phi(M, N)$.

Proof Assume that K is a finitely generated R-module such that $\mathrm{Supp}_R K \subsetneq \mathrm{Supp}_R M$. Using the same method in the proof of [2, Theorem B] we see that $\mathrm{cd}_\Phi(M, K) \leq \mathrm{cd}_\Phi(M, N)$. Hence, the claim follows from Proposition 2.3. □

The following result is implied immediately from Proposition 2.4.

Corollary 2.5 *Let M, N, K be three finitely generated R-modules such that* $\mathrm{Supp}_R N = \mathrm{Supp}_R K$. *Then* $\mathrm{cd}_\Phi(M, N) = \mathrm{cd}_\Phi(M, K)$.

Theorem 2.6 *Let M, N be two finitely generated R-modules. Then*

$$\mathrm{cd}_\Phi(M, N) = \sup\{\mathrm{cd}_\Phi(M, R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}_R N\}.$$

Moreover, $\mathrm{cd}_\Phi(M, N) = \mathrm{cd}_\Phi(M, R/\mathfrak{p})$ *for some prime ideal $\mathfrak{p}$ which is a minimal element of* $\mathrm{Supp}_R N$.

Proof We have by Proposition 2.4 that

$$\mathrm{cd}_\Phi(M, N) \geq \sup\{\mathrm{cd}_\Phi(M, R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}_R N\}.$$

Now let $m = \sup\{\mathrm{cd}_\Phi(M, R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}_R N\}$ and $n = \mathrm{cd}_\Phi(M, N)$. Suppose that $m < n$, and we look for a contradiction. It follows from [20, 6.4] that there is a filtration of submodules of N

$$0 = N_0 \subsetneq N_1 \subsetneq \ldots \subsetneq N_k = N$$

such that $N_i/N_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Supp}_R N$ for all $i = 1, 2, \ldots, k$. Let $i \geq 1$, the short exact sequence

$$0 \to N_{i-1} \to N_i \to R/\mathfrak{p}_i \to 0$$

induces a long exact sequence

$$H_\Phi^{n-1}(M, R/\mathfrak{p}_i) \to H_\Phi^n(M, N_{i-1}) \to H_\Phi^n(M, N_i) \to H_\Phi^n(M, R/\mathfrak{p}_i).$$

Note that $H_\Phi^n(M, R/\mathfrak{p}_i) = 0$ for all $i \geq 1$ and $H_\Phi^n(M, N_1) \cong H_\Phi^n(M, R/\mathfrak{p}_1) = 0$. It follows from the long exact sequence that $H_\Phi^n(M, N_i) = 0$ for all $1 \leq i \leq k$. In particular, $0 = H_\Phi^n(M, N_k) = H_\Phi^n(M, N)$, which is a contradition.

If $\mathfrak{p} \in \mathrm{Supp}_R N$, then there exists a prime ideal $\mathfrak{q}$ which is a minimal element of $\mathrm{Supp}_R N$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. According to Proposition 2.4, we have $\mathrm{cd}_\Phi(M, R/\mathfrak{p}) \leq \mathrm{cd}_\Phi(M, R/\mathfrak{q})$. This implies that $\mathrm{cd}_\Phi(M, R/\mathfrak{q}) = \mathrm{cd}_\Phi(M, N)$, and the proof is complete. □

3. Annihilators of top generalized local cohomology modules with respect to a system of ideals

We will present some results on the annihilators of top generalized local cohomology modules with respect to a system of ideals. Firstly, we need the following concept.

Definition 3.1 *Let M, N be two nonzero finitely generated R-modules. We denote by $T_\Phi(M, N)$ the largest submodule of N such that* $\mathrm{cd}_\Phi(M, T_\Phi(M, N)) < \mathrm{cd}_\Phi(M, N)$.

It is easy to check that

$$T_\Phi(M,N) = \bigcup\{K \mid K \text{ is a submodule of } N \text{ and } \operatorname{cd}_\Phi(M,K) < \operatorname{cd}_\Phi(M,N)\}.$$

The first result of this section gives a decomposition of $T_\Phi(M,N)$ based on a reduced primary decomposition of the zero submodule of N.

Theorem 3.2 *Let M, N be two nonzero finitely generated R-modules with cohomological dimension $c := \operatorname{cd}_\Phi(M,N)$. Assume that $0 = \bigcap_{i=1}^n N_i$ is a reduced primary decomposition of the zero submodule of N and N_i is a $\mathfrak{p}_i$-primary submodule of N. Then*

$$T_\Phi(M,N) = \bigcap_{\mathfrak{p}_i \in \operatorname{Ass}_R(N), \operatorname{cd}_\Phi(M,R/\mathfrak{p}_i)=c} N_i.$$

Proof Let

$$Q = \bigcap_{\mathfrak{p}_i \in \operatorname{Ass}_R(M), \operatorname{cd}_\Phi(M,R/\mathfrak{p}_i)=c} N_i \text{ and } K = \bigcap_{\mathfrak{p}_i \in \operatorname{Ass}_R(M), \operatorname{cd}_\Phi(M,R/\mathfrak{p}_i)<c} N_j.$$

By the hypothesis, we have $K \cap Q = 0$. Thus, there is an exact sequence

$$0 \to Q \to N/K.$$

It follows from Proposition 2.4 that $\operatorname{cd}_\Phi(M,Q) \leq \operatorname{cd}_\Phi(M,N/K)$. Note that

$$\operatorname{Ass}_R Q \subseteq \operatorname{Ass}_R(N/K) = \{\mathfrak{p} \in \operatorname{Ass}_R(N) \mid \operatorname{cd}_\Phi(M,R/\mathfrak{p}) < c\}.$$

By Theorem 2.6, we have $\operatorname{cd}_\Phi(M,N/K) < c$ and then $\operatorname{cd}_\Phi(M,Q) < c$. This implies that $Q \subseteq T_\Phi(M,N)$.

Now, let $x \in T_\Phi(M,N)$ and it is easy to see that $\operatorname{cd}_\Phi(M,Rx) < c$. Note that $H^c_\Phi(M,R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Ass}_R(Rx)$. Therefore, one gets that

$$\operatorname{Ass}_R(Rx) \subseteq \{\mathfrak{p} \in \operatorname{Ass}_R(N) \mid \operatorname{cd}_\Phi(M,R/\mathfrak{p}) < c\}.$$

Therefore,

$$\bigcap_{\mathfrak{p} \in \operatorname{Ass}_R(N), \operatorname{cd}_\Phi(M,R/\mathfrak{p})<c} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R(Rx)} \mathfrak{p} = \sqrt{\operatorname{Ann}_R(Rx)}.$$

Let $J = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R(N), \operatorname{cd}_\Phi(M,R/\mathfrak{p})<c} \mathfrak{p}$, there exists a positive integer m such that

$$J^m x = 0.$$

By the primary decomposition of the zero submodule of N, we have

$$J^m x \in N_i$$

for all $1 \leq i \leq n$. Assume that there exists an R-module N_j such that $x \notin N_j$ and $\operatorname{cd}_\Phi(M,R/\mathfrak{p}_j) = c$. Since N_j is $\mathfrak{p}_j$-primary, we can conclude that $J^m \subseteq \mathfrak{p}_j$. This implies that there is a prime ideal $\mathfrak{p}_k \in \operatorname{Ass}_R N$ such that $\operatorname{cd}_\Phi(M,R/\mathfrak{p}_k) < c$ and $\mathfrak{p}_k \subseteq \mathfrak{p}_j$. Consequently, we have

$$c = \operatorname{cd}_\Phi(M,R/\mathfrak{p}_j) \leq \operatorname{cd}_\Phi(M,R/\mathfrak{p}_k) < c,$$

which is a contradition. Hence, $x \in Q$ and then $T_\Phi(M,N) = Q$. □

We are going to state and prove the first main result of this paper. The following theorem is an extension of [3, 2.3].

Theorem 3.3 *Let* M, N *be two nonzero finitely generated* R*-modules with cohomological dimension* $c := \mathrm{cd}_\Phi(M,N)$. *Then*

$$\mathrm{Ann}_R H^c_\Phi(M,N) = \mathrm{Ann}_R(N/T_\Phi(M,N)).$$

Proof The short exact sequence

$$0 \to T_\Phi(M,N) \to N \to N/T_\Phi(M,N) \to 0$$

induces the following exact sequence

$$H^c_\Phi(M, T_\Phi(M,N)) \to H^c_\Phi(M,N) \to H^c_\Phi(M, N/T_\Phi(M,N)) \to 0.$$

Since $\mathrm{cd}_\Phi(M, T_\Phi(M,N)) < c$, there is an isomorphism

$$H^c_\Phi(M,N) \cong H^c_\Phi(M, N/T_\Phi(M,N)).$$

The proof is complete by showing that

$$\mathrm{Ann}_R H^c_\Phi(M, N/T_\Phi(M,N)) = \mathrm{Ann}_R(N/T_\Phi(M,N)).$$

Let $\overline{N} = N/T_\Phi(M,N)$ and it is clear that

$$\mathrm{Ann}_R \overline{N} \subseteq \mathrm{Ann}_R H^c_\Phi(M, \overline{N}).$$

Let $x \in \mathrm{Ann}_R H^c_\Phi(M,\overline{N})$, we will prove that $x \in \mathrm{Ann}_R \overline{N}$. The short exact sequence

$$0 \to 0 :_{\overline{N}} x \to \overline{N} \xrightarrow{x} x\overline{N} \to 0$$

deduces the long exact sequence

$$H^c_\Phi(M, 0 :_{\overline{N}} x) \to H^c_\Phi(M, \overline{N}) \xrightarrow{x} H^c_\Phi(M, x\overline{N}) \to 0.$$

Since $x \in \mathrm{Ann}_R H^c_\Phi(M,\overline{N})$, it follows that $H^c_\Phi(M, x\overline{N}) = xH^c_\Phi(M,\overline{N}) = 0$ and then $\mathrm{cd}_\Phi(M, x\overline{N}) < c$. By the definition of $T_\Phi(M,N)$, we can conclude that $x\overline{N} = 0$. Hence, $x \in \mathrm{Ann}_R \overline{N}$ and the proof is complete. □

Corollary 3.4 *Let* R *be a ring with cohomological dimension* $c := \mathrm{cd}_\Phi(M,R)$. *Then*

$$\mathrm{Ann}_R H^c_\Phi(M,R) = T_\Phi(M,R) = \bigcap_{\mathrm{cd}_\Phi(M,R/\mathfrak{p}_i)=c} \mathfrak{q}_i,$$

where $0 = \bigcap_{\mathfrak{p}_i \in \mathrm{Ass}_R R} \mathfrak{q}_i$ *is a reduced primary decomposition of the zero ideal of* R *and* $\mathfrak{q}_i$ *is a* $\mathfrak{p}_i$*-primary ideal of* R.

Proof It follows from Theorem 3.2 and Theorem 3.3. □

Corollary 3.5 *Let R be a ring of finite dimension d and $\operatorname{cd}_\Phi(M,R) = c$. Then the following conditions are equivalent:*

(i) $\operatorname{Ann}_R(H^c_\Phi(M,R)) = 0$.

(ii) $\operatorname{Ass}_R R = \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{cd}_\Phi(M, R/\mathfrak{p}) = c\}$.

Proof It follows from Corollary 3.4. □

Corollary 3.6 *Let R be a domain such that $\operatorname{cd}_\Phi(M,R) = \dim R$. Then*

$$\operatorname{Ann}_R H^{\dim R}_\Phi(M,R) = 0.$$

Proof If R is a domain, then we have $\operatorname{Ass}_R R = \{0\}$. The assertion follows from Corollary 3.4. □

Corollary 3.7 *[3, Corollary 2.10] Let R be a domain such that $\operatorname{cd}_\mathfrak{a}(R) = \dim R$. Then*

$$\operatorname{Ann}_R H^{\dim R}_\mathfrak{a}(R) = 0.$$

4. Attached primes of top generalized local cohomology modules with respect to a system of ideals

The set of attached prime ideals of an R-module M is denoted by $\operatorname{Att}_R M$. The attached prime ideals have been studied by Zöschinger [24]. In the case where M is a representable R-module, this definition agrees with the one of Macdonald [19].

Definition 4.1 (See [24]) *Let M be an R-module. A prime ideal $\mathfrak{p}$ of $\operatorname{Spec} R$ is called attached to M if there is a submodule N of M such that $\mathfrak{p} = \operatorname{Ann}_R(M/N)$.*

The following facts are basic properties of the set of attached prime ideals.

Lemma 4.2 (See [1]) *The following statements hold true.*

(i) *If $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules, then*

$$\operatorname{Att}_R C \subseteq \operatorname{Att}_R B \subseteq \operatorname{Att}_R C \cup \operatorname{Att}_R A.$$

(ii) *If N is a finitely generated R-module, then*

$$\operatorname{Att}_R(M \otimes_R N) = \operatorname{Att}_R M \cap \operatorname{Supp}_R N$$

for all R-module M.

Lemma 4.3 *Let R be a ring of finite cohomological dimension $c := \operatorname{cd}_\Phi(M,R)$. Then*

$$\operatorname{Att}_R(H^c_\Phi(M,R)) \subseteq \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{cd}_\Phi(M, R/\mathfrak{p}) = c\}.$$

Proof Let $\mathfrak{p} \in \mathrm{Att}_R(H^c_\Phi(M,R))$, we have by the right exactness of $H^c_\Phi(M,-)$ that

$$0 \neq H^c_\Phi(M,R)/\mathfrak{p}H^c_\Phi(M,R) \cong H^c_\Phi(M,R/\mathfrak{p}).$$

Hence, $\mathrm{cd}_\Phi(M,R/\mathfrak{p}) = c$, and the proof is complete. □

We are going to state and prove another main result of this paper which describes the set of attached prime ideals of $H_\Phi^{\mathrm{pd}M+\dim R}(M,N)$.

Theorem 4.4 *Let M,N be two nonzero finitely generated R-module such that $p := \mathrm{pd}M < \infty$ and $d := \dim N = \dim R < \infty$. Assume that $\mathrm{cd}_\Phi(M,N) = p+d$. Then*

$$\mathrm{Att}_R(H^{p+d}_\Phi(M,N)) = \{\mathfrak{p} \in \mathrm{Ass}_R N \mid \mathrm{cd}_\Phi(M,R/\mathfrak{p}) = p+d\}.$$

Moreover, $\mathrm{Att}_R H^{p+d}_\Phi(M,N) \subseteq \mathrm{Att}_R H^d_\Phi(N)$.

Proof Let $F := \mathrm{Hom}(M,-)$ and $G := \Gamma_\Phi(-)$ be two functors from the category of R-modules to itself. It is clear that $FG(N) \cong \Gamma_\Phi(M,N)$ for any R-module N and $R^iF(G(E)) = 0$ for any $i > 0$ and any injective R-module E. There is a spectral sequence by [22, Theorem 10.47]

$$E_2^{i,j} = \mathrm{Ext}^i_R(M, H^j_\Phi(N)) \underset{i}{\Rightarrow} H^{i+j}_\Phi(M,N).$$

Note that $E_2^{i,j} = 0$ for all $i > p$ and by [5, 2.7] we have $E_2^{i,j} = 0$ for all $j > d$. Therefore, $E_\infty^{i,j} = 0$ for all $i > p$ or $j > d$ since $E_\infty^{i,j}$ is a subquotient of $E_2^{i,j}$. There is a filtration of submodules of $H^{p+d} := H^{p+d}_\Phi(M,N)$

$$0 = \varphi^{p+d+1}H^{p+d} \subseteq \varphi^{p+d}H^{p+d} \subseteq \ldots \subseteq \varphi^1 H^{p+d} \subseteq \varphi^0 H^{p+d} = H^{p+d}$$

such that

$$E_\infty^{i,p+d-i} \cong \varphi^i H^{p+d}/\varphi^{i+1}H^{p+d}$$

for all $0 \leq i \leq p+d$. By the above analysis, we obtain that

$$\varphi^{p+1}H^{p+d} = \varphi^{p+2}H^{p+d} = \ldots = \varphi^{p+d+1}H^{p+d} = 0$$

and

$$\varphi^p H^{p+d} = \varphi^{p-1}H^{p+d} = \ldots = \varphi^0 H^{p+d} = H^{p+d}.$$

Consequently, there is an isomorphism

$$H^{p+d}_\Phi(M,N) \cong \mathrm{Ext}^p_R(M, H^d_\Phi(N)).$$

Let $\mathfrak{p} \in \mathrm{Att}_R H^{p+d}_\Phi(M,N)$, we have by the right exactness of the functor $H^{p+d}_\Phi(M,-)$ that

$$0 \neq H^{p+d}_\Phi(M,N)/\mathfrak{p}H^{p+d}_\Phi(M,N) \cong H^{p+d}_\Phi(M,R/\mathfrak{p}) \otimes_R N.$$

Hence, $H^{p+d}_\Phi(M,R/\mathfrak{p}) \neq 0$ and we get by Proposition 2.4 that $\mathrm{cd}_\Phi(M,R/\mathfrak{p}) = p+d$. Since $\mathrm{Ext}^p_R(M,-)$ is a right exact functor, we obtain

$$\mathrm{Ext}^p_R(M, H^d_\Phi(N)) \cong H^d_\Phi(N) \otimes_R \mathrm{Ext}^p_R(M,R).$$

According to Lemma 4.2, we have

$$\mathrm{Att}_R H_\Phi^{p+d}(M,N) = \mathrm{Att}_R(H_\Phi^d(N)) \cap \mathrm{Supp}_R \mathrm{Ext}_R^p(M,R)$$

and

$$0 \neq H_\Phi^{p+d}(M,N)/\mathfrak{p}H_\Phi^{p+d}(M,N) \cong \mathrm{Ext}_R^p(M,R) \otimes_R H_\Phi^d(N)/\mathfrak{p}H_\Phi^d(N).$$

This implies that $H_\Phi^d(N)/\mathfrak{p}H_\Phi^d(N) \neq 0$. Note that $H_\Phi^d(-)$ is a right exact functor by [5, 2.7]. It is clear that $H_\Phi^d(R/\mathfrak{p}) \neq 0$. Therefore, we can conclude that $\dim R/\mathfrak{p} = d$ and then $\mathfrak{p} \in \mathrm{Ass}_R N$.

Let $\mathfrak{q} \in \mathrm{Ass}_R N$ such that $\mathrm{cd}_\Phi(M, R/\mathfrak{q}) = p + d$. There exists a submodule K of N such that K is $\mathfrak{q}$-primary and $\mathrm{Ass}_R(N/K) = \{\mathfrak{q}\}$. It follows from Theorem 2.6 that $\mathrm{cd}_\Phi(M, N/K) = \mathrm{cd}_\Phi(M, R/\mathfrak{q}) = p + d$. By the above argument, we see that

$$\mathrm{Att}_R H_\Phi^{p+d}(M, N/K) \subseteq \{\mathfrak{p} \in \mathrm{Ass}_R N/K \mid \mathrm{cd}_\Phi(M, R/\mathfrak{p}) = p + d\} = \{\mathfrak{q}\}.$$

Now the short exact sequence

$$0 \to K \to N \to N/K \to 0$$

leads to the following exact sequence

$$H_\Phi^{p+d}(M,N) \to H_\Phi^{p+d}(M,N/K) \to 0.$$

By Lemma 4.2(i), $\mathrm{Att}_R H_\Phi^{p+d}(M,N/K) \subseteq \mathrm{Att}_R H_\Phi^{p+d}(M,N)$ and then $\mathfrak{q} \in \mathrm{Att}_R H_\Phi^{p+d}(M,N)$, which completes the proof. □

Corollary 4.5 *Let M, N be two nonzero finitely generated R-modules such that $p := \mathrm{pd} M < \infty$ and $d := \dim N = \dim R < \infty$. Assume that $\mathrm{cd}_\Phi(M,N) = p + d$. Then*

$$\mathrm{Att}_R(H_\Phi^{p+d}(M,N)) = \{\mathfrak{p} \in \mathrm{mAss}_R N \mid \mathrm{cd}_\Phi(M, R/\mathfrak{p}) = p + d\}.$$

Proof Let $\mathfrak{p} \in \mathrm{Ass}_R N$ such that $\mathrm{cd}_\Phi(M, R/\mathfrak{p}) = p + d$. It follows from the above proof that $\dim R/\mathfrak{p} = \dim N$ and then $\mathfrak{p} \in \mathrm{mAss}_R N$.

Now, let $\mathfrak{p} \in \mathrm{mAss}_R N$ such that $\mathrm{cd}_\Phi(M, R/\mathfrak{p}) = p + d$. It follows from Theorem 4.4 that

$$\mathrm{Att}_R H_\Phi^{p+d}(M, N/\mathfrak{p}N) = \{\mathfrak{q} \in \mathrm{Ass}_R N/\mathfrak{p}N \mid \mathrm{cd}_\Phi(M, R/\mathfrak{q}) = p + d\} = \{\mathfrak{p}\}.$$

The short exact sequence

$$0 \to \mathfrak{p}N \to N \to N/\mathfrak{p}N \to 0$$

induces the following exact sequence

$$H_\Phi^{p+d}(M,N) \to H_\Phi^{p+d}(M,N/\mathfrak{p}N) \to 0.$$

By Lemma 4.2(ii), we get $\mathrm{Att}_R H_\Phi^{p+d}(M,N/\mathfrak{p}N) \subseteq \mathrm{Att}_R H_\Phi^{p+d}(M,N)$ and then $\mathfrak{p} \in \mathrm{Att}_R H_\Phi^{p+d}(M,N)$. □

Corollary 4.6 *Let M, N be two nonzero finitely generated R-modules such that $p := \mathrm{pd} M < \infty$ and $d := \dim N = \dim R < \infty$. Assume that $\mathrm{cd}_\Phi(M,N) = p + d$. Then*

(i) *There exists a submodule* T *of* N *such that* $\dim N/T = d$,

(ii) $\operatorname{Ass}_R N/T = \{\mathfrak{p} \in \operatorname{Ass}_R N \mid \dim R/\mathfrak{p} = d\}$,

(iii) $\operatorname{Att}_R H_\Phi^{p+d}(M,N) = \operatorname{Ass}_R N/T$.

Proof According to Theorem 4.4, we have $\operatorname{Att}_R H_\Phi^{p+d}(M,N) \subseteq \operatorname{Ass}_R N$. It follows from [10, p. 263, Proposition 4] that there is a submodule T of N such that $\operatorname{Ass}_R N/T = \operatorname{Att}_R H_\Phi^{p+d}(M,N)$ and $\operatorname{Ass}_R T = \operatorname{Ass}_R N \setminus \operatorname{Att}_R H_\Phi^{p+d}(M,N)$. It is clear that $\dim N/T = d$, and the proof is complete. □

Corollary 4.7 *Let* N *be a nonzero finitely generated* R*-module of finite dimension* d*. Then*

$$\operatorname{Att}_R(H_\Phi^d(N)) = \{\mathfrak{p} \in \operatorname{Ass}_R N \mid \dim R/\mathfrak{p} = d\}.$$

Proof It follows from Theorem 4.4 that

$$\operatorname{Att}_R(H_\Phi^{\dim R}(R)) = \{\mathfrak{p} \in \operatorname{Ass}_R R \mid \dim R/\mathfrak{p} = \dim R\}.$$

Now, let $\overline{R} = R/\operatorname{Ann}_R N$, it is clear that $\dim \overline{R} = \dim N$. By [5, 2.5], there is an isomorphism

$$H_\Phi^d(N) \cong H_{\Phi\overline{R}}^d(N),$$

where $\Phi\overline{R} = \{\mathfrak{a}\overline{R} \mid \mathfrak{a} \in \Phi\}$ is a system of ideals in $\overline{R}$. On the other hand, there are isomorphisms

$$\begin{aligned} H_{\Phi\overline{R}}^d(N) &\cong H_{\Phi\overline{R}}^d(\overline{R} \otimes_{\overline{R}} N) \\ &\cong H_{\Phi\overline{R}}^d(\overline{R}) \otimes_{\overline{R}} N \end{aligned}$$

since $H_{\Phi\overline{R}}^d(-)$ is a right exact functor. It follows from Lemma 4.2(ii) that

$$\operatorname{Att}_{\overline{R}}(H_{\Phi\overline{R}}^d(N)) = \operatorname{Att}_{\overline{R}}(H_{\Phi\overline{R}}^d(\overline{R})) \cap \operatorname{Supp}_{\overline{R}} N = \operatorname{Att}_{\overline{R}}(H_{\Phi\overline{R}}^d(\overline{R})).$$

Note that

$$\operatorname{Att}_{\overline{R}}(H_{\Phi\overline{R}}^d(\overline{R})) = \{\mathfrak{p} \in \operatorname{Ass}_R \overline{R} \mid \operatorname{cd}(\Phi\overline{R}, \overline{R}/\mathfrak{p}) = d\}.$$

Consequently, one gets

$$\operatorname{Att}_R(H_\Phi^d(N)) = \{\mathfrak{p} \in \operatorname{Ass}_R N \mid \operatorname{cd}(\Phi, R/\mathfrak{p}) = d\},$$

and the proof is complete. □

In the case where $p := \operatorname{pd} M > 0$, we see that $H_\Phi^{p+\dim R}(M,N) = 0$ for all R-module N. Now we consider that M is a projective R-module.

Theorem 4.8 *Let* M *be a nonzero finitely generated projective* R*-module and* N *be a nonzero finitely generated* R*-module of finite dimension* $d := \dim N = \dim R$*. Assume that* $\operatorname{cd}_\Phi(M,N) = d$*. Then*

$$\operatorname{Att}_R(H_\Phi^d(M,N)) = \{\mathfrak{p} \in \operatorname{Supp}_R M \cap \operatorname{Ass}_R N \mid \dim R/\mathfrak{p} = d\}.$$

Proof Since M is a projective R-module, it follows by [22, Corollary 10.65] that

$$\begin{aligned} H^d_\Phi(M,N) &\cong \varinjlim_{\mathfrak{a}\in\Phi} H^d_{\mathfrak{a}}(M,N) \\ &\cong \varinjlim_{\mathfrak{a}\in\Phi} \varinjlim_{n} \operatorname{Ext}^d_R(M/\mathfrak{a}^n M, N) \\ &\cong \varinjlim_{\mathfrak{a}\in\Phi} \varinjlim_{n} \operatorname{Ext}^d_R(R/\mathfrak{a}^n, \operatorname{Hom}_R(M,N)) \\ &\cong \varinjlim_{\mathfrak{a}\in\Phi} H^d_{\mathfrak{a}}(\operatorname{Hom}_R(M,N)) \\ &\cong H^d_\Phi(\operatorname{Hom}_R(M,N)). \end{aligned}$$

Combining Corollary 4.7 with [10, p. 267, Proposition 10], we have

$$\begin{aligned} \operatorname{Att}_R H^d_\Phi(M,N) &= \operatorname{Att}_R H^d_\Phi(\operatorname{Hom}(M,N)) \\ &= \{\mathfrak{p} \in \operatorname{Ass}_R \operatorname{Hom}_R(M,N) \mid \dim R/\mathfrak{p} = d\} \\ &= \{\mathfrak{p} \in \operatorname{Supp}_R M \cap \operatorname{Ass}_R N \mid \dim R/\mathfrak{p} = d\}, \end{aligned}$$

as required. □

Corollary 4.9 *Let $(R,\mathfrak{m})$ be a local ring with finite dimension. Let M be a nonzero finitely generated projective R-module and N be a nonzero finitely generated R-module of finite dimension $d := \dim N = \dim R$. Assume that $\operatorname{cd}_\Phi(M,N) = d$. Then there is an ideal $\mathfrak{a} \in \Phi$ and $\mathfrak{p} \in \operatorname{Supp}_{\widehat{R}}\widehat{M} \cap \operatorname{Ass}_{\widehat{R}}\widehat{N}$ such that $\dim(\widehat{R}/(\mathfrak{a}\widehat{R}+\mathfrak{p})) = 0$.*

Proof Combining the hypothesis with Corollary 4.8, we can conclude that $H^d_\Phi(\operatorname{Hom}(M,N)) \neq 0$. The isomorphism

$$H^d_\Phi(\operatorname{Hom}(M,N)) \cong \varinjlim_{\mathfrak{a}\in\Phi} H^d_{\mathfrak{a}}(\operatorname{Hom}(M,N))$$

shows that there exists an ideal $\mathfrak{a} \in \Phi$ such that $H^d_{\mathfrak{a}}(\operatorname{Hom}(M,N)) \neq 0$. By the Lichtenbaum-Hartshorne Vanishing theorem, there is an ideal $\mathfrak{p} \in \operatorname{Supp}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(\widehat{M},\widehat{N})$ such that $\dim \widehat{R}/\mathfrak{p} = d$ and $\dim(\widehat{R}/(\mathfrak{a}\widehat{R}+\mathfrak{p})) = 0$. Note that $\dim \operatorname{Hom}_{\widehat{R}}(\widehat{M},\widehat{N}) = d$ and then $\mathfrak{p} \in \operatorname{Supp}_{\widehat{R}}\widehat{M} \cap \operatorname{Ass}_{\widehat{R}}\widehat{N}$, which completes the proof. □

Theorem 4.10 *Let R be a Cohen-Macaulay local ring with $d := \dim R > 0$. Assume that M is a nonzero Cohen Macaulay finitely generated R-module with $\operatorname{pd}M < \infty$ and $\operatorname{cd}_\Phi(M,R) = d$. Then*

$$\operatorname{Att}_R(H^{\dim R}_\Phi(M,R)) = \operatorname{Att}_R(H^{\dim M}_\Phi(M)).$$

Proof Let $p = \operatorname{pd}M$, by [9, Theorem 2.1.5], we have $p = \inf\{i \mid \operatorname{Ext}^i_R(M,R) \neq 0\}$. Let $F = \Gamma_\Phi(-)$ and $G = \operatorname{Hom}_R(M,-)$ be two functors from the category of R-modules to itself. We have a Grothendieck spectral sequence

$$E_2^{i,j} = H^i_\Phi(\operatorname{Ext}^j_R(M,R)) \underset{i}{\Rightarrow} H^{i+j}_\Phi(M,R).$$

It should be mentioned that $E_2^{i,j} = 0$ for all $j \neq p$. Hence, there is an isomorphism

$$H^d_\Phi(M,R) \cong H^{d-p}_\Phi(\operatorname{Ext}^p_R(M,R)).$$

Since $H^d_\Phi(M,R) \neq 0$, it follows from [5, 2.7] that $\dim \operatorname{Ext}^p_R(M,R) \geq d-p$. On the other hand, by [18, Lemma 3.1], we can assert that

$$\dim \operatorname{Ext}^p_R(M,R) = d-p = \operatorname{depth} R - \operatorname{pd} M = \operatorname{depth} M = \dim M.$$

Combining [9, Exercise 1.4.26] with Corollary 4.7, we have

$$\begin{aligned}\operatorname{Att}_R(H^d_\Phi(M,R)) &= \operatorname{Att}_R(H^{d-p}_\Phi(\operatorname{Ext}^p_R(M,R))) \\ &= \{\mathfrak{p} \in \operatorname{Ass}_R \operatorname{Ext}^p_R(M,R) \mid \operatorname{cd}_\Phi(R/\mathfrak{p}) = \dim M\} \\ &= \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}_\Phi(R/\mathfrak{p}) = \dim M\} \\ &= \operatorname{Att}_R(H^{\dim M}_\Phi(M)),\end{aligned}$$

as required. □

5. Attached primes of generalized local cohomology modules with respect to a pair of ideals

In [23], Takahashi et al. introduced an extension of local cohomology modules which is called the local cohomology modules with respect to a pair ideals. Let I, J be two ideals of R and

$$W(I,J) = \{\mathfrak{p} \in \operatorname{Spec} R \mid I^n \subseteq \mathfrak{p} + J \text{ for some integer } n\}.$$

The functor $\Gamma_{I,J}$ from the category of R-modules to itseft is defined by

$$\Gamma_{I,J}(N) = \{x \in N \mid \operatorname{Supp}_R(Rx) \subseteq W(I,J)\},$$

where N is an R-module. The functor $\Gamma_{I,J}$ is R-linear and left exact. For an integer i, the ith right derived functor of $\Gamma_{I,J}$ is called the ith local cohomology functor and denoted by $H^i_{I,J}$. Let M be an R-module, we call $H^i_{I,J}(M)$ the ith local cohomology module of M with respect to (I,J). Let

$$\tilde{W}(I,J) = \{\mathfrak{a} \text{ is an ideal of } R \mid I^n \subseteq \mathfrak{a} + J \text{ for some integer } n\}$$

and we define a partial order on $\tilde{W}(I,J)$ by letting $\mathfrak{a} \leq \mathfrak{b}$ if $\mathfrak{a} \supseteq \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in \tilde{W}(I,J)$. It follows from [23, Theorem 3.2] that

$$H^i_{I,J}(M) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(I,J)} H^i_\mathfrak{a}(M)$$

for all $i \geq 0$ and for any R-module M. It is clear that $\tilde{W}(I,J)$ is a system of ideals of R.

A natural generalization of local cohomology modules with respect to (I,J) was introduced in [21] as follows: Let M, N be two R-modules, the module $\Gamma_{I,J}(\operatorname{Hom}_R(M,N))$ is denoted by $\Gamma_{I,J}(M,N)$. For each finitely generated R-module M, the ith generalized local cohomology functor $H^i_{I,J}(M,-)$ is the ith right derived functor of the functor $\Gamma_{I,J}(M,-)$. When $M = R$, the generalized local cohomology module $H^i_{I,J}(R,N)$ is the local cohomology module $H^i_{I,J}(N)$ in [23].

It is clear that the local cohomology modules with respect to a pair of ideals are special cases of local cohomology modules with respect to a system of ideals.

We denote by

$$\mathrm{cd}_{I,J}(M,N) := \sup\{i \mid H^i_{I,J}(M,N) \neq 0\}$$

and call the cohomological dimension of M, N with respect to (I, J).

Theorem 5.1 *Let M, N be two nonzero finitely generated R-module such that $p := \mathrm{pd}M$ and $d := \dim N = \dim R$. Assume that $\mathrm{cd}_{I,J}(M,N) = p + d$. Then*

$$\mathrm{Att}_R(H^{p+d}_{I,J}(M,N)) = \{\mathfrak{p} \in \mathrm{Ass}_R N \mid \mathrm{cd}_{I,J}(M, R/\mathfrak{p}) = p + d\}.$$

Moreover, $\mathrm{Att}_R H^{p+d}_{I,J}(M,N) \subseteq \mathrm{Att}_R H^d_{I,J}(N)$.

Proof It follows from Theorem 4.4. □

Theorem 5.2 *Let R be a Cohen–Macaulay local ring with $d := \dim R > 0$. Assume that M is a nonzero Cohen–Macaulay finitely generated R-module with $\mathrm{pd}M < \infty$ and $\mathrm{cd}_{I,J}(M,R) = d$. Then*

$$\mathrm{Att}_R(H^{\dim R}_{I,J}(M,R)) = \{\mathfrak{p} \in \mathrm{Supp}_R M \cap V(J) \mid \mathrm{cd}_I(R/\mathfrak{p}) = \dim M\}.$$

Proof It follows from Theorem 4.10 that

$$\mathrm{Att}_R(H^{\dim R}_{I,J}(M,R)) = \mathrm{Att}_R(H^{\dim M}_{I,J}(M)).$$

The assertion follows from [11, Theorem 2.1]. □

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