The Meyer function on the handlebody group

Yusuke KUNO1, Masatoshi SATO2,*

1Department of Mathematics, Tsuda University, Tokyo, Japan
2Department of Mathematics, Tokyo Denki University, Tokyo, Japan

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Abstract: We give an explicit formula for the signature of handlebody bundles over the circle in terms of the homological monodromy. This gives a cobounding function of Meyer’s signature cocycle on the mapping class group of a 3-dimensional handlebody, i.e. the handlebody group. As an application, we give a topological interpretation for the generator of the first cohomology group of the hyperelliptic handlebody group.

Key words: Signature cocycle, handlebody group, mapping class groups

1. Introduction

Let \( \Sigma_g \) be a closed connected oriented surface of genus \( g \) and \( \text{Mod}(\Sigma_g) \) the mapping class group of \( \Sigma_g \), namely the group of isotopy classes of orientation-preserving self-diffeomorphisms of \( \Sigma_g \). Unless otherwise stated, we assume that (co)homology groups have coefficients in \( \mathbb{Z} \). The second cohomology of \( \text{Mod}(\Sigma_g) \) has been determined for all \( g \geq 1 \) by works of many people, in particular by the seminal work of Harer [6, 7] for \( g \geq 3 \). We have \( H^2(\text{Mod}(\Sigma_1)) \cong \mathbb{Z}/12\mathbb{Z}, \ H^2(\text{Mod}(\Sigma_2)) \cong \mathbb{Z}/10\mathbb{Z}, \) and

\[
H^2(\text{Mod}(\Sigma_g)) \cong \mathbb{Z} \quad \text{for} \ g \geq 3.
\]

There are various interesting constructions of nontrivial second cohomology class of \( \text{Mod}(\Sigma_g) \); the reader is referred to the survey article [13]. Among others, the remarkable approach of Meyer [16, 17] was to consider the signature of \( \Sigma_g \)-bundles over surfaces. The central object that Meyer used was a normalized 2-cocycle

\[
\tau_g: \text{Sp}(2g; \mathbb{Z}) \times \text{Sp}(2g; \mathbb{Z}) \to \mathbb{Z}
\]

on the integral symplectic group of degree \( 2g \).

Meyer showed that for \( g \geq 3 \) the pullback of the cohomology class of \( \tau_g \) by the homology representation \( \rho: \text{Mod}(\Sigma_g) \to \text{Sp}(2g; \mathbb{Z}) \) is of infinite order in \( H^2(\text{Mod}(\Sigma_g)) \). On the other hand, if \( g = 1, 2 \) then \( [\rho^* \tau_g] \) is torsion and there exists a (unique) rational valued cobounding function \( \phi_g: \text{Mod}(\Sigma_g) \to \mathbb{Q} \) of \( \rho^* \tau_g \). This means that

\[
\tau_g(\rho(\varphi_1), \rho(\varphi_2)) = \phi_g(\varphi_1) + \phi_g(\varphi_2) - \phi_g(\varphi_1 \varphi_2) \quad \text{for any} \ \varphi_1, \varphi_2 \in \text{Mod}(\Sigma_g).
\]

Since the case \( g = 1 \) was extensively studied by Meyer, such a cobounding function is called a Meyer function. Some number-theoretic and differential geometric aspects of the function \( \phi_1 \) were studied by Atiyah [2]. The

*Correspondence: msato@mail.dendai.ac.jp

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case \( g = 2 \) was studied by Matsumoto [15], Morifuji [18], and Iida [11]. For \( g \geq 3 \), there is no cobounding function of \( \rho^*\tau_g \) on the whole mapping class group \( \text{Mod}(\Sigma_g) \). However, if we restrict \( \rho^*\tau_g \) to a subgroup called the hyperelliptic mapping class group \( \mathcal{H}(\Sigma_g) \), then it is known that there is a (unique) cobounding function \( \phi^V_g: \mathcal{H}(\Sigma_g) \to \mathbb{Q} \) of \( \rho^*\tau_g \). Note that \( \mathcal{H}(\Sigma_g) = \text{Mod}(\Sigma_g) \) for \( g = 1, 2 \). This function \( \phi^V_g \) was studied by Endo [4] and Morifuji [18]. One motivation for studying Meyer functions comes from the localization phenomenon of the signature of fibered 4-manifolds. See, e.g., [1, 14].

In this paper, we study a new example of Meyer functions: the Meyer function on the handlebody group. The handlebody group of genus \( g \), which we denote by \( \text{Mod}(V_g) \), is defined as the group of isotopy classes of orientation-preserving self-diffeomorphisms of the 3-dimensional handlebody \( V_g \) of genus \( g \). It is well known that the natural homomorphism \( \text{Mod}(V_g) \to \text{Mod}(\Sigma_g) \), \( \varphi \mapsto \varphi|_{\Sigma_g} \) is injective since \( V_g \) is an irreducible 3-manifold. Therefore, we can think of \( \text{Mod}(V_g) \) as a subgroup of \( \text{Mod}(\Sigma_g) \). For a mapping class \( \varphi \in \text{Mod}(V_g) \), we denote by \( M_\varphi \) the mapping torus of \( \varphi \). It is a compact oriented 4-manifold. We define

\[
\phi^V_g(\varphi) := \text{Sign } M_\varphi \in \mathbb{Z}.
\]

We show in Lemma 4.2 that \( \phi^V_g \) is a cobounding function of the cocycle \( \rho^*\tau_g \) on the handlebody group \( \text{Mod}(V_g) \). If \( g \geq 3 \), this is the unique cobounding function since \( H_1(\text{Mod}(V_g)) \) is torsion (see [21, Theorem 20] and [12, Remark 3.5]).

The value \( \phi^V_g(\varphi) \) can be computed from the action of \( \varphi \) on the first homology \( H_1(\Sigma_g) \), and our first result gives its explicit description. To state it, we take a suitable basis of \( H_1(\Sigma_g) \) so that the homology representation \( \rho \) restricted to \( \text{Mod}(V_g) \) takes values in a subgroup \( \text{urSp}(2g;\mathbb{Z}) \subset \text{Sp}(2g;\mathbb{Z}) \). (See Section 2.3 for details.) Then, \( \rho(\varphi) \) is of the form \( \rho(\varphi) = \begin{pmatrix} P & Q \\ O_g & S \end{pmatrix} \), where \( P, Q, \) and \( S \) are \( g \times g \) matrices. We consider a \( \mathbb{Q} \)-linear space \( U_\varphi := \text{Ker}(S - I_g) \subset \mathbb{Q}^g \), and define a bilinear form \( \langle \cdot, \cdot \rangle_\varphi \) on it by

\[
\langle x, y \rangle_\varphi := x^t Q y, \quad \text{for } x, y \in U_\varphi.
\]

It turns out that \( \langle \cdot, \cdot \rangle_\varphi \) is symmetric, and we have the following:

**Theorem 1.1** The value \( \phi^V_g(\varphi) \) coincides with the signature of the symmetric bilinear form \( \langle \cdot, \cdot \rangle_\varphi \) on \( U_\varphi \).

In fact, we will show in Section 3.5 that the intersection form on \( H_2(M_\varphi) \) is equivalent to the bilinear form \( \langle \cdot, \cdot \rangle_\varphi \).

As a corollary, we see that the function \( \phi^V_g \) is bounded by \( g = \text{rank } H_1(V_g) \). We also give sample calculations of \( \phi^V_g \) in Lemmas 4.4 and 4.5. Walker also constructed a function \( j: \text{Mod}(\Sigma_g) \to \mathbb{Q} \) whose restriction to \( \text{Mod}(V_g) \) coincides with \( \phi^V_g \). Our description of \( \phi^V_g \) in Theorem 1.1 is similar to but different from a description of \( j \) given by Gilmer and Masbaum [5, Proposition 6.9]. See, for details, Remark 3.6.

As an application of the function \( \phi^V_g \), we obtain a nontrivial first cohomology class in the intersection \( \mathcal{H}(\Sigma_g) \cap \text{Mod}(V_g) \) called the hyperelliptic handlebody group, denoted by \( \mathcal{H}(V_g) \). The group \( \mathcal{H}(V_g) \) is an extension by \( \mathbb{Z}/2\mathbb{Z} \) of a subgroup of the mapping class group of a 2-sphere with \((2g+2)\)-punctures, called the Hilden group. The Hilden group was introduced in [8], and it is related to the study of links in 3-manifolds. In
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[10], Hirose and Kin studied the minimal dilatation of pseudo-Anosov elements in $\mathcal{H}(V_g)$, and gave a presentation of $\mathcal{H}(V_g)$.

We consider the difference

$$\phi^H_g - \phi^V_g \in \text{Hom}(\mathcal{H}(V_g), \mathbb{Q}) = H^1(\mathcal{H}(V_g); \mathbb{Q})$$

of the Meyer functions on $\mathcal{H}(\Sigma_g)$ and on $\text{Mod}(V_g)$. From the abelianization of $\mathcal{H}(V_g)$ obtained in [10, Corollary A.9], we see that the rank of $H^1(\mathcal{H}(V_g))$ is one. Let us denote a generator of $H^1(\mathcal{H}(V_g))$ by $\mu$.

Our second result is:

**Theorem 1.2** Let $g \geq 1$. We have

$$\phi^H_g - \phi^V_g = \begin{cases} \frac{2}{2g+1} \mu & \text{if } g \text{ is even}, \\ \frac{1}{2g+1} \mu & \text{if } g \text{ is odd}. \end{cases}$$

When $g = 1, 2$, we have $\mathcal{H}(V_g) = \text{Mod}(V_g)$, and $\phi^H_g - \phi^V_g$ gives an abelian quotient of $\text{Mod}(V_g)$.

There is an interpretation of the cohomology class $\phi^H_g - \phi^V_g$ in terms of a kind of connecting homomorphism. We assume that $g \geq 3$. From the diagram

$$\begin{array}{ccc}
\mathcal{H}(V_g) & \xrightarrow{j_2} & \text{Mod}(V_g) \\
\downarrow{i_1} & & \downarrow{j_2} \\
\mathcal{H}(\Sigma_g) & \xrightarrow{j_1} & \text{Mod}(\Sigma_g).
\end{array}$$

defined as follows. For $[c] \in H^2(\text{Mod}(\Sigma_g); \mathbb{Q})$, there are cobounding functions $f^H : \mathcal{H}(\Sigma_g) \to \mathbb{Q}$ of $j_1^*c$ and $f^V : \text{Mod}(V_g) \to \mathbb{Q}$ of $j_2^*c$, respectively. The cochain $i_1^*f^H - i_2^*f^V$ is actually a homomorphism on $\mathcal{H}(V_g)$. It does not depend on the choices of the representatives $c$, $f^H$, and $f^V$ since $H^1(\text{Mod}(V_g); \mathbb{Q}) = H^1(\mathcal{H}(\Sigma_g); \mathbb{Q}) = 0$ when $g \geq 3$. Then $\Upsilon([c])$ is defined to be $i_1^*f^H - i_2^*f^V$. In this setting, our cohomology class is written as $\Upsilon([\tau_g]) = \phi^H_g - \phi^V_g \in H^1(\mathcal{H}(V_g); \mathbb{Q})$.

The outline of this paper is as follows. In Section 2, we review the definition of Meyer’s signature cocycle and the handlebody group $\text{Mod}(V_g)$. We also review the abelianization of the hyperelliptic handlebody group obtained in [10], and describe a generator of the cohomology group $H^1(\mathcal{H}(V_g))$ in Corollary 2.6. In Section 3, we investigate the intersection form of the mapping torus of $\varphi \in \text{Mod}(V_g)$, and prove Theorem 1.1. As it turns out, we can explicitly describe $\phi^V_g$ as a function on a subgroup $\text{urSp}(2g; \mathbb{Z})$ of the integral symplectic group. In Section 4, we prove Theorem 1.2 by using explicit calculations of the Meyer function $\phi^V_g : \text{Mod}(V_g) \to \mathbb{Z}$ in Lemmas 4.4 and 4.5.
2. Preliminaries on mapping class groups

Fix a nonnegative integer \( g \).

2.1. Mapping class group of a surface

Let \( \Sigma_g \) be a closed connected oriented surface of genus \( g \). The mapping class group of \( \Sigma_g \), denoted by \( \text{Mod}(\Sigma_g) \), is the group of isotopy classes of orientation-preserving self-diffeomorphisms of \( \Sigma_g \). To simplify notation, we will use the same letter for a self-diffeomorphism of \( \Sigma_g \) and its isotopy class.

The first homology group \( H_1(\Sigma_g) \) is equipped with a nondegenerate skew-symmetric pairing \( \langle \cdot, \cdot \rangle \), namely the intersection form. Thus, we can take a symplectic basis \( \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \) for \( H_1(\Sigma_g) \). This means that \( \langle \alpha_i, \beta_j \rangle = \delta_{ij} \) and \( \langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0 \) for any \( i, j \in \{1, \ldots, g\} \), where \( \delta_{ij} \) is the Kronecker symbol.

Once a symplectic basis for \( H_1(\Sigma_g) \) is fixed, we obtain the homology representation

\[
\rho: \text{Mod}(\Sigma_g) \to \text{Sp}(2g; \mathbb{Z}), \quad \varphi \mapsto \varphi_*.
\]

Here, the target is the integral symplectic group

\[
\text{Sp}(2g; \mathbb{Z}) = \{ A \in \text{GL}(2g; \mathbb{Z}) \mid {}^t A^t A = J \},
\]

where \( J = \begin{pmatrix} O_g & I_g \\ -I_g & O_g \end{pmatrix} \), and \( \rho(\varphi) = \varphi_* \) is the matrix presentation of the action of \( \varphi \) on \( H_1(\Sigma_g) \) with respect to the fixed symplectic basis. We use block matrices to denote elements in \( \text{Sp}(2g; \mathbb{Z}) \), e.g., \( A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \) with \( g \times g \) integral matrices \( P, Q, R, \) and \( S \).

2.2. Meyer’s signature cocycle

Let \( A, B \in \text{Sp}(2g; \mathbb{Z}) \). We consider an \( \mathbb{R} \)-linear space

\[
V_{A,B} := \{(x, y) \in \mathbb{R}^{2g} \oplus \mathbb{R}^{2g} \mid (A^{-1} - I_{2g})x + (B - I_{2g})y = 0\}
\]

and a bilinear form on \( V_{A,B} \) given by

\[
\langle (x, y), (x', y') \rangle_{A,B} := {}^t (x + y)J(I_{2g} - B)y'.
\]

The form \( \langle \cdot, \cdot \rangle_{A,B} \) turns out to be symmetric, and thus its signature is defined; we set

\[
\tau_g(A, B) := \text{Sign}(V_{A,B}, \langle \cdot, \cdot \rangle_{A,B}).
\]

The map \( \tau_g: \text{Sp}(2g; \mathbb{Z}) \times \text{Sp}(2g; \mathbb{Z}) \to \mathbb{Z} \) is called Meyer’s signature cocycle [16, 17]. It is a normalized 2-cocycle of the group \( \text{Sp}(2g; \mathbb{Z}) \).

Let \( P \) be a compact oriented surface of genus \( 0 \) with three boundary components, i.e., a pair of pants. We denote by \( C_1, C_2, \) and \( C_3 \) the boundary components of \( P \). Choose a base point in \( P \), and let \( \ell_1, \ell_2, \) and \( \ell_3 \) be based loops in \( P \) such that \( \ell_i \) is parallel to the negatively oriented boundary component \( C_i \) for any \( i \in \{1, 2, 3\} \) and \( \ell_1\ell_2\ell_3 = 1 \) holds in the fundamental group \( \pi_1(P) \).

For given two mapping classes \( \varphi_1, \varphi_2 \in \text{Mod}(\Sigma_g) \), there is an oriented \( \Sigma_g \)-bundle \( E(\varphi_1, \varphi_2) \to P \) such that the monodromy along \( \ell_i \) is \( \varphi_i \) for \( i = 1, 2 \). It is unique up to bundle isomorphisms. The total space \( E(\varphi_1, \varphi_2) \) is a compact 4-manifold equipped with a natural orientation; hence, its signature is defined.
Proposition 2.1 (Meyer [16, 17]) \[ \text{Sign}(E(\varphi_1, \varphi_2)) = \tau_g(\rho(\varphi_1), \rho(\varphi_2)). \]

Remark 2.2 Turaev [20] independently found the signature cocycle. He also studied its relation to the Maslov index.

2.3. Handlebody group

Let \( V_g \) be a handlebody of genus \( g \). That is, \( V_g \) is obtained by attaching \( g \) one-handles to the 3-ball \( D^3 \). We identify \( \Sigma_g \) and the boundary of \( V_g \) by choosing an orientation-preserving diffeomorphism between them. We have the following short exact sequence

\[
0 \rightarrow H_2(V_g, \Sigma_g) \xrightarrow{\partial} H_1(\Sigma_g) \xrightarrow{i_*} H_1(V_g) \rightarrow 0
\]

which is a part of the homology exact sequence of the pair \((V_g, \Sigma_g)\). There are properly embedded, oriented and pairwise disjoint disks \( D_1, \ldots, D_g \) in \( V_g \) whose homology classes (denoted by the same letters) constitute a basis for \( H_2(V_g, \Sigma_g) \). We set \( i_* := \partial_* (D_i) \in H_1(\Sigma_g) \) for \( i \in \{1, \ldots, g\} \). Then \( i_* \)'s extend to a symplectic basis \( \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \) for \( H_1(\Sigma_g) \). In what follows, we fix a symplectic basis obtained in this way. The image of the homology classes \( \beta_1, \ldots, \beta_g \) by the map \( i_* \) constitute a basis for \( H_1(V_g) \). For simplicity, we denote them by the same letters \( \beta_1, \ldots, \beta_g \).

We denote by \( \text{Mod}(V_g) \) the handlebody group of genus \( g \). It can be considered a subgroup of \( \text{Mod}(\Sigma_g) \). For any \( \varphi \in \text{Mod}(V_g) \), the matrix \( \rho(\varphi) \) lies in the subgroup of \( \text{Sp}(2g; \mathbb{Z}) \) defined by

\[
\text{urSp}(2g; \mathbb{Z}) := \left\{ A \in \text{Sp}(2g; \mathbb{Z}) \mid A = \begin{pmatrix} P & Q \\ O_g & S \end{pmatrix} \right\},
\]

cf. [3, 9] for details. The matrices \( P, Q, \) and \( S \) satisfy the following relations:

\[ ^tPS = I_g, \quad ^tQS = ^tSQ. \]  \hspace{1cm} (2.2)

Remark 2.3 The group \( \text{Mod}(V_g) \) acts naturally on the groups in (2.1), and the maps \( \partial_* \) and \( i_* \) are \( \text{Mod}(V_g) \)-module homomorphisms. The matrix presentation of the action \( \varphi_* \) on \( H_1(V_g) \) is \( S \).

2.4. Hyperelliptic handlebody group

An involution of \( \Sigma_g \) is called hyperelliptic if it acts on \( H_1(\Sigma_g) \) as \( -\text{id} \). We fix an hyperelliptic involution \( \iota \) which extends to an involution of \( V_g \), as in Figure 1.

![Figure 1. The involution \( \iota \) of \( V_g \) and the curves \( C_1, C_2, C_3 \).](image)

The hyperelliptic mapping class group \( \mathcal{H}(\Sigma_g) \) is the centralizer of \( \iota \) in \( \text{Mod}(\Sigma_g) \):

\[
\mathcal{H}(\Sigma_g) := \{ \varphi \in \text{Mod}(\Sigma_g) \mid \varphi \iota = \iota \varphi \}.
\]
Definition 2.4 ([10]) The hyperelliptic handlebody group $\mathcal{H}(V_g)$ is defined by

$$\mathcal{H}(V_g) := \mathcal{H}(\Sigma_g) \cap \text{Mod}(V_g).$$

Hirose and Kin [10, Appendix A] gave a finite presentation of the group $\mathcal{H}(V_g)$. Moreover, they determined the abelianization of $\mathcal{H}(V_g)$ as

$$\mathcal{H}(V_g)^{\text{abel}} \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad \text{for } g \geq 2.$$  

In fact, using their presentation, it is easy to make this result more explicit. Let $C_1$, $C_2$, and $C_3$ be simple closed curves on $\Sigma_g$ as in Figure 1. For each $i \in \{1, 2, 3\}$ denote by $t_i$ the right handed Dehn twist along $C_i$. Following [10], set $r_1 = t_2^{-1}t_3^{-1}t_1t_2$ and $s_1 = t_2t_3t_1t_2$. (Note that in [10], $t_C$ denotes the left handed Dehn twist along $C$.)

Lemma 2.5 When $g = 1$, one has $\mathcal{H}(V_1) \cong \mathbb{Z}[t_1s_1] \oplus \mathbb{Z}_2 \langle t_1^2s_1 \rangle$. If $g \geq 2$, then

$$\mathcal{H}(V_g)^{\text{abel}} \cong \begin{cases} \mathbb{Z}[s_1] \oplus \mathbb{Z}_2 \langle t_1s_1^2 \rangle \oplus \mathbb{Z}_2 \langle r_1 \rangle & \text{if } g \text{ is even}, \\ \mathbb{Z}[t_1s_1^{\frac{g+1}{2}}] \oplus \mathbb{Z}_2 \langle t_1^2s_1 \rangle \oplus \mathbb{Z}_2 \langle r_1 \rangle & \text{if } g \text{ is odd}. \end{cases}$$

Here, $[s_1]$ is the class of $s_1$ in $\mathcal{H}(V_g)^{\text{abel}}$, and $\mathbb{Z}[s_1]$ is the infinite cyclic group generated by $[s_1]$, etc.

Proof The case $g = 1$ follows from the fact that $\mathcal{H}(V_1) \cong \text{Mod}(V_1)$ and a result of Wajnryb [21, Theorem 14].

Assume that $g \geq 2$. Using [10, Theorem A.8], one sees that $\mathcal{H}(V_g)^{\text{abel}}$ is generated by $[r_1]$, $[s_1]$, and $[t_1]$ with the relations

$$2[r_1] = 0, \quad 4[t_1] + 2g[s_1] = 0, \quad 2(g+1)[t_1] + g(g+1)[s_1] = 0.$$  

The assertion follows from these relations by a direct computation. \hfill \Box

The following corollary to Lemma 2.5 will be used in Section 4.4 to prove Theorem 1.2.

Corollary 2.6 Let $g \geq 1$. There is a unique homomorphism $\mu: \mathcal{H}(V_g) \to \mathbb{Z}$ satisfying the following property:

1. If $g$ is even, $\mu(s_1) = 1$ and $\mu(t_1) = -g/2$;

2. If $g$ is odd, $\mu(t_1) = -g$, $\mu(s_1) = 2$, and thus $\mu(t_1s_1^{\frac{g+1}{2}}) = 1$.

Moreover, the first cohomology group $H^1(\mathcal{H}(V_g)) = \text{Hom}(\mathcal{H}(V_g), \mathbb{Z})$ is an infinite cyclic group generated by $\mu$.

3. Handlebody bundles over $S^1$

3.1. Mapping torus

Let $I = [0, 1]$ be the unit interval. By identifying the endpoints of $I$, we obtain the circle $S^1 = [0, 1]/0 \sim 1$. Let $\ell: I \to S^1$ be the natural projection. For $t \in I$, we set $[t] := \ell(t)$. Choose $[0]$ as a base point of $S^1$. Then the fundamental group $\pi_1(S^1)$ is an infinite cyclic group generated by the homotopy class of $\ell$. 

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In what follows, we use the following cell decomposition of $S^1$: the 0-cell is $e^0 = [0]$ and the 1-cell is $e^1 = S^1 \setminus e^0$. The map $\ell$ induces an orientation of $e^1$.

Let $\varphi \in \text{Mod}(V_g)$. The mapping torus of $\varphi$ is the quotient space

$$M_\varphi := (I \times V_g)/(0, x) \sim (1, \varphi(x)).$$

For $(t, x) \in I \times V_g$, its class in $M_\varphi$ is denoted by $[t, x]$. The natural projection $\pi: M_\varphi \to S^1$, $[t, x] \mapsto [t]$ is an oriented $V_g$-bundle, and the total space $M_\varphi$ is a compact 4-manifold with boundary equipped with a natural orientation. The pullback of $M_\varphi \to S^1$ by $\ell$ is a trivial $V_g$-bundle over $I$, and its trivialization is given by the map

$$\Phi: I \times V_g \to M_\varphi, \quad (t, x) \mapsto [t, x]. \quad (3.1)$$

The following composition of maps coincides with $\varphi$:

$$V_g \cong \{0\} \times V_g \xrightarrow{\Phi(0, \cdot)} \pi^{-1}(\{0\}) = \pi^{-1}(\{1\}) \xrightarrow{\Phi(1, \cdot)} \{1\} \times V_g \cong V_g.$$ 

Therefore, the monodromy of $M_\varphi \to S^1$ along $\ell$ is equal to the mapping class $\varphi$. As was mentioned in Remark 2.3, the groups $H_2(V_g, \Sigma_g)$, $H_1(\Sigma_g)$, and $H_1(V_g)$ are $\text{Mod}(V_g)$-modules. Thus, these groups become $\pi_1(S^1)$-modules; the homotopy class of $\ell$, which is a generator of $\pi_1(S^1)$, acts as the monodromy $\varphi \in \text{Mod}(V_g)$.

### 3.2. Second homology of the mapping torus

For a nonnegative integer $q \geq 0$, let $\mathcal{H}_q(V_g)$ be the local system on $S^1$ which comes from the $V_g$-bundle $\pi: M_\varphi \to S^1$, and whose fiber at $x \in S^1$ is the $q$-th homology group $H_g(\pi^{-1}(x))$. Similarly, we consider the local system $\mathcal{H}_q(V_g, \Sigma_g)$ whose fiber at $x \in S^1$ is the $q$-th relative homology group $H_q(\pi^{-1}(x), \partial \pi^{-1}(x))$.

Consider the Serre homology spectral sequence of the $V_g$-bundle $M_\varphi \to S^1$. It degenerates at the $E^2$ page, which is given by $E^2_{p, q} = H_p(S^1; \mathcal{H}_q(V_g))$. Since $H_2(V_g) = 0$ and the base space $S^1$ is 1-dimensional, we obtain

$$H_2(M_\varphi) \cong E^\infty_{1,1} \cong E^2_{1,1} = H_1(S^1; \mathcal{H}_1(V_g)).$$

Moreover, using the cellular homology of $S^1$ with coefficients in $\mathcal{H}_1(V_g)$, we have

$$H_1(S^1; \mathcal{H}_1(V_g)) \cong \text{Ker}(\partial): C_1(S^1; \mathcal{H}_1(V_g)) \to C_0(S^1; \mathcal{H}_1(V_g))$$

$$= \text{Ker}(\partial): \mathbb{Z}e^1 \otimes H_1(V_g) \to \mathbb{Z}e^0 \otimes H_1(V_g) = H_1(V_g),$$

where the boundary map is given by

$$\partial(e^1 \otimes x) \equiv \ell_*(\alpha) - \alpha = (\Phi(0, \cdot)^{-1} \circ \Phi(1, \cdot))_* x(\alpha) - \alpha = \varphi^{-1}_*(\alpha) - \alpha.$$

In summary, we have proved the following lemma. In the statement, $H_1(V_g)^{\pi_1(S^1)}$ is the space of invariants under the action of $\pi_1(S^1)$, i.e., $H_1(V_g)^{\pi_1(S^1)} = \{\alpha \in H_1(V_g) \mid \varphi_*(\alpha) = \alpha\}$.

**Lemma 3.1** We have $H_2(M_\varphi) \cong H_1(S^1; \mathcal{H}_1(V_g)) \cong H_1(V_g)^{\pi_1(S^1)}$. 

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Similarly, for the relative homology of the pair \((M_\varphi, \partial M_\varphi)\), there is a spectral sequence converging to \(H_*(M_\varphi, \partial M_\varphi)\) such that \(E^2_{p,q} = H_p(S^1; \mathcal{K}_q(V_g, \Sigma_g))\). This degenerates at the \(E^2\) page, too. Since \(H_1(V_g, \Sigma_g) = 0\), we obtain
\[
H_2(M_\varphi, \partial M_\varphi) \cong E^{\infty}_{0,2} \cong E^2_{0,2} = H_0(S^1; \mathcal{K}_2(V_g, \Sigma_g)).
\]

By the same argument as above, we obtain the following lemma. In the statement, \(H_2(V_g, \Sigma_g)_{\pi_1(S^1)}\) is the space of coinvariants under the action of \(\pi_1(S^1)\), i.e. the quotient of \(H_2(V_g, \Sigma_g)\) by the subgroup generated by the set \(\{\varphi_*(\delta) - \delta \mid \delta \in H_2(V_g, \Sigma_g)\}\).

**Lemma 3.2** We have \(H_2(M_\varphi, \partial M_\varphi) \cong H_0(S^1; \mathcal{K}_2(V_g, \Sigma_g)) \cong H_2(V_g, \Sigma_g)_{\pi_1(S^1)}\).

### 3.3. Description of the inclusion homomorphism

Recall that the short exact sequence (2.1) is \(\text{Mod}(V_g)\)-equivariant. Let \(\alpha \in H_1(V_g)_{\pi_1(S^1)}\) be a \(\varphi_\ast\)-invariant homology class. Pick an element \(\tilde{\alpha} \in H_1(\Sigma_g)\) such that \(i_\ast(\tilde{\alpha}) = \alpha\). Then \(\varphi_\ast(\tilde{\alpha}) - \tilde{\alpha} \in \text{Ker}(i_\ast) = \text{Im}(\partial_\ast)\).

**Definition 3.3** \(d(\alpha) := [\partial_\ast^{-1}(\varphi_\ast(\tilde{\alpha}) - \tilde{\alpha})] \in H_2(V_g, \Sigma_g)_{\pi_1(S^1)}\).

It is easy to see that \(d(\alpha)\) is independent of the choice of \(\tilde{\alpha}\). Thus, we obtain a well-defined map \(d: H_1(V_g)_{\pi_1(S^1)} \rightarrow H_2(V_g, \Sigma_g)_{\pi_1(S^1)}\).

**Proposition 3.4** The following diagram is commutative:

\[
\begin{array}{ccc}
H_1(V_g)_{\pi_1(S^1)} & \xrightarrow{d} & H_2(V_g, \Sigma_g)_{\pi_1(S^1)} \\
\cong & & \cong \\
H_2(M_\varphi) & \xrightarrow{i_\ast} & H_2(M_\varphi, \partial M_\varphi),
\end{array}
\]

where the bottom horizontal arrow is the inclusion homomorphism, and the vertical arrows are the isomorphisms in Lemmas 3.1 and 3.2.

### 3.4. Proof of Proposition 3.4

In this section, for a topological space \(X\), we denote by \(S_n(X)\) and \(Z_n(X)\) the groups of singular \(n\)-chains and singular \(n\)-cycles, respectively.

Let \(\alpha \in H_1(V_g)_{\pi_1(S^1)}\). Pick its lift \(\tilde{\alpha} \in H_1(\Sigma_g)\) such that \(i_\ast(\tilde{\alpha}) = \alpha\). Take a singular 1-cycle \(\tilde{a} \in Z_1(\Sigma_g)\) representing the homology class \(\tilde{\alpha}\). Then, \(\varphi_\ast^{-1}(\tilde{a}) - \tilde{a}\) is a singular 1-boundary in \(V_g\) since \(\varphi_\ast^{-1}(\tilde{\alpha}) - \tilde{\alpha} \in \text{Ker}(i_\ast)\). Therefore, there exists \(\sigma_{\varphi,\alpha} \in S_2(V_g)\) such that \(\partial \sigma_{\varphi,\alpha} = \varphi_\ast^{-1}(\tilde{a}) - \tilde{a}\).

First we compute the composition of \(d\) and the right vertical map. We claim that \(d(\alpha)\) is represented by the relative 2-cycle \(-\sigma_{\varphi,\alpha} \in Z_2(V_g, \Sigma_g)\). This follows from the equality \(\varphi_\ast(\tilde{a}) - \tilde{a} = -(\varphi_\ast^{-1}(\tilde{\alpha}) - \tilde{\alpha})\) in \(H_1(\Sigma_g)_{\pi_1(S^1)}\) and the relation \(\partial \sigma_{\varphi,\alpha} = \varphi_\ast^{-1}(\tilde{a}) - \tilde{a}\). Hence, the right vertical map sends \(d(\alpha)\) to the homology class represented by the relative 2-cycle \(-e^0 \times \sigma_{\varphi,\alpha} \in Z_2(M_\varphi, \partial M_\varphi)\), where the symbol \(\times\) means the cross product.
Next we compute the composition of the left vertical map and $i_*$. For this purpose, we set

$$Z_\alpha := \Phi_2(I \times \tilde{a}) - e^0 \times \sigma_{\varphi, \alpha} \in S_2(M).$$

Here, $\Phi$ is the map defined in (3.1), and the unit interval is regarded as a singular 1-chain in the obvious way. Actually, $Z_\alpha$ is a 2-cycle in $M$.

**Lemma 3.5** The isomorphism in Lemma 3.1 sends $\alpha$ to the homology class of $Z_\alpha$.

**Proof** We need to inspect the spectral sequence involved in Lemma 3.1. For simplicity we denote $M = M$, and for every nonnegative integer $q \geq 0$ let $M^{(q)}$ be the inverse image of the $q$-skeleton of $S^1$ by the projection map $\pi$. Thus, we have $\emptyset \subset M^{(0)} = \pi^{-1}([0]) \subset M^{(1)} = M$. Accordingly, the singular chain complex $S_*(M)$ has an increasing filtration: $\{\emptyset\} \subset S_*(M^{(0)}) \subset S_*(M^{(1)}) = S_*(M)$. The associated spectral sequence is the one that we consider.

Now let $\alpha \in H_1(V_g)^{\pi_1(S^1)}$. There is an isomorphism

$$E^2_{1,1} = H_1(S^1; \mathcal{K}_g) \cong \text{Ker}(\partial_1 : H_2(M, M^{(0)}) \to H_1(M^{(0)})),$$

under which the homology class $[e^1 \otimes \alpha]$ is mapped to the homology class of the relative 2-cycle $\Phi_2(I \times \tilde{a})$. However, since $e^0 \times \sigma_{\varphi, \alpha} \in S_2(M^{(0)})$, it holds that

$$[\Phi_2(I \times \tilde{a})] = [\Phi_2(I \times \tilde{a}) - e^0 \times \sigma_{\varphi, \alpha}] = [Z_\alpha] \in H_2(M, M^{(0)}).$$

Thus, the homology class under consideration is now represented by a genuine 2-cycle in $M$. Finally, we observe that the natural map

$$H_2(M) \cong E_{1,1}^{\infty} \xrightarrow{\cong} E^2_{1,1} \subset H_2(M, M^{(0)})$$

coinsides with the inclusion homomorphism. This completes the proof.

By Lemma 3.5, it is enough to compute $i_*([Z_\alpha])$. Since $\tilde{a}$ is a 1-cycle in $\Sigma_g = \partial V_g$, the 2-chain $\Phi_2(I \times \tilde{a})$ lies in $\partial M$. Hence,

$$Z_\alpha = -e^0 \times \sigma_{\varphi, \alpha} \in Z_2(M, \partial M).$$

This shows that $i_*([Z_\alpha])$ is represented by the relative 2-cycle $-e^0 \times \sigma_{\varphi, \alpha}$. This completes the proof of Proposition 3.4.

### 3.5. Proof of Theorem 1.1

We describe the intersection form of $M$ and prove Theorem 1.1.

First we claim that the second homology group $H_2(M)$ is naturally isomorphic to $U^2_\varphi := \text{Ker}(S - I_g) \subset \mathbb{Z}^g$. In fact, by Lemma 3.1 we have $H_2(M) \cong H_1(V_g)^{\pi_1(S^1)}$, and the action of $\varphi$ on $H_1(V_g) \cong \mathbb{Z}^g$ is given by the matrix $S$. Thus, the claim follows.

We next claim that under the isomorphism $H_2(M) \cong U^2_\varphi$, the intersection form on $H_2(M)$ is transferred to the bilinear form $(\cdot, \cdot)_\varphi$. Since $\phi^\prime_g(\varphi) = \text{Sign} M\varphi$, this will complete the proof of Theorem 1.1. The proof of this claim consists of two steps.
**Step 1.** We give a description of the bilinear form on $H_1(V_g)^{\pi_1(S^1)}$ that is obtained by transferring the intersection form on $H_2(M_\varphi)$. Let $\langle \cdot, \cdot \rangle_V: H_2(V_g, \Sigma_g) \times H_1(V_g) \to \mathbb{Z}$ be the intersection product of the compact oriented 3-manifold $V_g$. We have

$$
(D_i, \beta_j)_V = \delta_{ij} \quad \text{for any } i, j \in \{1, \ldots, g\}.
$$

(3.2)

Let

$$
H_0(S^1; \mathcal{H}_2(V_g, \Sigma_g)) \times H_1(S^1; \mathcal{H}_1(V_g)) \to \mathbb{Z}
$$

be the intersection product of $H_0(S^1; \mathcal{H}_2(V_g, \Sigma_g))$ and $H_1(S^1; \mathcal{H}_1(V_g))$ followed by the contraction of the coefficients by the form $\langle \cdot, \cdot \rangle_V$. Under the isomorphisms in Lemmas 3.1 and 3.2, this is equivalent to the intersection product $H_2(M_\varphi) \times H_2(M_\varphi, \partial M_\varphi) \to \mathbb{Z}$. By composing (3.3) and the homomorphism

$$
H_1(V_g)^{\pi_1(S^1)} \times H_1(V_g)^{\pi_1(S^1)} \xrightarrow{d \otimes \text{id}} H_2(V_g, \Sigma_g)^{\pi_1(S^1)} \times H_1(V_g)^{\pi_1(S^1)}
$$

$$
\cong H_0(S^1; \mathcal{H}_2(V_g, \Sigma_g)) \times H_1(S^1, \mathcal{H}_1(V_g)),
$$

we obtain a bilinear form on $H_1(V_g)^{\pi_1(S^1)}$. Proposition 3.4 implies that this is equivalent to the intersection form on $H_2(M_\varphi)$. 

**Step 2.** We prove that the bilinear form on $H_1(V_g)^{\pi_1(S^1)}$ described in the previous paragraph is equivalent to $\langle \cdot, \cdot \rangle_\varphi$ under the identification $H_1(V_g)^{\pi_1(S^1)} \cong U_\varphi^2$. Let $x = (x_1, \ldots, x_g)$, $y = (y_1, \ldots, y_g) \in U_\varphi^g \subset \mathbb{Z}^g$. We regard $x$ as an element of $H_1(V_g)^{\pi_1(S^1)}$. Then, we can take $\tilde{x} = \sum_{i=1}^g x_i\beta_i \in H_1(\Sigma_g)$ as a lift of $x$ which we need to compute $d(x)$. Thus, we have

$$
\varphi_*(\tilde{x}) - \tilde{x} = (\alpha_1, \ldots, \alpha_g) Q^t(x_1, \ldots, x_g) = (x_1, \ldots, x_g)^t Q^t(\alpha_1, \ldots, \alpha_g),
$$

and hence $d(x) = (x_1, \ldots, x_g)^t Q^t(D_1, \ldots, D_g)$. Therefore, the pairing of $x$ and $y$ by the bilinear form on $H_1(V_g)^{\pi_1(S^1)}$ described above is equal to

$$
\langle (x_1, \ldots, x_g)^t Q^t(D_1, \ldots, D_g), (\beta_1, \ldots, \beta_g)^t (y_1, \ldots, y_g) \rangle_V = \langle x, y \rangle_\varphi.
$$

Here we used the equality (3.2). This completes the proof of Theorem 1.1.

**Remark 3.6** There is a 2-cocycle $m_\lambda$ on $\text{Sp}(2g; \mathbb{Z})$ constructed by Turaev [20] which satisfies $\langle m_\lambda \rangle = \langle \sigma_g \rangle \in H^2(\text{Sp}(2g; \mathbb{Z}))$, and Walker, in page 124 of his note*, constructed a (unique) cobounding function $j: \text{Mod}(\Sigma_g) \to \mathbb{Q}$ of the sum $\tau_g^* + \rho^* m_\lambda$ of 2-cocycles. The 2-cocycle $m_\lambda$ and the function $j$ depend on the choice of a lagrangian $\lambda \subset H_1(\Sigma_g; \mathbb{Q})$. If we choose a suitable lagrangian $\lambda$, the restriction of $j$ to $\text{Mod}(V_g)$ is known to be a cobounding function of $\rho^* \tau_g$, and coincides with our function $\delta_V^V$. Gilmer and Masbaum [5, Proposition 6.9] described $j$ explicitly in a way which is similar to but different from ours.

**Remark 3.7** Since $Sy = y$ for any $y \in U_\varphi$, we have $\langle x, y \rangle_\varphi = \langle x, y \rangle_\varphi = \langle x, y \rangle_\varphi$. Since $t Q^t S y$ is symmetric by (2.2), this gives a purely algebraic explanation for the symmetric property of the form $\langle \cdot, \cdot \rangle_\varphi$ on $U_\varphi$.

---

Remark 3.8 By Theorem 1.1, one can regard $\phi_g^V$ as a 1-cochain on $\text{urSp}(2g; \mathbb{Z})$. For $g \geq 3$, it is the unique 1-cochain which cobounds $\tau_g$ on $\text{urSp}(2g; \mathbb{Z})$ since $H^1(\text{urSp}(2g; \mathbb{Z})) = 0$; see [19, Corollary 4.4].

4. Evaluation of Meyer functions

4.1. The Meyer function on the hyperelliptic mapping class group

There is a unique 1-cochain $\phi_g^H : \mathcal{H}(\Sigma_g) \to \mathbb{Q}$ such that for any $\varphi_1, \varphi_2 \in \mathcal{H}(\Sigma_g)$,

$$\phi_g^H(\varphi_1) + \phi_g^H(\varphi_2) - \phi_g^H(\varphi_1 \varphi_2) = \tau_g(\rho(\varphi_1), \rho(\varphi_2)).$$

(4.1)

The 1-cochain $\phi_g^H$ is called the Meyer function on the hyperelliptic mapping class group of genus $g$; see [4, 18].

Recall the element $s_1 = t_2t_3t_1t_2 \in \mathcal{H}(V_g) \subset \mathcal{H}(\Sigma_g)$ which was defined in Section 2.4.

Lemma 4.1 $\phi_g^H(s_1) = (2g + 3)/(2g + 1)$.

Proof Set $T_i = \rho(t_i)$ for every $i \in \{1, 2, 3\}$. Using (4.1), we have

$$\phi_g^H(s_1) = \phi_g^H(t_2) + \phi_g^H(t_3) + \phi_g^H(t_1) + \phi_g^H(t_2) - \tau_g(T_1, T_2) - \tau_g(T_3, T_1T_2) - \tau_g(T_2, T_3T_1T_2).$$

As was shown in [4, Lemma 3.3] and [18, Proposition 1.4], we have $\phi_g^H(t_i) = (g + 1)/(2g + 1)$ for all $i \in \{1, 2, 3\}$. Also, by a direct computation we obtain $\tau_g(T_1, T_2) = 0$, $\tau_g(T_3, T_1T_2) = 0$, and $\tau_g(T_2, T_3T_1T_2) = 1$. The result follows from these equalities.

4.2. The Meyer function on the handlebody group

Recall from the introduction that we defined $\phi_g^V : \text{Mod}(V_g) \to \mathbb{Z}$ by $\varphi \mapsto \text{Sign} M_\varphi$, where $M_\varphi$ is the mapping torus of $\varphi$.

Lemma 4.2 The function $\phi_g^V : \text{Mod}(V_g) \to \mathbb{Z}$ cobounds the cocycle $\rho^* \tau_g$ in the handlebody group $\text{Mod}(V_g)$. If $g \geq 3$, $\phi_g^V$ is the unique cobounding function of $\rho^* \tau_g$.

Proof The uniqueness follows from the fact that $H_1(\text{Mod}(V_g))$ is torsion when $g \geq 3$.

For given two mapping classes $\varphi, \psi \in \text{Mod}(V_g)$, there is an oriented $V_g$-bundle $W(\varphi, \psi) \to P$ such that the monodromy along $\ell_1$, $\ell_2$, and $\ell_3$ are $\varphi$, $\psi$, and $(\varphi \psi)^{-1}$, respectively. The boundary of $W(\varphi, \psi)$ is written as

$$\partial W(\varphi, \psi) = E(\varphi, \psi) \cup (M_{\varphi^{-1}} \sqcup M_{\psi^{-1}} \sqcup M_{\varphi\psi}).$$

Note that $M_{\varphi^{-1}}$ is diffeomorphic to $-M_\varphi$ under an orientation-preserving diffeomorphism, where $-M_\varphi$ denotes the mapping torus $M_\varphi$ with orientation reversed. Since the signature of $\partial W(\varphi, \psi)$ is zero, Novikov additivity implies that

$$\text{Sign} E(\varphi, \psi) = \text{Sign} M_\varphi - \text{Sign} M_\psi + \text{Sign} M_{\varphi\psi} = 0.$$

This shows that $\phi_g^V$ is a cobounding function of $\rho^* \tau_g$ restricted to $\text{Mod}(V_g)$. 

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Since \( \dim V_{A,B} \leq 4g \) for any \( A, B \in \text{Sp}(2g; \mathbb{Z}) \), the signature cocycle \( \tau_g \) is a bounded 2-cocycle. Therefore, it represents a class in the second bounded cohomology group \( H^2_b(\text{Mod}(\Sigma_g)) \). The image of \([\tau_g]\) under the natural homomorphism \( H^2_b(\text{Mod}(\Sigma_g); \mathbb{Q}) \to H^2_b(\mathcal{H}(\Sigma_g); \mathbb{Q}) \) is nontrivial since the Meyer function \( \phi_g^H \) is unbounded. In contrast, we have:

**Proposition 4.3** Under the natural homomorphism \( H^2_b(\text{Mod}(\Sigma_g); \mathbb{Q}) \to H^2_b(\text{Mod}(V_g); \mathbb{Q}) \), the image of the cohomology class \([\tau_g]\) vanishes.

**Proof** The restriction of the signature cocycle \( \tau_g \) to \( \text{Mod}(V_g) \) is cobounded by the function \( \phi_g^V \), and \( \phi_g^V \) is a bounded function since the rank of \( H_2(M_\phi) \) is at most \( g \). \( \square \)

### 4.3. Computation of the Meyer function on the handlebody group

Theorem 1.1 shows that the bilinear form \( \langle \ , \ , \ \rangle_\varphi \) on \( U_\varphi \), whose signature coincides with \( \phi_g^V(\varphi) \), can be computed from the homological monodromy \( \rho(\varphi) \in \text{urSp}(2g; \mathbb{Z}) \). In more detail, if \( \rho(\varphi) = \begin{pmatrix} P & Q \\ O_g & S \end{pmatrix} \), then \( U_\varphi = \text{Ker}(S - I_g) \subset \mathbb{Q}^g \) and \( \langle x, y \rangle_\varphi = \tau x^t Q y \) for \( x, y \in U_\varphi \).

The 1-cochain \( \phi_g^V \), regarded as the one defined on \( \text{urSp}(2g; \mathbb{Z}) \), is stable with respect to \( g \) in the following sense. For every nonnegative integer \( g \geq 0 \), there is a natural embedding \( \iota \colon \text{urSp}(2g; \mathbb{Z}) \hookrightarrow \text{urSp}(2(g + 1); \mathbb{Z}) \):

\[
A = \begin{pmatrix} P & Q \\ O_g & S \end{pmatrix} \mapsto \iota(A) = \begin{pmatrix} \tilde{P} & \tilde{Q} \\ O_{g+1} & \tilde{S} \end{pmatrix},
\]

where
\[
\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}.
\]

Then \( \phi_g^V(\iota(A)) = \phi_g^V(A) \) for any \( A \in \text{urSp}(2g; \mathbb{Z}) \).

**Lemma 4.4** For any positive integer \( m \), we have \( \phi_g^V(t_1^m) = 1 \).

**Proof** Since the action of \( \rho(t_1^m) \) on \( H_1(\Sigma_g) \) is given by
\[
\rho(t_1) \colon \alpha_i \mapsto \alpha_i \quad (i = 1, \ldots, g), \quad \beta_1 \mapsto m\alpha_1 + \beta_1, \quad \beta_i \mapsto \beta_i \quad (i = 2, \ldots, g),
\]
we may assume that \( g = 1 \). Then \( \rho(t_1^m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \), and \( \text{Ker}(S - I_1) = \mathbb{Z} \) on which the pairing is given by the \( 1 \times 1 \) matrix \( (m) \). Hence, \( \phi_g^V(t_1^m) = 1 \), as required. \( \square \)

**Lemma 4.5** \( \phi_g^V(s_1) = 1 \).

**Proof** The proof proceeds as in the same way as the previous lemma. In this case we may assume that \( g = 2 \). Then
\[
\rho(s_1) = \begin{pmatrix} P & Q \\ O_2 & S \end{pmatrix} \quad \text{with} \quad P = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

The rest of computation is straightforward, so we omit it. \( \square \)
4.4. Proof of Theorem 1.2

Since both the 1-cochains $\phi_g^H$ and $\phi_g^V$ cobound the signature cocycle, their difference becomes a $\mathbb{Q}$-valued homomorphism on $\mathcal{H}(V_g) = \mathcal{H}(\Sigma_g) \cap \text{Mod}(V_g)$.

We compare the homomorphism $\phi_g^H - \phi_g^V$ with the generator $\mu \in H^1(\mathcal{H}(V_g))$ in Corollary 2.6. It is sufficient to evaluate $\phi_g^H - \phi_g^V$ on $s_1$ if $g$ is even, and on $t_1 s_1^{2g+1}$ if $g$ is odd. By Lemmas 4.1 and 4.5 we immediately obtain

$$ (\phi_g^H - \phi_g^V)(s_1) = \frac{2}{2g+1}. \quad (4.2) $$

This settles the case where $g$ is even. When $g$ is odd, we compute

$$ (\phi_g^H - \phi_g^V)(t_1 s_1^{2g+1}) = (\phi_g^H - \phi_g^V)(t_1) + \frac{g+1}{2}(\phi_g^H - \phi_g^V)(s_1) $$

$$ = \left( \frac{g+1}{2g+1} - 1 \right) + \frac{g+1}{2} \cdot \frac{2}{2g+1} $$

$$ = \frac{1}{2g+1}. $$

Here, we used the fact that $\phi_g^H - \phi_g^V$ is a homomorphism on $\mathcal{H}(V_g)$ in the first line; we used the fact that $\phi_g^H(t_1) = (g+1)/(2g+1)$ (see the proof of Lemma 4.1), Lemma 4.4 and (4.2) in the second line. This completes the proof of Theorem 1.2.

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