On Holomorphic poly-Norden Manifolds

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Abstract: In this paper, we investigated a new manifold with a poly-Norden structure, which is inspired by the positive root of the equation \( x^2 - mx - 1 = 0 \). We call this new manifold as holomorphic poly-Norden manifolds. We examine some properties of the Riemann curvature tensor and give an example of this manifold. Then, we define a different connection on this manifold which is named the semisymmetric metric poly F-connection and study some properties of the curvature and torsion tensor field according to this connection.

Key words: Poly-Norden structure, semisymmetric metric connection, Tachibana operator, bronze ratio

1. Introduction

The theory of differential structures on manifolds is studied with great interest. In [19], the authors have extensively investigated complex, product, contact, and f-structures. Later, a very interesting structure was defined on the manifolds, which is called the golden-structure [1]. In fact, the golden-structure is inspired by the equation \( x^2 - x - 1 = 0 \), whose positive root \( \eta = \frac{1 + \sqrt{5}}{2} = 1.61803398874989... \) is the golden ratio. If the equation \( \varphi^2 - \varphi - 1 = 0 \) is provided on manifold \( M \), then \((M, \varphi)\) is called golden manifold, where \( \varphi \) is the tensor field of type \((1,1)\) on the manifold.

In [9], the authors defined metallic structures which are the generalization of the golden structure. For integers \( p \) and \( q \), the metallic ratio \( \sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2} \) is the root of the equation \( x^2 - px - q = 0 \). Also, a manifold \( M \) endowed with the tensor field \( J \) of type \((1,1)\), such that \( J^2 - pJ - qI = 0 \), is named metallic manifold. Many authors have made interesting studies on golden and metallic manifolds. In one of them [4], they defined a semisymmetric metric \( F \)-connection on golden manifolds and made studies on it. A semisymmetric connection \( \nabla \) is a connection whose torsion tensor checks the equation \( S(U,V) = w(U)V - w(U)V \), where \( U, V \) are vector fields and \( w \) is a covector field. In addition, if this connection holds the requirements \( \nabla g = 0 \) and \( \nabla F = 0 \), then this connection is called semisymmetric metric \( F \)-connection. See [2, 3, 5, 11, 13, 17, 18] studies for more information.

The new bronze ratio is defined by

\[ B_m = \frac{m + \sqrt{m^2 - 4}}{2}, \]

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which is the positive root of the equation $x^2 - mx + 1 = 0$ [10]. In [14], by inspiring from the ratio, the author introduced a new structure on a manifold, which is called a poly-Norden structure. In his work, the author examined some geometric properties of the poly-Norden manifold and investigated certain maps between poly-Norden manifolds and other manifolds endowed with different structures.

A poly-Norden structure on a differentiable manifold $M$ is a $(1,1)$-type tensor field (affinor) $F$, which satisfies the relation $F^2 = mF - I$, where $I$ is the identity operator on the Lie algebra of vector fields on the manifold. Thus, the pair $(M,F)$ is named an almost poly-Norden manifold. We say that a semi-Riemann metric $g$ is pure (or self-adjoint) with respect to a poly-Norden structure $F$ if $g(FU, V) = g(U, FV)$ for any vector fields $U$, $V$. Also, if $g(FU, FV) = mg(FU, V) - g(U, V)$, then the semi-Riemann metric $g$ is called a $F$-compatible metric (see [6, 7]). So, an almost poly-Norden manifold $(M,F)$ endowed with a semi-Riemann metric $g$ is called an almost poly-Norden semi-Riemann manifold and is represented by $(M,g,F)$ [14]. Also, see [12] study for more information on almost poly-Norden manifolds.

In this paper, we derive the integrability condition of the almost poly-Norden structure $F$ on $(M,g,F)$ with the help of a different operator whose name is $\varphi$ operator (or Tachibana operator) [16]. Then, we named this manifold $(M,g,F)$ as holomorphic poly-Norden manifold because it satisfies the condition $\varphi_F g = 0$ and by examining the curvature property, we gave an example of such manifold. After that, we introduced a connection $^{\varphi}\nabla$ with semisymmetric torsion endowed with poly-Norden structure $F$ on this manifold and proved that this new connection satisfies the equations $^{\varphi}\nabla g = 0$ and $^{\varphi}\nabla F = 0$, that is, $^{\varphi}\nabla$ is a semisymmetric metric $F$-connection. Finally, by using the operator $\varphi$, we investigated the curvature and torsion properties of this connection $^{\varphi}\nabla$.

2. Preliminaries

Let $M_n$ be $(n = 2k)$ differentiable manifold of class $C^\infty$. Throughout this paper, all connections and tensor fields on the manifold will be assumed to be of class $C^\infty$. In addition, the set of tensor fields of type $(p, q)$ will be represented by $\mathfrak{S}^p_q(M_n)$. For example, the set of vector and covector fields will be indicated by $\mathfrak{S}^0_0(M_n)$ and $\mathfrak{S}^1_0(M_n)$, respectively. Now, let us give some definitions that we will use in this article.

**Definition 2.1 ([16])** Let $M_n$ be differentiable manifold. For any $K \in \mathfrak{S}^0_q(M_n)$, if the following condition holds, then the tensor field $K$ is called a pure tensor field.

$$K(JV_1, V_2, ..., V_q) = K(V_1, JV_2, ..., V_q)$$

$$= \ldots = K(V_1, V_2, ..., JV_q),$$

where $V_1, V_2, ..., V_q \in \mathfrak{S}^1_0(M_n)$ and $J \in \mathfrak{S}^1_1(M_n)$.

**Definition 2.2 ([16])** Let $M_n$ be differentiable manifold. If $K$ is a pure tensor field, then the operator $\varphi$ (or Tachibana operator) applied to this tensor is given by

$$ (\varphi_J K)(X, V_1, V_2, ..., V_q) $$

$$ = (JX)(K(V_1, V_2, ..., V_q)) - X(K(JV_1, V_2, ..., V_q)) $$

$$ + \sum_{i=1}^q K(V_1, ..., (LV_i)X, ..., V_q), \quad (2.1) $$
where \( X \in \mathfrak{g}_0^1(M_n) \) and \( L_V \) represents the Lie differentiation according vector field \( V \).

Let \( J \) be a complex structure, that is, \( J^2 = -I \). In the equation (2.1), if \( \varphi_J K = 0 \), then the vector field \( K \) is called a holomorphic (or analytic) tensor field. The Riemann metric \( g \) on an almost complex manifold \((M_n, J)\) is called a Norden (or anti-Hermitian) manifold if it satisfies the condition

\[
g(JU, V) = g(U, JV) \quad \text{or} \quad g(JU, JV) = -g(U, V),
\]

where \( U, V \in \mathfrak{g}_0^1(M_n) \). It is easy to see that \( g \) is a semi-Riemannian metric \footnote{6}. Then, the triplet \((M_n, g, J)\) is named almost Norden manifold. Besides, if \( \nabla J = 0 \), then the triplet \((M_n, g, J)\) becomes a Norden (anti-Kähler) manifold, where \( \nabla \) is the Riemannian connection of \( g \).

On almost Norden manifold \((M_n, g, J)\), if \( \varphi_J g = 0 \), then \( g \) is holomorphic and this manifold is called almost holomorphic Norden manifold.

### 3. Holomorphic poly-Norden manifolds

In \footnote{14}, the author (Propositions 3.4 and 3.5) shows that complex and poly-Norden structures will be written in terms of each other, such that

\[
F_\pm = \frac{m}{2} I \pm \frac{\sqrt{4 - m^2}}{2} J
\]

and

\[
J_\pm = \pm \left( \frac{-m}{\sqrt{4 - m^2}} I + \frac{2}{\sqrt{4 - m^2}} F \right),
\]

where \(-2 < m < 2\). From the equation (2.1) and (3.1), we obtain

\[
\varphi_F K = \frac{\sqrt{4 - m^2}}{2} \varphi_J K
\]

and from here, we can easily say that if \( \varphi_F K = 0 \), then the tensor \( K \) is holomorphic. This means that we can study holomorphicity on the almost poly-Norden semi-Riemann manifold \((M_n, g, F)\).

**Theorem 3.1** Let \((M_n, g, F)\) be an almost poly-Norden semi-Riemann manifold. If \( \nabla \) denotes the Levi-Civita connection of the metric \( g \), then \( \nabla F = 0 \) if and only if \( \varphi_F g = 0 \).

**Proof** From the covariant derivation of the \( g(FU, V) = g(U, FV) \) with respect to Riemann connection \( \nabla \), we obtain

\[
g((\nabla_X F)U, V) = g(U, (\nabla_X F)V).
\]

Applying \( \varphi \) to the Riemannian tensor \( g \) and from \( L_U V = [U, V] = \nabla_U V - \nabla_V U \), we get

\[
(\varphi_{FX} g)(U, V) = (FX)g(U, V) - Xg(FU, V)
\]

\[
+ g((L_U F)X, V) + g(U, (L_V F)X)
\]

\[
= -g((\nabla_X F)U, V) + g((\nabla_U F)X, V) + g(X, (\nabla_F V)U)
\]

and

\[
(\varphi_{VF} g)(U, X) = -g((\nabla_F V)U, X) + g((\nabla_U F)X, V) + g(V, (\nabla_X F)U).
\]

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From the last two equations, we have

\[(\varphi_{F,X} g)(U, V) + (\varphi_{F,V} g)(U, X) = 2 g(\nabla_U F) V, X).\] (3.4)

It is clear that in the equation (3.3), if \(\nabla F = 0\), then \(\varphi_F g = 0\) and in the equation (3.4), if \(\varphi_F g = 0\), then \(\nabla F = 0\).

Also, from the equation (3.2), we have

\[\varphi_F g = \frac{\sqrt{4 - m^2}}{2} \varphi_J g.\]

Then, if \(\varphi_F g = 0\) (or \(\nabla F = 0\)), then the triplet \((M_n, g, F)\) is called holomorphic poly-Norden manifold.

Twin metric \(G\) of the almost poly-Norden semi-Riemann manifold \((M_n, g, F)\) is defined by

\[G(U, V) = g(FU, V),\] (3.5)

for \(X, Y \in \mathcal{I}_0(M_n)\). Then,

\[G(U, V) = g(FU, V) = g(V, FU) = g(FY, U) = G(V, U)\]

and

\[G(FU, V) = g(F^2 U, V) = g(FU, FV) = G(U, FV),\]

that is, twin metric \(G\) is both symmetric and pure according to poly-Norden structure \(F\). From the covariant derivation of the equation (3.5) with respect to Riemann connection \(\nabla\), we obtain

\[(\nabla_X G)(U, V) = (\nabla_X g)(FU, V) + g(\nabla_X F)U, V) = g(\nabla_X F)U, V)\]

and then,

**Proposition 3.2** Let \((M_n, g, F)\) be a holomorphic poly-Norden manifold. The Riemann connection of the metric \(g\) equals to the Riemann connection of the twin metric \(G\), i.e., \(G\nabla = \nabla\).

Let \(gR\) and \(G\) be Riemann curvature tensors of the metric \(g\) and the twin metric \(G\), respectively. From the proposition 3.2, we can easily see that \(gR = g \cdot F \cdot g\). The Ricci identity for poly-Norden structure \(F\) on holomorphic poly-Norden manifold \((M_n, g, F)\) is as follows:

\[gR(U, V, FZ) - F(gR(U, V, Z)) = 0.\] (3.6)
Also, for the \((0,4)\)-type of the curvature tensor \(g^{\text{R}}\), we get
\[
g^{\text{R}}(U, V, FZ, W) = g(F^{\text{R}}(U, V, Z), W)
\]
that is, the curvature tensor \(g^{\text{R}}\) is pure according to \(Z\) and \(W\). Besides, from the \(g^{\text{R}}(U, V, Z, W) = g^{\text{R}}(Z, W, U, V)\) property of the curvature tensor \(g^{\text{R}}\), we have
\[
g^{\text{R}}(FU, V, Z, W) = g^{\text{R}}(U, FV, Z, W).
\]

Finally, for \(g^{\text{R}} = G^{\text{R}}\) and (3.5), the curvature tensor \(G^{\text{R}}\) of the twin metric \(G\) is as follows:
\[
G^{\text{R}}(U, V, Z, W) = G(G^{\text{R}}(U, V, Z), W)
\]
and
\[
\]

From the last two equations, we obtain
\[
\]
After all, we say that the curvature tensor \(g^{\text{R}}\) is pure with regard to poly-Norden structure \(F\), i.e.
\[
\]

Then,

**Theorem 3.3**  Let \((M_n, g, F)\) be a holomorphic poly-Norden manifold. Then, \(\varphi F^{\text{R}} = 0\), that is, the curvature tensor \(g^{\text{R}}\) is a holomorphic tensor.

**Proof**  From the covariant derivation of the equation (3.6), we have,
\[
(\nabla_X g^{\text{R}})(FU_1, U_2, U_3) = F(\nabla_X g^{\text{R}})(U_1, U_2, U_3).
\]

If the operator \(\varphi\) is applied to the Riemann curvature tensor \(g^{\text{R}}\), we obtain
\[
(\varphi F_X g^{\text{R}})(U_1, U_2, U_3, U_4) = (\nabla_F X g^{\text{R}})(U_1, U_2, U_3, U_4) - (\nabla_X g^{\text{R}})(FU_1, U_2, U_3, U_4).
\]
Substituting (3.7) in (3.8) and using the Bianchi’s 2nd identity for the tensor field $^gR$, we obtain

\[
\varphi_{FX}^gR(U_1, U_2, U_3, U_4) = g(\nabla_{FX}^gR(U_1, U_2, U_3) - (\nabla_X^gR)(FU_1, U_2, U_3), U_4)
\]

\[
= g(\nabla_{UX}^gR(U_1, U_2, U_3) - (\nabla_X^gR)(U_2, FX, U_3) + F(\nabla_X^gR)(U_1, U_2, U_3), U_4)
\]

\[
= -g(\nabla_{U_2}^gR)(FX, U_1, U_3) + (\nabla_{U_1}^gR)(U_2, X, U_3) + F(\nabla_X^gR)(U_1, U_2, U_3), U_4)
\]

\[
= -g(\sigma (\nabla_{U_2}^gR)(FX, U_1, U_3), U_4) = 0
\]

where $\sigma$ represents the cyclic sum over $X, U_1$, and $U_2$. Finally, from the equation (3.2), we have

\[
\varphi_{F^gR} = \frac{\sqrt{4 - m^2}}{2}\varphi_{J^gR},
\]

namely, the curvature tensor $^gR$ is a holomorphic tensor.

**Example 3.4** Let $\mathbb{R}_{2n}$ be a semi-Euclidean space endowed with semi-Euclidean metric $g$, that is,

\[
g = \begin{pmatrix}
\delta^i_j & 0 \\
0 & -\delta^\bar{i}_\bar{j}
\end{pmatrix}
\]

where $i, j = 1, ..., n$, $\bar{i}, \bar{j} = n + 1, ..., 2n$. Also, let $\mathbb{C}_n$ be a complex space with $\mathbb{R}_{2n}$ such that

\[
s : z \in \mathbb{C}_n \rightarrow s(z) = \bar{Z} \in \mathbb{R}_{2n},
\]

where $z = (z_1, z_2, ..., z_n)$, $s(z) = \bar{Z} = (x_1, x_2, ..., x_n, y_1, y_2, ..., y_n)$ and $z_t = x_t + iy_t$, $t = 1, 2, ..., n$. Then, the complex structure $J$ on $\mathbb{R}_{2n}$ is given by

\[
J = \begin{pmatrix}
0 & \delta^i_j \\
-\delta^\bar{j}_\bar{i} & 0
\end{pmatrix}.
\]

From here, we easily see that $g_{im}F_{mj}^n = g_{mj}F_{imi}^n = , i.e. the structure $J$ is compatible (purity) with metric $g$ and then $(\mathbb{R}_{2n}, J, g)$ is a holomorphic Norden Euclidean space. Also, poly-Norden structures $F_{\pm}$ on $\mathbb{R}_{2n}$ obtained from complex structure $J$ are as follows:

\[
F_{\pm} = \begin{pmatrix}
\frac{m}{2} \delta^i_j & \pm \frac{\sqrt{4 - m^2}}{2} \delta^\bar{j}_\bar{i} \\
\mp \frac{\sqrt{4 - m^2}}{2} \delta^\bar{i}_\bar{j} & \frac{m}{2} \delta^i_j
\end{pmatrix}
\]

and the triple $(\mathbb{R}_{2n}, F, g)$ is called a holomorphic poly-Norden Euclidean space.
4. Semisymmetric metric poly $F$-connection

In this section, we are going to study the holomorphic poly-Norden manifold endowed with another connection rather than the metric connection.

**Theorem 4.1** Let $(M_n, g, F)$ be a holomorphic poly-Norden manifold and $^p\nabla$ be a connection with torsion $^pT$ on that manifold such that

$$^pT(U, V) = \gamma(V)(U) - \gamma(U)(V) - \gamma(FV)(FU) + \gamma(FU)(FV)$$

where $U, V \in \mathfrak{X}_0(M_n)$ and $\gamma \in \mathfrak{X}_1(M_n)$. If this connection satisfies $^p\nabla g = 0$ and $^p\nabla F = 0$, then

$$^p\nabla_{U} V = \nabla_{U} V + \gamma(V)(U) - g(U, V)(W) - \gamma(FV)(FU) + g(FU, V)(FW),$$

where $\nabla$ stands for the Levi-Civita connection of the metric $g$ and $g(W, Y) = \gamma(Y), W \in \mathfrak{X}_0(M_n)$.

**Proof** It is well known that a new connection $^p\nabla$ can be formed with

$$^p\nabla_U V = \nabla_U V + D(U, V),$$

where $D$ is the deformation tensor field of type $(1,2)$. Then, from $^pT(U, V) = ^p\nabla_U V - ^p\nabla_V U - [U, V]$ and the method of Hayden [8], we obtain

$$^pT(U, V) = D(U, V) - D(V, U)$$

If $^p\nabla g = 0$, we have

$$D(U, V, Z) + D(U, Z, V) = 0.$$ (4.5)

From the equations (4.4) and (4.5), we have

$$^pT(U, V, Z) = D(U, V, Z) - D(V, U, Z)$$

$$^pT(Z, U, V) = D(Z, U, V) - D(U, Z, V)$$

$$^pT(Z, V, U) = D(Z, V, U) - D(V, Z, U)$$

and then

$$^pT(U, V, Z) + ^pT(Z, U, V) + ^pT(Z, V, U) = 2D(U, V, Z),$$

where $^pT(U, V, Z) = g(^pT(U, V), Z)$.

Substituting (4.1) in the last equation, we get

$$D(U, V) = \gamma(V)(U) - g(U, V)(W) - \gamma(FV)(FU) + g(FU, V)(FW).$$

Also, the connection given by (4.2) satisfies the condition $^p\nabla F = 0$. So, this proof is complete. \(\square\)

From now on, the connection $^p\nabla$ will be called semisymmetric metric poly $F$-connection.

With a simple calculation, we can see that the torsion tensor $^pT$ is pure according to poly-Norden structure $F$, i.e.

$$^pT(FU, V) = ^pT(U, FV) = F^pT(U, V).$$
Also, in [15], the author has proved that an $F$–connection is pure if and only if its torsion tensor is pure. Then, we can easily write as following equation:

$$p\nabla_{FU}V = p\nabla_U(FV) = Fp\nabla_UV.$$ 

Then,

**Theorem 4.2** Let $(M_n, g, F)$ be a holomorphic poly-Norden manifold. If the covector $\gamma$ in (4.1) is holomorphic, then the torsion tensor $pT$ is also a holomorphic tensor, i.e. $\varphi_F\gamma = 0$ and $\varphi_F pT = 0$.

**Proof** By applying the $\varphi$ operator to the torsion tensor $pT$, we get

$$(\varphi_{FX} pT)(U, V) = (p\nabla_{FX} pT)(U, V) - (p\nabla_X pT)(FU, V) \quad (4.6)$$

Also, for the covector field $\gamma$ in the equation (4.1), we obtain

$$(\varphi_{FX}\gamma)(U) = (p\nabla_{FX}\gamma)(U) - (p\nabla_X\gamma)(FU) \quad (4.7)$$

Finally, from the equation (4.6) and (4.7), we get

$$(\varphi_{FX} pT)(U, V) = (\varphi_{FX}\gamma)(U) - (\varphi_{FX}\gamma)(FU)$$

$$+ (\varphi_{FX}\gamma)(FU)(FU) - (\varphi_{FX}\gamma)(FU)(FU) = 0$$

Then, we write the following corollary:

**Remark 4.3** 1. From the equation (3.2), it is obvious that

$$\varphi_F \gamma = \frac{\sqrt{1 - m^2}}{2} \varphi_J \gamma,$$

and

$$\varphi_F pT = \frac{\sqrt{1 - m^2}}{2} \varphi_J pT,$$

2. If $\varphi_F pT = 0$, from $(\varphi_{FX} pT)(U, V) = (p\nabla_{FX} pT)(U, V) - (p\nabla_X pT)(FU, V)$, we can write

$$(p\nabla_{FX} pT)(U, V) = (p\nabla_X pT)(FU, V)$$

$$= (p\nabla_X pT)(U, FV) = F(p\nabla_X pT)(U, V),$$

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that is, the covariant derivation of the torsion tensor $pT$ according to $p\nabla$ is pure according to poly-Norden structure $F$.

3. The last theorem can also be proved for Riemann connection $\nabla$, that is, we write $(\varphi_{FX}pT)(U,V) = (\nabla_{FX}pT)(U,V) - (\nabla_{X}pT)(FU,V)$ and

$$
(\nabla_{FX}pT)(U,V) = (\nabla_{X}pT)(FU,V) = (\nabla_{X}pT)(U,FV) = F(\nabla_{X}pT)(U,V).
$$

Throughout this article, we will assume that $\varphi F = 0$, that is,

$$
(\nabla_{FX}p\gamma)(U) - (\nabla_{X}p\gamma)(FU) = 0.
$$

5. The curvature tensor of semisymmetric metric poly $F$-connection

It is well-known that the curvature tensor of any linear connection $\nabla$ for all vector fields is as follows:

$$
R(U,V,Z) = (\nabla_{U}V - \nabla_{V}U - \nabla_{[U,V]}V)Z.
$$

Then, (0,4)-type of the curvature tensor for the connection (4.2) has the following form:

$$
pR(U,V,Z,W) = \varphi R(U,V,Z,W)
+ \varsigma(U,Z)g(V,W) - \varsigma(V,Z)g(U,W)
+ \varsigma(V,W)g(U,Z) - \varsigma(U,W)g(V,Z)
+ \varsigma(V,FZ)g(FU,W) - \varsigma(U,FZ)g(FV,W)
+ \varsigma(U,FW)g(FV,Z) - \varsigma(V,FW)g(FU,Z),
$$

where

$$
\varsigma(U,V) = (\nabla_{U}\gamma)(V) - \gamma(U)\gamma(V) + \frac{1}{2}\gamma(W)g(U,V)
+ \gamma(FU)\gamma(FV) - \frac{1}{2}\gamma(W)g(U,V).
$$

It is said that the curvature tensor $pR$ is hold:

$$
$$

that is, $pR$ is antisymmetric according to the first and last two components. Also,

$$
\varsigma(U,V) - \varsigma(V,U) = (\nabla_{U}\gamma)(V) - (\nabla_{V}\gamma)(U)
$$

and for the exterior differential operator $d$ applied to the covector field $\omega$, we get

$$
2(d\gamma)(U,V) = U\gamma(V) - V\gamma(U) - \gamma([U,V])
= (\nabla_{U}\gamma)V + \gamma(\nabla_{V}V) - (\nabla_{V}\gamma)U - \gamma([U,V])
= (\nabla_{U}V) - (\nabla_{V}U) + \gamma(\nabla_{V}V - \nabla_{U}U) - \gamma([U,V])
= (\nabla_{U}\gamma)(V) - (\nabla_{V}\gamma)(U).
$$
From the equations (5.3) and (5.4), we obtain
\[
\varsigma(U,V) - \varsigma(V,U) = (\nabla_U \gamma)(V) - (\nabla_V \gamma)(U) \\
= 2(d\gamma)(U,V).
\] (5.5)

Then, we write the following corollary.

**Corollary 5.1**

1. The covector field $\gamma$ is closed if and only if the tensor field $\varsigma$ is symmetric.

2. If the covector field $\gamma$ is a gradient, that is $\gamma = \partial f$, then the tensor $\varsigma$ is symmetric.

For the tensor $\varsigma$ given by (5.2) is pure with regard to poly-Norden structure $F$, that is,
\[
\varsigma(U, FU) - \varsigma(FU, V) = 0,
\]

and from the equation (5.5), we get
\[
^p R(U, V, Z, W) - ^p R(Z, W, U, V) \\
= 2(d\gamma)(U, Z)g(V, W) - 2(d\gamma)(V, Z)g(U, W) \\
+ 2(d\gamma)(V, W)g(U, Z) - 2(d\gamma)(U, W)g(V, Z) \\
+ 2(d\gamma)(FU, W)g(FV, Z) - 2(d\gamma)(FU, Z)g(FV, W) \\
+ 2(d\gamma)(FU, W)g(FV, Z) - 2(d\gamma)(V, W)g(FU, Z),
\]

then,

**Proposition 5.2** For the curvature tensor $^p R$, if the covector field $\gamma$ is closed ($d\gamma = 0$), then $^p R(U, V, Z, W) - ^p R(Z, W, U, V) = 0$.

By applying the $\varphi$ operator to the tensor $\varsigma$, we get
\[
(\varphi_{FX}\varsigma)(U, V) = (p\nabla_{FX}\varsigma)(U, V) - (\nabla_{FX}\varsigma)(FU, V) \\
= (\nabla_{FX}\varsigma)(U, V) - (\nabla_{FX}\varsigma)(FU, V)
\] (5.6)

From the equation (5.2), we have
\[
(\varphi_{FX}\varsigma)(U, V) = (\nabla_{FX} \nabla_U \gamma)(V) - (\nabla_X \nabla_{FU} \gamma)(V).
\] (5.7)

In the last equation, if we apply the Ricci identity to the 1-form $\gamma$, we obtain
\[
(\nabla_{FX} \nabla_U \gamma)(V) = (\nabla_U \nabla_{FX} \gamma)(V) - \frac{1}{4} \gamma(pR(FX, U, V))
\]
and

\[(\nabla_X \nabla_{FU} \gamma)(V) = (\nabla_X \nabla_U \gamma)(FV)\]
\[= (\nabla_U \nabla_X \gamma)(FV) - \frac{1}{2} \gamma (gR(X, U, FV))\]
\[= (\nabla_U \nabla_{FX} \gamma)(V) - \frac{1}{2} \gamma (gR(X, FU, V))\]

Substituting (5.2) in the equation (5.6), we get

\[\varphi_{FX}(U, V) = -\frac{1}{2} \gamma [gR(FX, U, V) - gR(X, U, FV)]\]
\[= 0.\]

Then,

**Proposition 5.3** Let \((M_n, g, F)\) be a holomorphic poly-Norden manifold. The tensor \(\varsigma\) given by the equation (5.2) is a holomorphic tensor, that is, \(\varphi_F \varsigma = \frac{\sqrt{4-m^2}}{2} \varphi_J \varsigma\) and

\[(p \nabla_{FX}) (U, V) = (p \nabla_X \varsigma)(FU, V) = (p \nabla_X \varsigma)(U, FV).\] (5.8)

Because of the purity of the tensor \(\varsigma\), we say that the curvature tensor of the semisymmetric metric poly \(F\)-connection is a pure tensor, namely,

\[= p R(U, V, FZ, W) = p R(U, V, Z, FW)\]

and from (2.1) and (5.1), we obtain

\[(\varphi_{FX} p R)(U_1, U_2, U_3, U_4) = (p \nabla_{FX} p R)(U_1, U_2, U_3, U_4)\]
\[-(p \nabla_X p R)(FU_1, U_2, U_3, U_4).\] (5.9)

Substituting (5.1) in the last equation, we have

\[(\varphi_{FX} p R)(U_1, U_2, U_3, U_4)\]
\[= (p \nabla_{FX} g R)(U_1, U_2, U_3, U_4) - (p \nabla_X g R)(FU_1, U_2, U_3, U_4)\]
\[+ (\varphi_{FX})(U_1, U_3)g(U_2, U_4) - (\varphi_{FX})(U_2, U_3)g(U_1, U_4)\]
\[+ (\varphi_{FX})(U_2, U_4)g(Y_1, U_3) - (\varphi_{FX})(U_1, Y_4)g(U_2, U_3)\]
\[+ (\varphi_{FX})(FU_1, U_3)g(FU_2, U_4) - (\varphi_{FX})(FU_1, U_3)g(FU_2, U_4)\]
\[+ (\varphi_{FX})(FU_1, U_3)g(FU_2, U_3) - (\varphi_{FX})(FU_2, U_4)g(FU_1, U_3)\]

From the proposition 5.3 and theorem 3.3, we obtain

\[(\varphi_{FX} p R)(U_1, U_2, U_3, U_4) = (p \nabla_{FX} g R)(U_1, U_2, U_3, U_4)\]
\[-(p \nabla_X g R)(FU_1, U_2, U_3, U_4)\]
\[= (\varphi_{FX} g R)(U_1, U_2, U_3, U_4)\]
\[= 0.\]
Finally,

**Theorem 5.4** Let \((M_n, g, F)\) be a holomorphic poly-Norden manifold. The curvature tensor \(pR\) of the semi-symmetric metric poly \(F\)-connection is a holomorphic tensor, i.e.

\[ \varphi_F \ pR = \frac{\sqrt{4-m^2}}{2} \varphi \ pR \quad \text{and} \quad (p\nabla_{FX} pR)(U_1, U_2, U_3, U_4) = (p\nabla_X pR)(FU_1, U_2, U_3, U_4) = (p\nabla_X pR)(U_1, FU_2, U_3, U_4) = (p\nabla_X pR)(U_1, U_2, FU_3, U_4) = (p\nabla_X pR)(U_1, U_2, U_3, FU_4) = F(p\nabla_X pR)(U_1, U_2, U_3, U_4). \]

**References**


