Solvability and maximal regularity results for a differential equation with diffusion coefficient

Kordan OSPANOV∗, Adilet YESBA YEV
Faculty of Mechanics and Mathematics, L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan

Received: 20.02.2020 • Accepted/Published Online: 14.05.2020 • Final Version: 08.07.2020

Abstract: We consider a second-order differential equation with rapidly growing intermediate coefficients. We obtain a solvability result in the cases that the diffusion coefficient of equation is unbounded or it tends to zero at the infinity. Under additional conditions, we prove the $L^p$- maximal regularity estimate for the solution of this equation.

Key words: Second-order differential equation, unbounded intermediate coefficient, existence of a solution, uniqueness, maximal regularity estimate

1. Introduction
In this paper, we consider the following equation:

$$-\rho(x) (\rho(x)y')' + r(x)y' + s(x)y = F(x),$$

where $x \in \mathbb{R} = (-\infty, +\infty)$, all the coefficient functions are defined on $\mathbb{R}$, $\rho(x)$ is a positive and twice continuously differentiable function, $r(x)$ is a continuously differentiable function, and $s(x)$ is a continuous function and $F \in L_p := L_p(\mathbb{R})$, $1 < p < +\infty$.

Let $C^{(2)}_0(\mathbb{R})$ be the set of twice continuously differentiable functions with compact support. We define the operator $\tilde{l}_0$ on $C^{(2)}_0(\mathbb{R})$ as $\tilde{l}_0y := -\rho(x) (\rho(x)y')' + r(x)y' + s(x)y$. We denote the closure of the operator $\tilde{l}_0$ by $\tilde{l}$ in the space $L_p$. By solution of the equation (1.1) we mean a function $y \in D(\tilde{l})$ such that $\tilde{l}y = F$.

In this work, we study questions of the existence and uniqueness of the solutions of (1.1) and conditions, which for a solution $y(x)$ of (1.1) the following estimate holds:

$$\|\rho(\rho'y')\|_p + \|ry'\|_p + \|sy\|_p \leq C\|F\|_p,$$

where $\| \cdot \|_p$ is the norm in $L_p$. If the estimate (1.2) holds, then we call that the solution $y(x)$ of (1.1) is maximally $L_p$-regular, and call (1.2) is an maximal $L_p$-regularity estimate.

In the applications of well-known projection methods (e.g., Fourier or Laplace transformations) to multidimensional differential equations and with coefficients depending on a single variable, we usually obtain ordinary differential equations. Therefore, the investigation of solvability questions for the one-dimensional equation (1.1) is important for the study of partial differential equations with unbounded coefficients. Since

∗Correspondence: ospanov_kn@enu.kz
2010 AMS Mathematics Subject Classification: 34A30, 34C11
equation (1.1) with \( \rho(x) \geq \delta > 0 \) and its multidimensional generalizations are used in quantum mechanics and stochastic analysis, they have been studied intensively (see [2] and references therein). In [3, Ch.9, Theorem 2.4], [15, Ch.10.17], [12, Ch.7, Theorem 6], the authors considered the case that \( s(x) \) is positive, and the growth at infinity of the absolute value of the intermediate coefficient \( r(x) \) is limited by some power of \( s(x) \). In [6–8, 10], it is assumed that the coefficient \( r(x) \) is independent of the function \( s(x) \), but \( r(x) \) cannot grow faster than \( |x| \ln |x| \) \((|x| \gg 1)\). The question arises whether there exists a unique solution of equation (1.1), if \(|r(x)|\) grows more rapidly than \(|x| \ln |x|\) \((|x| \gg 1)\) and cannot be controlled by the coefficient \( s(x) \). It is also interesting to consider the case when the coefficient \( \rho(x) \) in the leading term of the equation (1.1) tends to zero as \( x \to +\infty \) or \( x \to -\infty \).

If \( \rho(x) = 1 \) and \(|r(x)|\) does not depend on the coefficient \( s(x) \) and grows rapidly, then equation (1.1) has unique solution and an estimate of the maximal regularity for the solution holds (see [14, Theorem 1.1]). If coefficient \( s(x) \) does not have a lower bound and the growth of \(|s(x)|\) depends on the growth of \(|r(x)|\), then equation (1.1) is well-posed ([13, Lemma 2.5]. The above gives that rapid and independent growth of the absolute value of the intermediate coefficient \( r(x) \) have close relationship with the well-posedness of equation (1.1).

In contrast with [13, 14], in the current paper we consider the equation (1.1) with the coefficient \( \rho(x) \) in the leading term. The study of (1.1) is not only of theoretical interest. It is known that an operator \( l \) above arises as generator of the transition semigroup of a stochastic Ornstein-Uhlenbeck process that determines a Brownian motion with a variable covariance matrix connected with \( \rho(x) \). Studying (1.1) with the coefficient \( \rho(x) \), we overcame new difficulties compared to [14], such as the choice and estimation of the linear functional in Theorem 3.1, as well as the construction of the operators \( B_\lambda \) and \( M_\lambda \), and the estimation of the norm \( B_\lambda \) in Theorem 3.4. Furthermore, if \( \rho(x) \) tends to zero at infinity, then we may consider the degeneracy case. For example, by Theorem 3.5, the following equation

\[
- \frac{1}{3 + x^2} \left( \frac{1}{3 + x^2} y' \right)' + (17 + 3x^2)^{10} y' - x^7 y = f(x), \quad x \in \mathbb{R},
\]

is uniquely solvable in \( L_2 \). Under additional conditions on \( \rho(x) \) and \( r(x) \), we can obtain the maximal regularity estimate of the solution \( y \) of the equation (1.1). We remark that the question of maximal regularity is an important tool in the theory of nonlinear PDEs (see, e.g., [1, 5]).

2. One-weighted integral inequality

Let \( g(x) \) and \( h(x) \neq 0 \) be given continuous functions, and \( q = \frac{p}{p-1} \). We denote

\[
\tilde{\alpha}_{g,h}(t) := \|g\|_{L_p(0,t)} \|h^{-1}\|_{L_q(t,+\infty)} \ (t > 0),
\]

\[
\tilde{\beta}_{g,h}(\tau) := \|g\|_{L_p(\tau,0)} \|h^{-1}\|_{L_q(-\infty,\tau)} \ (\tau < 0),
\]

\[
\alpha_{g,h} := \sup_{t > 0} \tilde{\alpha}_{g,h}(t), \quad \beta_{g,h} := \sup_{\tau < 0} \tilde{\beta}_{g,h}(\tau), \quad \gamma_{g,h} := \max(\alpha_{g,h}, \beta_{g,h}).
\]
Lemma 2.1 If \( g(x) \) and \( h(x) \neq 0 \) are continuous functions with \( \gamma_{g,h} < +\infty \), then

\[
\int_{-\infty}^{+\infty} |g(x)y(x)|^p \, dx \leq C \int_{-\infty}^{+\infty} |h(x)y'(x)|^p \, dx, \quad \forall y \in C_0^{(1)}(\mathbb{R}).
\]

Moreover, if \( C \) is the smallest constant for which this inequality holds, then

\[
(\min (\alpha_{g,h}, \beta_{g,h}))^p \leq C \leq \left(p^{\frac{1}{p}} q^{\frac{1}{q}} \gamma_{g,h}\right)^p.
\]

Proof Let \( y \in C_0^{(1)}(\mathbb{R}) \). We define

\[
y_1 := y\chi_{[0, +\infty)}, \quad y_2 := y\chi_{(-\infty, 0]}, \quad g_1 := g\chi_{[0, +\infty)}), \quad h_1 := h\chi_{[0, +\infty)}),
\]

\[
g_2 := g\chi_{(-\infty, 0]}, \quad h_2 := h\chi_{(-\infty, 0]},
\]

where \( \chi_A \) is the characteristic function of the set \( A \). By [11, Theorem 2], we have that

\[
\int_{0}^{+\infty} |g_1(x)y_1(x)|^p \, dx \leq C_1 \int_{0}^{+\infty} |h_1(x)y'_1(x)|^p \, dx,
\]

and if \( C_1 \) is the smallest constant for which this inequality holds, then

\[
(\alpha_{g_1,h_1})^p \leq C_1 \leq \left(p^{\frac{1}{p}} q^{\frac{1}{q}} \alpha_{g_1,h_1}\right)^p.
\]  

Let \( t > 0, \tau = -t \). Then \( \tilde{\alpha}_{g_2(-x), h_2(-x)}(t) = \tilde{\beta}_{g_2(-x), h_2(-x)}(\tau) \leq \gamma_{g,h} < +\infty \). Applying (2.1), we obtain that

\[
\int_{0}^{+\infty} |g_2(-x)y_2(-x)|^p \, dx \leq C_2 \int_{0}^{+\infty} |h_2(-x)y'_2(-x)|^p \, dx,
\]

i.e.

\[
\int_{-\infty}^{0} |g_2(x)y_2(x)|^p \, dx \leq C_2 \int_{-\infty}^{0} |h_2(x)y'_2(x)|^p \, dx.
\]  

If \( C_2 \) is the smallest constant for which this inequality holds, then (2.2) gives that

\[
(\beta_{g_2,h_2})^p \leq C_2 \leq \left(p^{\frac{1}{p}} q^{\frac{1}{q}} \beta_{g_2,h_2}\right)^p.
\]  

Putting (2.1) and (2.3) together, we get

\[
\int_{-\infty}^{+\infty} |g(x)y(x)|^p \, dx \leq C \int_{-\infty}^{+\infty} |h(x)y'(x)|^p \, dx.
\]

From (2.2) and (2.4), it follows that

\[
(\min (\alpha_{g,h}, \beta_{g,h}))^p \leq C \leq \left(p^{\frac{1}{p}} q^{\frac{1}{q}} \gamma_{g,h}\right)^p.
\]

\[\Box\]
3. Main results
We consider the following equation

\[-\rho(x) (\rho(x)y')' + r(x)y' = f(x),\]

(3.1)

where \( f \in L_p \). Let \( l_0 y := -\rho(x) (\rho(x)y')' + r(x)y' \) for \( y \in C_0^{(2)}(\mathbb{R}) \). We denote the closure of the operator \( l_0 \) by \( l \) in the space \( L_p \). By solution of equation (3.1) we mean a function \( y \in D(l) \) such that \( ly = f \).

**Theorem 3.1** Let \( 1 < p < +\infty \), and \( \rho(x) > 0 \) be twice continuously differentiable function, and \( r(x) \geq 1 \) be continuously differentiable function. Suppose that

\[ 1 \leq \frac{r(x)}{\rho^2(x)} \leq C \left( \frac{r(x)}{\rho(x)} \right)^p, \]

(3.2)

\[ \gamma_1, \left( \frac{1}{\rho} \right)^{1/p} < +\infty, \]

(3.3)

and there exists \( \xi \in \mathbb{R} \) such that

\[ \sup_{x < \xi} \left\{ \rho(x) \exp \left( - \int_{\xi}^{x} \frac{r(t)}{\rho^2(t)} dt \right) \right\} < +\infty. \]

(3.4)

Then equation (3.1) has a unique solution \( y \) for any \( f \in L_p \) and the following estimate holds

\[ \left\| \left( \frac{r}{\rho^2} \right)^{1/p} \rho y' \right\|_p + \| y \|_p \leq C \| f \|_p. \]

**Proof** Let \( y \in D(l_0) \). Set \( z(x) := y'(x) \) and \( Lz := l_0 y = -\rho(x)(\rho(x)z)' + r(x)z \). Let \( \gamma \in \mathbb{R}, \gamma > -1 \). Using integration by parts it is easy to verify that

\[ \int_{\mathbb{R}} z(x)((\rho(x)z(x))^2)^{\gamma/2} \, Lz \, dx = \int_{\mathbb{R}} z(x) \left[ (\rho(x)z(x))^2 \right]^{\gamma/2} \left[ -\rho(x)(\rho(x)z(x))' + r(x)z(x) \right] \, dx \]

\[ = \int_{\mathbb{R}} r(x)\rho^\gamma(x) \left[ z^2(x) \right]^{\frac{\gamma+1}{2}} \, dx. \]

(3.5)

Let \( \gamma = p - 2 \). Then \( \gamma + 1 = \frac{q+2}{q} \), where \( q = \frac{p}{p-1} \). Applying Hölder’s inequality and (3.2), we obtain that

\[ \int_{\mathbb{R}} z(x)((\rho(x)z)^2)^{\gamma/2} \, Lz \, dx \leq \left( \int_{\mathbb{R}} r^{-\frac{1}{2}}(x)\rho^\gamma(x) \rho^2 Lz \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}} \left[ r^{\frac{1}{2}}(x)\rho^\gamma(x) |z(x)|^{\frac{q+2}{q}} \right]^q \, dx \right)^{\frac{1}{q}}, \]

i. e.

\[ \int_{\mathbb{R}} z(x)((\rho z)^2)^{\gamma/2} \, Lz \, dx \leq \left\| \left( \frac{r}{\rho^2} \right)^{-\frac{1}{2}} \frac{1}{\rho} Lz \right\|_p \left( \int_{\mathbb{R}} r(x)\rho^\gamma(x) |z(x)|^{\gamma+2} \, dx \right)^{\frac{1}{q}}. \]

(3.6)
From (3.5) and (3.6) it follows that
\[
\left\| \left( \frac{r}{\rho^2} \right)^{\frac{1}{p}} \rho y' \right\|_p \leq \left\| \left( \frac{r}{\rho^2} \right)^{\frac{1}{p}} \frac{1}{\rho} ly \right\|_p.
\] (3.7)

By (3.2),
\[
\frac{1}{\rho(x)} \left[ \frac{r(x)}{\rho^2(x)} \right]^{-\frac{1}{p}} = \left( \frac{r(x)}{\rho^2(x)} \right)^{\frac{1}{p}} \rho(x) \leq C^{1/p},
\]
and so
\[
\left\| \left( \frac{r}{\rho^2} \right)^{\frac{1}{p}} \frac{1}{\rho} ly \right\|_p \leq C^{1/p} \| ly \|_p.
\] (3.8)

Applying (3.3) and Lemma 2.1, we obtain
\[
\| y \|_p \leq C_1 \left\| \left( \frac{r}{\rho^2} \right)^{\frac{1}{p}} \rho y' \right\|_p.
\] (3.9)

Using (3.7), (3.8), and (3.9), we get that
\[
\left\| \left( \frac{r}{\rho^2} \right)^{\frac{1}{p}} \rho y' \right\|_p + \| y \|_p \leq C_2 \| ly \|_p.
\] (3.10)

Now, we suppose that \( y \in D(l) \). Then there exists a sequence \( \{ y_n(x) \}_{n=1}^{\infty} \subset C^0_0(\mathbb{R}) \) such that \( \| y_n - y \|_p \to 0 \) and \( \| ly_n - ly \|_p \to 0 \) (\( n \to +\infty \)). Let \( W^1_p(r, \rho) \) be the completion of the linear space
\[
\left\{ y \in L_p : \| y \|_{W^1_p(r, \rho)} := \left\| \left( \frac{r}{\rho^2} \right)^{\frac{1}{p}} \rho y' \right\|_p + \| y \|_p < +\infty \right\}
\]
with respect to the norm \( \| y \|_{W^1_p(r, \rho)} \). Hence, from (3.10) it follows that
\[
\| y_n \|_{W^1_p(r, \rho)} \leq C_2 \| ly_n \|_p, \quad n = 1, 2, \ldots.
\] (3.11)

(3.11) implies that the sequence \( \{ y_n \}_{n=1}^{\infty} \) is a Cauchy sequence. Since \( W^1_p(r, \rho) \) is the Banach space, we deduce that \( y(x) \in W^1_p(r, \rho) \) and \( \| y_n - y \|_{W^1_p(r, \rho)} \to 0 \) (\( n \to +\infty \)). Letting \( n \to +\infty \) in (3.11), we get
\[
\| y \|_{W^1_p(r, \rho)} \leq C_2 \| ly \|_p,
\]
i.e. (3.10) holds for all \( y \in D(l) \). Hence, the operator \( l \) is invertible and \( R(l) \) is closed.

Next we prove that \( R(l) = L_p \). We denote the adjoint operator of \( l \) by \( l^* \). Let \( R(l) \neq L_p \). Since \( R(l) = N(l^*)^\perp = \{ w(x) \in L_p : \langle w, y^* \rangle = 0 \ \forall y^* \in N(l^*) \} \), there exists a nonzero element \( v(x) \in L_p \setminus R(l) \) such that \( l^* v = \rho(x)(\rho(x)v)' + r(x)v = 0 \) [16, p. 205, Theorem]. Assume that \( x_0 \in \mathbb{R} \) and \( v(x_0) \neq 0 \). Since the last equation is shift invariant, we can choose \( x_0 = \xi \). It is easy to verify that
\[
\rho(x)(\rho(x)v)' + r(x)v = C.
\] (3.12)

1308
We check the conditions of Theorem 3.1. Let \( C \neq 0 \). Since equation (3.12) is linear, without loss of generality, we can assume that \( C = -1 \) and \( v(x_0) > 0 \). Then
\[
\left( \rho(x)v(x) \exp \int_{x_0}^{x} \frac{r(t)}{\rho^2(t)} dt \right)' < 0.
\]
Therefore,
\[
\rho(x) \exp \left( -\int_{x_0}^{x} \frac{r(t)}{\rho^2(t)} dt \right) v(x) > \rho(x_0)v(x_0), \quad x < x_0.
\]
By (3.4), there exists \( K > 0 \) such that
\[
v(x) > \rho(x_0)v(x_0) \frac{1}{K}, \quad x < x_0.
\]
It follows that \( v(x) \notin L_p \).

If \( C = 0 \), then
\[
\rho(x)v(x) \exp \int_{x_0}^{x} \frac{r(t)}{\rho^2(t)} dt = C_1.
\]
It is clear that \( C_1 \neq 0 \). Thus, for \( x < x_0 \),
\[
|v(x)| = \frac{|C_1|}{\rho(x)} \exp \left( -\int_{x_0}^{x} \frac{r(t)}{\rho^2(t)} dt \right) > \frac{|C_1|}{K}.
\]
Hence, for \( x < x_0 \) the inequality \( |v(x)| \geq C_4 > 0 \) holds, consequently \( v \notin L_p \). This is contradiction, so we obtain that \( R(l) = L_p \).

**Example 3.2** Let \( r(x) = (1 + x^2)^{\frac{n}{p}} \) and \( \rho(x) = (1 + x^2)^{\frac{k}{p}} \) in (3.1), \( k, n > 0 \).

We check the conditions of Theorem 3.1. \( \left( \frac{r(x)}{\rho(x)} \right)^{1/p} \rho(x) = (1 + x^2)^{m/2} \) where \( m = \frac{n-2k}{p} + k \). Then
\[
\tilde{\alpha}_1 \left( \frac{r(x)}{\rho(x)} \right)^{1/p} \rho(t) = \tilde{\beta} \left( \frac{r(x)}{\rho(x)} \right)^{1/p} (-t) = t^{1/p} \left( \frac{\int_{t}^{+\infty} \frac{dx}{(1+x^2)^{\frac{2n}{p}}} - \frac{1}{\frac{2n}{p} - \frac{q}{p}}}{\int_{t}^{+\infty} \frac{dx}{(1+x^2)^{\frac{2n}{p} + \frac{q}{p}}}} \right)^{\frac{1}{p}} \leq t^{1/p} \left( \frac{1}{1 + \epsilon} \right)^{\frac{1}{p}} \left( \int_{t}^{+\infty} \frac{dx}{(1+x^2)^{\frac{2n}{p} + \frac{q}{p}}} \right)^{\frac{1}{p}} \leq \frac{1}{(1 + \epsilon) \frac{2n}{p} - \frac{q}{p}},
\]
where \( \epsilon > 0 \). Thus, \( \sup_{t>0} \tilde{\alpha}_1 \left( \frac{r(x)}{\rho(x)} \right)^{1/p} \rho(t) = \sup_{\tau<0} \tilde{\beta} \left( \frac{r(x)}{\rho(x)} \right)^{1/p} (\tau) < +\infty \), if \( \frac{m}{p} - \frac{1}{\frac{2n}{p} - \frac{q}{p}} \leq 0 \). The last inequality is equivalent to \( m \geq 1 + \frac{2n}{q} \). Hence, if \( n > 2k + p - kp \), then (3.3) holds. Also, if \( n \geq 2k \), then (3.2) and (3.4) hold.
Therefore, if \( k \geq 1 \) and \( n \geq 2k \), or \( 0 \leq k < 1 \) and \( n > 2k + p(1 - k) \), then the equation
\[
-(1 + x^2)^k (1 + x^2)^{\frac{k}{2}} y' + (1 + x^2)^{\frac{3k}{2}} y' = f_0(x)
\]
for any \( f_0 \in L_p \) has a unique solution \( y(x) \) and the following estimate holds
\[
\left\| (1 + x^2)^{\frac{n-2k}{2p}} y' \right\|_p + \| y \|_p \leq C \| f_0 \|_p.
\]

**Example 3.3** Consider the following differential equation in \( L_2 \)
\[
-\rho_0(x) (\rho_0(x)y')' + (4x^2 + 3)^2 y' = f(x),
\]
where \( x \in \mathbb{R} \), \( f \in L_2 \), and
\[
\rho_0(x) = \begin{cases} \frac{2x^2-x+1}{(1+x^2)(1-x)}, & -\infty < x < 0, \\ 1 + x^2, & 0 \leq x < +\infty. \end{cases}
\]
The function \( \rho_0(x) \) is twice continuously differentiable, since \( \lim_{x \to 0^-} \rho_0(x) = \lim_{x \to 0^+} \rho_0(x) = \rho_0(0) = 1 \), \( \lim_{x \to 0^-} \rho_0'(x) = \lim_{x \to 0^+} \rho_0'(x) = \rho_0'(0) = 0 \), and \( \lim_{x \to 0^+} \rho_0''(x) = \lim_{x \to 0^-} \rho_0''(x) = \rho_0''(0) = 2 \).

If \( r_0(x) := (4x^2 + 3)^2 \), then in the case \( p = 2 \), \( \rho_0(x) \) and \( r_0(x) \) satisfy (3.2), (3.3), and (3.4). Indeed, we have that
\[
\frac{r_0(x)}{\rho_0'(x)} = \begin{cases} \left( \frac{4x^2+3(1+x^2)(1-x)}{2x^2-x+1} \right)^2, & -\infty < x < 0, \\ \left( \frac{4x^2+3}{1+x^2} \right)^2, & 0 \leq x < +\infty. \end{cases}
\]
Since \( \left( \frac{r_0(x)}{\rho_0'(x)} \right)' < 0 \) for \( x < 0 \) and \( \left( \frac{r_0(x)}{\rho_0'(x)} \right)' > 0 \) for \( x > 0 \), we obtain that \( \frac{r_0(x)}{\rho_0'(x)} \geq \frac{r_0(0)}{\rho_0'(0)} = 9 \) for any \( x \in \mathbb{R} \).
On the other hand, since \( p = 2 \) and \( r_0(x) \geq 3 \), we deduce that (3.2) holds.

Notice that \( \left( \frac{r_0(x)}{\rho_0'(x)} \right)^{1/2} \rho_0(x) = 4x^2 + 3 \) is even function. From
\[
\beta_{1,4x^2+3}(t) \leq \sup_{t > 0} \left( \frac{t}{1+t^2} \right)^{1/2} \left( \int_0^{+\infty} \frac{dx}{x^2 + 1} \right)^{1/2} \leq \frac{1}{\sqrt{2}} \sqrt{\frac{\pi}{2}} < +\infty,
\]
we obtain that (3.3) holds. For any \( \xi < 0 \), one easily checks that
\[
\sup_{x < \xi} \frac{2x^2-x+1}{(1+x^2)(1-x)} \exp \left( -\int_{x}^{\xi} \frac{(4x^2+3)(1+t^2)(1-t)}{2t^2-t+1} \right) dt \leq \frac{3}{2} \sup_{x < \xi} \exp \left( -\int x \right) = \frac{3}{2} \sup_{x < \xi} e^{-\xi + x} = \frac{3}{2},
\]
i.e., (3.4) holds.

Therefore, the equation (3.13) is uniquely solvable and for its solution \( y(x) \), \( \|(4x^2 + 3)^2 y'\|_2 + \| y \|_2 \leq C \| f \|_2 \) holds.
Theorem 3.4 Let $1 < p < +\infty$, $\rho(x) > 0$ be twice continuously differentiable function, and $r(x) \geq 1$ be continuously differentiable function. Suppose that (3.2), (3.3), and (3.4) hold. If $\theta r^{\frac{1}{r-1}}(x) \leq \rho(x) < +\infty$ for $p > 2$ and $\theta > 0$, and there exist $C > 1$ and $C_1 > 1$ such that

$$C^{-1} \leq \frac{\rho(x)}{\rho(\nu)} \leq C, \quad C_1^{-1} \leq \frac{r(x)}{r(\nu)} \leq C_1, \quad x, \nu \in \mathbb{R} : |x - \nu| \leq 1,$$

then for the solution $y(x)$ of the equation (3.1) the following estimate holds

$$\|\rho(y')'\|_p + \|ry'\|_p \leq C_0 \|f\|_p.$$  \hfill (3.15)

Proof Let $\lambda \geq 0$. First we consider the differential operator $l_{\lambda, 0} = -\rho(x)(\rho(x)y')' + [r(x) + \lambda]y'$ with $D(l_{\lambda, 0}) = C_0^2(\mathbb{R})$. Let $z = y'$ and $l_{\lambda, 0} z = -\rho(x)(\rho(x)z)' + [r(x) + \lambda]z$. We denote the closure in $L_p$ of the operator $l_{\lambda, 0}$ by $L_{\lambda}$. By Theorem 3.1 $L_{\lambda}$ is invertible, and its inverse $L_{\lambda}^{-1}$ is defined on whole $L_p$. The following inequality holds for $z \in D(L_{\lambda})$ (see (3.7)):

$$\left\| \left( \frac{r + \lambda}{\rho^2} \right)^{\frac{1}{2}} \rho z \right\|_p \leq \left\| \left( \frac{r + \lambda}{\rho^2} \right)^{-\frac{1}{2}} \frac{1}{\rho} L_{\lambda} z \right\|_p,$$  \hfill (3.16)

and by (3.8)

$$\|z\|_p \leq C_1 \|L_{\lambda} z\|_p.$$  \hfill (3.17)

Let $\Delta_j := (j - 1, j + 1)$ ($j \in \mathbb{Z}$), and $\{\varphi_j(x)\}_{j=-\infty}^{+\infty}$ be a sequence in $C_0^\infty(\Delta_j)$ such that

$$0 \leq \varphi_j(x) \leq 1 \quad (j \in \mathbb{Z}), \quad \sum_{j=-\infty}^{+\infty} \varphi_j^2(x) = 1.$$  \hfill (3.18)

We denote $z_j(x) = \varphi_j(x)z(x)$ and $\|\cdot\|_{p, \Delta_j} = \|\cdot\|_{L_p(\Delta_j)}$, where $j \in \mathbb{Z}$. By (3.16) and (3.14), we have that

$$\|L_{\lambda} z_j\|_{p, \Delta_j} \geq \inf_{x \in \Delta_j} \left[ \left( \frac{r(x) + \lambda}{\rho^2(x)} \right)^{\frac{1}{2}} \rho(x) \right] \|z_j\|_{p, \Delta_j},$$

$$\geq C_2 \sup_{x \in \Delta_j} \left( \frac{r(x) + \lambda}{\rho^2(x)} \right)^{\frac{1}{2}} \rho(x) \|z_j\|_{p, \Delta_j} = C_2 \sup_{x \in \Delta_j} (r_j(x) + \lambda) \|z_j\|_p.$$  \hfill (3.19)

For any $f \in L_p$, we define

$$B_{\lambda} f := - \sum_{j=-\infty}^{+\infty} \rho^2(x) \varphi_j'(x) L_{\lambda}^{-1} \varphi_j f, \quad M_{\lambda} f := \sum_{j=-\infty}^{+\infty} \varphi_j(x) L_{\lambda}^{-1} \varphi_j f.$$  

It is clear that for any $x \in \mathbb{R}$, the above two series have at most two nonzero terms. Hence, $B_{\lambda}$ and $M_{\lambda}$ are well-defined.
We consider the operator $L_\lambda M_\lambda$. Using the equality $L_\lambda(gz) = g(x)L_\lambda z - \rho^2(x)g'(x)z$ and the properties (3.18) of $\varphi_j(x)$ ($j \in \mathbb{Z}$), we get

$$L_\lambda M_\lambda f = \sum_{j=-\infty}^{+\infty} L_\lambda(\varphi_j L^{-1}_\lambda \varphi_j f) = \sum_{j=-\infty}^{+\infty} (\varphi_j(x)L_\lambda L^{-1}_\lambda \varphi_j f - \rho^2(x)\varphi'_j(x)L^{-1}_\lambda \varphi_j f) = (E + B_\lambda)f,$$

where $E$ is the identity map on $L_p$. So,

$$L_\lambda M_\lambda = E + B_\lambda. \quad (3.20)$$

On the other hand,

$$\|B_\lambda f\|_p^p = \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} \rho^2(x)\varphi'_j(x)L^{-1}_\lambda \varphi_j f \right|^p dx$$

$$\leq \sum_{j=-\infty}^{+\infty} \int \left| \int_{-\infty}^{+\infty} \rho^2(x)\varphi'_j(x)L^{-1}_\lambda \varphi_j f \right|^p dx$$

$$\leq \sum_{k=-\infty}^{+\infty} \int \left| \int_{-\infty}^{+\infty} \rho^2(x)\varphi'_k(x)L^{-1}_\lambda \varphi_k f \right|^p dx$$

$$= \sum_{k=-\infty}^{+\infty} \int \left| \int_{-\infty}^{+\infty} \rho^2(x)\varphi'_k(x)L^{-1}_\lambda \varphi_k f + \rho^2(x)\varphi'_k(x)L^{-1}_\lambda \varphi_k f \right|^p dx$$

$$\leq C_3 \sum_{k=-\infty}^{+\infty} \int \left| \rho^2(x)\varphi'_k(x)L^{-1}_\lambda \varphi_k f \right|^p dx.$$

Using (3.19), we obtain that

$$\|\rho^2 \varphi'_k L^{-1}_\lambda \varphi_k f\|_{L^p,\Delta_k} \leq \frac{C_4 \sup_{x \in \Delta_k} \rho^2(x)}{\inf_{x \in \Delta_k} (r(x) + \lambda)} \|\varphi_k f\|_{L^p,\Delta_k} \leq \frac{C_5}{1 + \lambda} \|\varphi_k f\|_{L^p,\Delta_k}. \quad (3.21)$$

By properties of the function $\varphi_k(x)$ ($k \in \mathbb{Z}$), it follows that

$$\sum_{k=-\infty}^{+\infty} \int \left| \rho^2(x)\varphi'_k(x)L^{-1}_\lambda \varphi_k f \right|^p dx \leq \frac{C_6}{1 + \lambda} \|f\|_p^p.$$

Thus, $\|B_\lambda\| \to 0$ as $\lambda \to \infty$. Therefore, there exists $\lambda_0 > 0$ such that $\|B_\lambda\| \leq \frac{1}{2}$ for $\lambda \geq \lambda_0$. Hence, from (3.20) it follows that

$$L^{-1}_\lambda = M_\lambda(E + B_\lambda)^{-1}, \quad \|(E + B_\lambda)^{-1}\| \leq 2, \quad \lambda \geq \lambda_0. \quad (3.22)$$
Now, we prove the estimate (3.15). According to (3.19) and (3.22), we get
\[
\|(r + \lambda)z\|_p^p = \|(r + \lambda)L^{-1}_x f\|_p^p \\
\leq 2 \|(r + \lambda)M_\lambda f\|_p^p \\
\leq C_7 \sum_{j=\infty}^{+\infty} \|(r + \lambda)L^{-1}_x \varphi_j f\|_{p,\Delta_j}^p \\
\leq C_7 \sum_{j=\infty}^{+\infty} \sup_{x \in \Delta_j} (r(x) + \lambda)^p \|L^{-1}_x \varphi_j f\|_{p,\Delta_j}^p \\
\leq C_7 \sum_{j=\infty}^{+\infty} \sup_{x \in \Delta_j} (r(x) + \lambda)^p \frac{1}{\inf_{x \in \Delta_j} (r(x) + \lambda)^p} \|\varphi_j f\|_{p,\Delta_j}^p \\
\leq C_8 \sum_{j=\infty}^{+\infty} \|\varphi_j f\|_{p,\Delta_j}^p \\
\leq C_9 \|f\|_p^p, \quad \forall z \in D(L_\lambda).
\]

Therefore,
\[
\|\rho(\rho z)'\|_p + \|(r + \lambda)z\|_p \leq C_{10} \|f\|_p.
\]

Taking \(z(x) = y'(x)\), by estimate (3.17), we obtain (3.15). \(\square\)

For the equation (1.1), we have the following result.

**Theorem 3.5** Let \(1 < p < \infty\), and functions \(\rho(x)\) and \(r(x)\) satisfy the conditions of Theorem 3.4. If \(s(x)\) is a continuous function such that \(\gamma_{s,r} < +\infty\), then for any \(F \in L_p\), the equation (1.1) has a unique solution \(y(x)\), which satisfies the following inequality:
\[
\|\rho(\rho y)'\|_p + \|ry'\|_p + \|sy\|_p \leq C \|F\|_p, \quad (3.23)
\]
where \(C\) depends only on \(\gamma_{s,r}\) and \(p\).

**Proof**

For \(a > 0\), we denote
\[
\tilde{y}(t) = y(at), \quad \tilde{\rho}(t) = \rho(at), \quad \tilde{r}(t) = r(at), \quad \tilde{s}(t) = s(at), \quad \tilde{f}(t) = a^{-1}F(at).
\]

The change of variable \(x \mapsto at\) changes the equation (1.1) to the following:
\[
-\tilde{\rho}(t)(\tilde{\rho}(t)\tilde{y})' + \tilde{r}(t)\tilde{y}' + a^{-1}\tilde{s}(t)\tilde{y} = \tilde{f}(t). \quad (3.24)
\]

We denote the closure in \(L_p\) of the differential operator \(-\tilde{\rho}(t)(\tilde{\rho}(t)\tilde{y})' + \tilde{r}(t)\tilde{y}\) defined on \(\tilde{y} \in C^2_0(\mathbb{R})\) by \(\lambda_a\). It is easy to show that \(\tilde{\rho}(t)\) and \(\tilde{r}(t)\) satisfy the conditions of Theorem 3.4. Therefore,
\[
\|\tilde{\rho}(\tilde{\rho}y)'\|_p + \|\tilde{r}y'\|_p \leq C_{1a} \|\lambda_a \tilde{y}\|_p, \quad \tilde{y} \in D(\lambda_a). \quad (3.25)
\]

According to Lemma 2.1,
\[
\|a^{-1}\tilde{s}\tilde{y}\|_p \leq a^{-1}p^{\frac{1}{q}}q^{\frac{1}{q}}\gamma_{s,r} \|\tilde{r}\tilde{y}'\|_p \leq a^{-1}p^{\frac{1}{q}}q^{\frac{1}{q}}\gamma_{s,r} C_{1a} \|\lambda_a \tilde{y}\|_p. \quad (3.26)
\]

1313
Set \( a = \frac{1}{2} p^{\frac{3}{2}} q^{\frac{1}{2}} \gamma_{s,r} C_{l_a} \). By [9, Ch. 4, Theorem 1.16], we get that the operator \( l_a + a^{-1} \hat{s}(x) E \) is bounded invertible. Using (3.24) and (3.25), we obtain that

\[
\| -\hat{p}(\hat{p} y')' \|_p + \| \hat{p} y' \|_p + \| a^{-1} \hat{s} \|_p \leq \left( C_{l_a} + \frac{1}{2} \right) \| l_a \hat{y} \|_p.
\]

It follows from (3.26) that

\[
\| l_a \hat{y} \|_p \leq \| (l_a + a^{-1} \hat{s} E) \hat{y} \|_p + \| a^{-1} \hat{s} \|_p \leq \| (l_a + a^{-1} \hat{s} E) \hat{y} \|_p + \frac{1}{2} \| l_a \hat{y} \|_p.
\]

The above two inequalities imply that

\[
\| -\hat{p}(\hat{p} y')' \|_p + \| \hat{p} y' \|_p + \| a^{-1} \hat{s} \|_p \leq C \| \hat{y} \|_p.
\]

Making the change of variable \( t \mapsto a^{-1} x \), we get the desired estimate (3.23). \( \square \)

**Example 3.6** We consider the following equation in \( L_2 \)

\[
-\frac{x^2 + 5}{(x^2 + 3)^2} \left( \frac{x^2 + 5}{(x^2 + 3)^2} y' \right)' + (x^2 + 3)^2 y' - 5xy = f(x). \tag{3.27}
\]

Let \( r_1(x) = (x^2 + 3)^2 \), \( \rho_1(x) = \frac{x^2 + 5}{(x^2 + 3)^2} \) and \( s_1(x) = -5x \).

1) \( \frac{r_1(x)}{\rho_1(x)} = \frac{(x^2 + 3)^6}{(x^2 + 3)^2} \geq 1 \), \( r_1(x) \geq 1 \) and \( p = 2 \). Hence, (3.2) holds.

2) Since \( r_1(x) = x^2 + 3 \) is even function, and

\[
\sup_{\tau < 0} \beta_{1,r_1}(\tau) = \sup_{t > 0} \left( \frac{t}{3 + t^2} \right) \left( \int_0^t \frac{dx}{x^2 + 3} \right)^{\frac{1}{2}} \leq \frac{\sqrt{\pi}}{2} < +\infty,
\]

(3.3) holds.

3) We have that

\[
\sup_{x < \xi} \rho_1(x) \exp \left( -\int_x^\xi \frac{r_1(x) dt}{\rho_1^2(x)} \right) = \sup_{x < \xi} \frac{x^2 + 5}{x^2 + 3} \exp \left( -\int_x^\xi \frac{(t^2 + 3)^6}{(t^2 + 5)^2} dt \right) \leq 2 \sup_{x < \xi} \left( -\int_x^\xi \frac{dt}{x^2 + 3} \right) = 2.
\]

Therefore, (3.4) holds.

4) If \( |x - \nu| < 1 \), then

\[
\frac{r_1(x)}{r_1(\nu)} = \left[ \frac{x^2 + 3}{\nu^2 + 3} \right]^2 \leq \sup \left( \frac{(\nu + 1)^2 + 3}{\nu^2 + 3} \right)^2 \leq 3.
\]

Consequently, \( \frac{r_1(\nu)}{r_1(x)} \geq \frac{1}{2} \). Furthermore, since

\[
\frac{\rho_1(x)}{\rho_1(\nu)} = \frac{1 + \frac{2}{\nu + 3}}{1 + \frac{2}{x + 3}} \frac{\frac{x + 3}{\nu + 3}}{\frac{x + 3}{x + 3}}.
\]
we have that
\[
\frac{\nu^2 + 3}{2(x^2 + 3)} \leq \frac{\rho_1(x)}{\rho_1(\nu)} \leq \frac{2(\nu^2 + 3)}{x^2 + 3}.
\]

It follows that
\[
\frac{1}{6} \leq \frac{\rho_1(x)}{\rho_1(\nu)} \leq 6.
\]

Thus, (3.14) holds.

5) It is clear that \( \rho_1(x) \) is a bounded function. We calculate \( \gamma_{s_1, r_1} \). Notice that \( \| -5x \|_{L_2(0,t)} = \| -5x \|_{L_2(-t,0)} \) for \( t > 0 \), and \( (x^2 + 3)^2 \) is even function. Hence,
\[
\sup_{\tau < 0} \tilde{\beta}_{s_1, r_1}(\tau) = \sup_{t > 0} \tilde{\alpha}_{s_1, r_1}(t) = \sup_{t > 0} \left( \int_{0}^{t} (-5x)^2 \, dx \right)^{\frac{1}{2}} \left( \int_{t}^{+\infty} \frac{dx}{(x^2 + 3)^{\tau}} \right)^{\frac{1}{2}} \\
\leq 5 \sup_{t > 0} t^{\frac{1}{2}} \left( \int_{t}^{+\infty} \frac{dx}{(x^2 + 3)^{\tau}} \right)^{\frac{1}{2}} \\
\leq 5 \sup_{t > 0} \frac{t^{\frac{1}{2}}}{(1+t^2)^{\frac{1}{2}}} \left( \int_{t}^{+\infty} \frac{dx}{1+x^2} \right)^{\frac{1}{2}} \leq 5 \sqrt{\frac{\pi}{2}},
\]
i.e., \( \gamma_{s_1, r_1} \leq 5 \sqrt{\frac{\pi}{2}} \).

Thus, the coefficients of equation (3.27) satisfy the conditions of Theorem 3.5. Hence, for each \( f \in L_2 \) there exists a unique solution \( y \) of equation (3.27), which satisfies the following:
\[
\left\| -\frac{x^2 + 5}{x^2 + 3} \left( \frac{(x^2 + 5)}{(x^2 + 3)^2} y' \right)' \right\|_2 + \left\| (x^2 + 3)^2 y' \right\|_2 + \| 5xy \|_2 \leq C \| f \|_2.
\]

Acknowledgment

The authors would like to express their sincere gratitude to the anonymous referee for his/her helpful comments that helped to improve the quality of the manuscript. The first author is supported by project AP05131649 of the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan and by the L.N. Gumilyov Eurasian National University Research Fund.

References


