A metric invariant of Möbius transformations

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Abstract: The complex unit disk $D = \{z \in \mathbb{C}: |z| < 1\}$ is endowed with Möbius addition $\oplus_M$ defined by

$$w \oplus_M z = \frac{w + z}{1 + wz}.$$ 

We prove that the metric $d_T$ defined on $D$ by $d_T(w, z) = \tan^{-1}|w \oplus_M z|$ is an invariant of Möbius transformations carrying $D$ onto itself. We also prove that $(D, d_T)$ and $(D, d_P)$, where $d_P$ denotes the Poincaré metric, have the same isometry group and then classify the isometries of $(D, d_T)$.

Key words: Möbius transformation, Poincaré metric, transformation invariant, isometry group, gyrogroup

1. Introduction

Recall that the Poincaré disk model (also called the conformal disk model) consists of the (open) complex unit disk,

$$D = \{z \in \mathbb{C}: |z| < 1\}, \quad (1.1)$$

naturally associated with the Poincaré metric. From a complex-analysis point of view, the Poincaré metric is the most natural metric on the complex unit disk (cf. [13, p. 53]). The right transformations of the Poincaré disk model are the Möbius transformations that carry $D$ onto itself (also called conformal self-maps of $D$). Several characterizations of Möbius transformations are obtained; see, for instance, [3–7, 9–12, 14–16, 21].

From an algebraic point of view, the complex unit disk has a nonassociative group-like structure when it is equipped with Möbius addition $\oplus_M$ defined by the equation

$$w \oplus_M z = \frac{w + z}{1 + wz} \quad (1.2)$$

for all $w, z \in D$ [20]. More precisely, $(D, \oplus_M)$ satisfies the following properties:

I. (identity) The zero number $0$ satisfies $0 \oplus_M z = z = z \oplus_M 0$ for all $z \in D$. 

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II. (Inverse) For each \(z \in \mathbb{D}\), the negative of \(z\) belongs to \(\mathbb{D}\) and satisfies
\[-z \oplus_M z = 0 = z \oplus_M -z.\]

III. (The Gyroassociative Law) For all \(a, b \in \mathbb{D}\), define a map \(\text{gyr}[a, b]\) by
\[
\text{gyr}[a, b]z = \frac{1 + ab}{1 + \overline{a}b}z
\]
for all \(z \in \mathbb{D}\). Then \(\text{gyr}[a, b]\) is an algebraic automorphism of \(\mathbb{D}\) with respect to Möbius addition for all \(a, b \in \mathbb{D}\). Furthermore,
\[
(a \oplus_M b) \oplus_M c = a \oplus_M (b \oplus_M c)\]
(1.4)
for all \(a, b, c \in \mathbb{D}\).

IV. (The Loop Property) For all \(a, b \in \mathbb{D}\),
\[
\text{gyr}[a \oplus_M b, b] = \text{gyr}[a, b]\quad \text{and} \quad \text{gyr}[a, a \oplus_M b] = \text{gyr}[a, b].
\]

V. (The Gyrocommutative Law) For all \(a, b \in \mathbb{D}\),
\[
a \oplus_M b = \text{gyr}[a, b](b \oplus_M a).
\]
(1.6)

We remark that Möbius addition is not associative so that \((\mathbb{D}, \oplus_M)\) does not form a group. However, it shares common properties with groups. The space \((\mathbb{D}, \oplus_M)\) is known as a gyrogroup, the term coined by Ungar [19]. The map \(\text{gyr}[a, b]\) in item III is called a gyroautomorphism due to the fact that it represents a rotation of the complex unit disk and preserves Möbius addition. Equation (1.4) resembles the associative law in groups, called the gyroassociative law, and Equation (1.6) resembles the commutative law in abelian groups, called the gyrocommutative law. From now on, the space \((\mathbb{D}, \oplus_M)\) is referred to as the (complex) Möbius gyrogroup.

The next theorem collects some algebraic identities of the Möbius gyrogroup extended from group-theoretic identities.

**Theorem 1.1** (See [17, 19].) The following properties are true in \((\mathbb{D}, \oplus_M)\):

1. \(-a \oplus_M (a \oplus_M b) = b;\) \texttt{(LEFT CANCELLATION LAW)}
2. \(-(a \oplus_M b) = \text{gyr}[a, b](-b \oplus_M -a);\)
3. \((-a \oplus_M b) \oplus_M \text{gyr}[a, b](-b \oplus_M c) = -a \oplus_M c;\)
4. \(\text{gyr}[-a, -b] = \text{gyr}[a, b];\) \texttt{(EVEN PROPERTY)}
5. \(\text{gyr}[a, b] = \text{gyr}^{-1}[a, b], \text{where} \text{gyr}^{-1}[a, b] \text{denotes the inverse of gyr}[a, b] with respect to composition of functions;}\) \texttt{(INVERSIVE SYMMETRY)}
6. \(L_a: z \mapsto a \oplus_M z \text{ is a bijective self-map of } \mathbb{D} \text{ and } L_a^{-1} = L_{-a}.\)

Theorem 1.1 will prove useful in studying the geometric structure of the complex unit disk in Sections 2 and 3. The map \(L_a\) described in Theorem 1.1(6) is in fact a Möbius transformation of \(\mathbb{D}\), which is called a hyperbolic translation or a left gyrotranslation as it resembles a Euclidean translation.
2. The metric induced by Möbius addition

In this section, we prove that the complex unit disk possesses another metric, denoted by \( d_T \) (here, \( "T" \) stands for \( \arctan^{-1} \)). We then establish that the Möbius transformations of \( D \) preserve \( d_T \). Hence, \( d_T \) becomes a numerical invariant of Möbius transformations of \( D \). We also describe a few fundamental classes of isometries of \( (D, d_T) \) as well as some geometry of \( (D, d_T) \). Let us begin with the fact that the Cauchy–Schwarz inequality in \( \mathbb{R}^2 \) may be expressed via complex multiplication and conjugation.

**Lemma 2.1** For all \( w, z \in D \),

\[
-2|w||z| \leq w\overline{z} + \overline{w}z \leq 2|w||z|. \tag{2.1}
\]

**Proof** Let \( w = a + bi \) and \( z = c + di \), where \( a, b, c, d \in \mathbb{R} \). Then

\[
\frac{w\overline{z} + \overline{w}z}{2} = ac + bd = \langle w, z \rangle, \quad \text{where } w = (a, b) \text{ and } z = (c, d).
\]

According to the Cauchy–Schwarz inequality in \( \mathbb{R}^2 \), we obtain \( |\langle w, z \rangle| \leq ||w|| ||z|| = |w||z| \), which implies \( -|w||z| \leq \langle w, z \rangle \leq |w||z| \) and so (2.1) holds. \( \square \)

**Theorem 2.2** The inequality

\[
\frac{|w| - |z|}{1 + |w||z|} \leq |w \oplus_M z| \leq \frac{|w| + |z|}{1 - |w||z|} \tag{2.2}
\]

holds in the complex unit disk.

**Proof** Let \( w, z \in D \). Using the triangle inequality and the reverse triangle inequality, we obtain

\[
(|w| - |z|)^2 \leq |w + z|^2 \leq (|w| + |z|)^2.
\]

Note that \( |1 + \overline{w}z|^2 = (1 + \overline{w}z)(1 + \overline{w}z)^\ast = 1 + w\overline{z} + \overline{w}z + |w|^2 |z|^2 \), which implies \( (1 - |w||z|)^2 \leq |1 + \overline{w}z|^2 \leq (1 + |w||z|)^2 \) by Lemma 2.1. It follows that

\[
|w \oplus_M z|^2 = \frac{|w + z|^2}{1 + \overline{w}z} \leq \frac{(|w| + |z|)^2}{(1 - |w||z|)^2}
\]

and so \( |w \oplus_M z| \leq \frac{|w| + |z|}{1 - |w||z|} \). Similarly, we have

\[
|w \oplus_M z|^2 = \frac{|w + z|^2}{1 + \overline{w}z} \geq \frac{(|w| - |z|)^2}{(1 + |w||z|)^2}
\]

and hence \( |w \oplus_M z| \geq \frac{|w| - |z|}{1 + |w||z|} \). \( \square \)

Define a function \( |\cdot|_T \) by

\[
|z|_T = \arctan^{-1} |z| \tag{2.3}
\]

for all \( z \in D \). The next theorem summarizes elementary properties of \( |\cdot|_T \).

**Theorem 2.3** \( |\cdot|_T \) satisfies the following properties:

1. \( |z|_T \geq 0 \) and \( |z|_T = 0 \) if and only if \( z = 0 \)

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2. \(| - z|_T = |z|_T\)
3. \(|w|_T - |z|_T \leq |w \oplus_M z|_T \leq |w|_T + |z|_T\)
4. \(|\text{gyr}[a, b]z|_T = |z|_T\)

for all \(a, b, w, z \in \mathbb{D}\).

**Proof** Item (1) follows from the fact that \(\tan^{-1}\) is increasing and injective. Item (2) is clear. To prove item (3), set \(r = |w|_T = \tan^{-1}|w|\) and \(s = |z|_T = \tan^{-1}|z|\). By Theorem 2.2,

\[
\frac{|w| - |z|}{1 + |w||z|} \leq |w \oplus_M z| \leq \frac{|w| + |z|}{1 - |w||z|}
\]

and so \(\tan(r - s) \leq |w \oplus_M z| \leq \tan(r + s)\). Since \(\tan^{-1}\) is increasing, it follows that

\[r - s \leq \tan^{-1}|w \oplus_M z| \leq r + s,
\]
as claimed. Item (4) follows from the fact that \(\text{gyr}[a, b]\) preserves the complex modulus. \(\square\)

In view of Theorem 2.3, one can define a new metric on the complex unit disk using \(\cdot|_T\). In fact, define \(d_T\) by

\[d_T(w, z) = |-w \oplus_M z|_T\]  \hspace{1cm} (2.4)

for all \(w, z \in \mathbb{D}\). We remark that \(d_T\) is a bounded metric on \(\mathbb{D}\), whereas the Poincaré metric is unbounded.

**Theorem 2.4** \(d_T\) is a bounded metric on the complex unit disk.

**Proof** By Theorem 2.3(1), \(d_T(w, z) \geq 0\). Furthermore, \(d_T(w, z) = 0\) if and only if \(-w \oplus_M z = 0\) if and only if \(w = z\). Let \(x, y, z \in \mathbb{D}\). Using some properties of Möbius addition mentioned in Theorem 1.1, we obtain

\[| - y \oplus_M x|_T = | - (-y \oplus_M x)|_T = |\text{gyr}[-y, x][-x \oplus_M y]|_T = | - x \oplus_M y|_T.
\]

Hence, \(d_T(y, x) = d_T(x, y)\). Furthermore, direct computation shows that

\[d_T(x, z) = | - x \oplus_M z|_T
\]

\[= |(-x \oplus_M y) \oplus_M \text{gyr}[-x, y][-y \oplus_M z]|_T
\]

\[\leq | - x \oplus_M y|_T + |\text{gyr}[-x, y][-y \oplus_M z]|_T
\]

\[= | - x \oplus_M y|_T + | - y \oplus_M z|_T
\]

\[= d_T(x, y) + d_T(y, z).
\]

This proves that \(d_T\) defines a metric on \(\mathbb{D}\).

Note that \(d_T(0, z) = |z|_T = \tan^{-1}|z| < \tan^{-1}1 = \frac{\pi}{4}\) for all \(z \in \mathbb{D}\). Hence,

\[d_T(w, z) \leq d_T(w, 0) + d_T(0, z) < \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}
\]

for all \(w, z \in \mathbb{D}\). \(\square\)
There are two important classes of isometries of $\mathbb{D}$ with respect to $d_T$: (i) modulus-preserving automorphisms of $(\mathbb{D}, \oplus_M)$ and (ii) hyperbolic translations; see Theorems 2.5 and 2.6. This allows us to prove that $(\mathbb{D}, d_T)$ is a homogeneous space and to construct point-reflections of $(\mathbb{D}, d_T)$.

**Theorem 2.5** If $\tau \in \text{Aut}(\mathbb{D}, \oplus_M)$ and $\tau$ preserves the complex modulus, then $\tau$ is an isometry of $(\mathbb{D}, d_T)$. In particular, every rotation of the complex unit disk is an isometry of $(\mathbb{D}, d_T)$.

**Proof** Let $\tau \in \text{Aut}(\mathbb{D}, \oplus_M)$ and suppose that $\tau$ preserves $| \cdot |$. Then

$$d_T(\tau(w), \tau(z)) = -|\tau(w) \oplus_M \tau(z)|_T = |\tau(-w \oplus_M z)|_T = |w - w \oplus_M z|_T = d_T(w, z)$$

for all $w, z \in \mathbb{D}$.

Let $\rho$ be a rotation of $\mathbb{D}$; that is, $\rho(z) = \omega z$ for all $z \in \mathbb{D}$, where $\omega$ is a unimodular complex number. Clearly, $\rho$ preserves $| \cdot |$ and is bijective. Next, we prove that $\rho$ preserves Möbius addition. Let $w, z \in \mathbb{D}$.

Direct computation shows that $\rho(w) \oplus_M \rho(z) = \frac{\omega (w + z)}{1 + |\omega|^2} = \omega \frac{w + z}{1 + \overline{w}z} = \rho(w \oplus_M z)$. It follows that $\rho$ is an isometry of $(\mathbb{D}, d_T)$.

**Theorem 2.6** The hyperbolic translation $L_w : z \mapsto w \oplus_M z$ is an isometry of $(\mathbb{D}, d_T)$ for all $w \in \mathbb{D}$.

**Proof** Let $w \in \mathbb{D}$. The hyperbolic translation $L_w$ is bijective since $L_w \circ L_{-w}$ and $L_{-w} \circ L_w$ are the identity transformation of $\mathbb{D}$ by the left cancellation law. Using some properties of Möbius addition mentioned in Theorem 1.1, we obtain

$$d_T(L_w(a), L_w(b)) = d_T(a, b).$$

Hence, $d_T(L_w(a), L_w(b)) = d_T(a, b)$.

**Theorem 2.7** (Homogeneity) For each pair of points $w$ and $z$ in $\mathbb{D}$, there is an isometry $\psi$ of $(\mathbb{D}, d_T)$ such that $\psi(w) = z$. In particular, $(\mathbb{D}, d_T)$ is homogeneous.

**Proof** Let $w, z \in \mathbb{D}$. Define $\psi = L_z \circ L_{-w}$. Then $\psi$ is an isometry of $\mathbb{D}$, being the composite of isometries of $\mathbb{D}$. Clearly, $\psi(w) = z \oplus_M (-w \oplus_M w) = z$.

With the aid of Möbius addition, a point-reflection symmetry of $\mathbb{D}$ is easy to construct, as shown in the following theorem.

**Theorem 2.8** (Symmetry) For each $z \in \mathbb{D}$, there is a point-reflection symmetry $\sigma_z$ of $(\mathbb{D}, d_T)$; that is, $\sigma_z$ is an isometry of $(\mathbb{D}, d_T)$ such that $\sigma_z^2$ is the identity transformation of $\mathbb{D}$ and $z$ is the unique fixed point of $\sigma_z$. 2880
Proof Let $\iota$ be the negative map of $\mathbb{D}$; that is, $\iota(z) = -z$ for all $z \in \mathbb{D}$. By Theorem 2.5, $\iota$ is an isometry of $\mathbb{D}$. Define $\sigma_z = L_z \circ \iota \circ L_z^{-1}$. Then $\sigma_z$ is a nonidentity isometry of $\mathbb{D}$, being the composite of isometries of $\mathbb{D}$. Since $L_z^{-1} = L_z^* = 1$, it follows that $\sigma_z^2 = (L_z \circ \iota \circ L_z^{-1}) \circ (L_z \circ \iota \circ L_z^{-1}) = L_z \circ \iota^2 \circ L_z^{-1} = L_z \circ L_z^* = I$, where $I$ is the identity transformation of $\mathbb{D}$. By construction, $z$ is a fixed point of $\sigma_z$. Suppose that $w$ is a fixed point of $\sigma_z$; that is, $\sigma_z(w) = w$. It follows that $z \oplus_M \iota(-z \oplus_M w) = w$ and hence $\iota(-z \oplus_M w) = -z \oplus_M w$. Since $0$ is the unique fixed point of $\iota$, we obtain that $-z \oplus_M w = 0$ and so $w = z$. \hfill \Box

3. Connections between the Poincaré metric and $d_T$-metric

In this section, we indicate some fruitful connections between the Poincaré metric, the metric $d_T$, and Möbius transformations.

3.1. The metric structure of the complex unit disk

Recall that the Poincaré metric on $\mathbb{D}$ is given by

$$d_P(w,z) = 2 \tanh^{-1} \left| \frac{z-w}{1-\overline{w}z} \right| = 2 \tanh^{-1} |w \oplus_M z|$$

(3.1)

and the pseudo-hyperbolic distance $\delta_M$ (also called the Möbius gyrometric) is given by

$$\delta_M(w,z) = \left| \frac{z-w}{1-\overline{w}z} \right| = |w \oplus_M z|$$

(3.2)

for all $w, z \in \mathbb{D}$. The pseudo-hyperbolic distance $\delta_M$ is indeed a metric on $\mathbb{D}$ [2, Theorem 2.4]. Although the metric $d_T$ is quite different from the Poincaré metric, they are topologically equivalent. To see this, note that $d_T(w,z) \leq d_P(w,z)$ for all $w, z \in \mathbb{D}$ since $f(x) = 2 \tanh^{-1} x - \tan^{-1} x$ defines a strictly increasing function on the open interval $(0,1)$. Hence, the topology induced by $d_P$ is finer than the topology induced by $d_T$. If $w \in \mathbb{D}$ and if $\epsilon > 0$, then set $\delta = \tanh^{-1} \left( \frac{\epsilon}{2} \right)$ so that $B_{d_T}(w, \delta) \subseteq B_{d_P}(w, \epsilon)$. This proves that the topology induced by $d_T$ is finer than the topology induced by $d_P$ and so they coincide. The topology of $\mathbb{D}$ induced by $d_P$ is in fact the usual Euclidean metric topology; see, for instance, Proposition 3 on p. 48 of [13].

It is well known that the metric space $(\mathbb{D}, d_P)$ is complete; see, for instance, Proposition 4 on p. 49 of [13]. Hence, every Cauchy sequence in $(\mathbb{D}, d_P)$ converges. This leads to a natural question of determining whether $(\mathbb{D}, d_T)$ is complete since $(\mathbb{D}, d_P)$ and $(\mathbb{D}, d_T)$ are topologically equivalent. We begin with the fact that $(\mathbb{D}, d_T)$ is not compact.

Let $B_P(z, \epsilon)$ be the open ball of $(\mathbb{D}, d_P)$ of radius $\epsilon$ centered at $z$ and let $B_T(z, \epsilon)$ be the open ball of $(\mathbb{D}, d_T)$ of radius $\epsilon$ centered at $z$. For all $\epsilon > 0$, note that

$$d_P(z,w) < \epsilon \quad \text{if and only if} \quad d_T(z,w) < \tanh^{-1} \left( \frac{\epsilon}{2} \right) .$$

(3.3)

This shows that $B_P(z, \epsilon) = B_T(z, \epsilon')$, where $\epsilon' = \tanh^{-1} \left( \frac{\epsilon}{2} \right)$, for all $\epsilon > 0$. This also implies that $B_T(z, \epsilon) = B_P(z, \epsilon')$, where $\epsilon' = 2 \tanh^{-1} (\tan \epsilon)$, for all $\epsilon$ with $0 < \epsilon < \frac{\pi}{4}$ because the domain of $\tanh^{-1}$ is $(-1,1)$.
**Theorem 3.1** \((\mathbb{D}, d_T)\) is not compact.

**Proof** Define \(V = \left\{ B_T \left(0, \frac{\pi}{4} - \frac{1}{n}\right) : n \in \mathbb{N} \text{ and } n \geq 2 \right\}\). Then \(V\) is an open cover of \((\mathbb{D}, d_T)\) without finite subcollections covering \(\mathbb{D}\). In fact, suppose that

\[
U = \left\{ B_T \left(0, \frac{\pi}{4} - \frac{1}{n_1}\right), B_T \left(0, \frac{\pi}{4} - \frac{1}{n_2}\right), \ldots, B_T \left(0, \frac{\pi}{4} - \frac{1}{n_r}\right) \right\}
\]

is a finite subcollection of \(V\). Set \(m = \max\{n_1, n_2, \ldots, n_r\}\). Then

\[
\bigcup_{i=1}^{r} B_T \left(0, \frac{\pi}{4} - \frac{1}{n_i}\right) = B_T \left(0, \frac{\pi}{4} - \frac{1}{m}\right) \subset \mathbb{D}.
\]

Therefore, \((\mathbb{D}, d_T)\) is not compact. \(\square\)

A similar argument used in the proof of Theorem 3.1 shows that \((\mathbb{D}, d_P)\) is not compact. In fact, \(V = \{B_P(0,n) : n \in \mathbb{N}\}\) is an open cover of \((\mathbb{D}, d_P)\) without finite subcollections covering \(\mathbb{D}\) since

\[
\bigcup_{i=1}^{r} B_P(0,n_i) = B_P(0,m) \subset \mathbb{D},
\]

where \(m = \max\{n_1, n_2, \ldots, n_r\}\). Since \((\mathbb{D}, d_P)\) is complete, it follows that \((\mathbb{D}, d_P)\) is not totally bounded as a metric space is compact if and only if it is complete and totally bounded. This in turn implies that \((\mathbb{D}, d_T)\) is not totally bounded, as shown in the next theorem.

**Theorem 3.2** \((\mathbb{D}, d_T)\) is not totally bounded.

**Proof** Suppose to the contrary that \((\mathbb{D}, d_T)\) is totally bounded. Let \(\epsilon > 0\). For \(\epsilon' = \tan^{-1} \left( \tanh \frac{\epsilon}{2} \right)\), there exist open balls \(B_T(z_1, \epsilon'), B_T(z_2, \epsilon'), \ldots, B_T(z_k, \epsilon')\) of \((\mathbb{D}, d_T)\) such that \(\bigcup_{i=1}^{k} B_T(z_i, \epsilon') = \mathbb{D}\). As proved above, \(B_T(z_i, \epsilon') = B_P(z_i, \epsilon)\) for all \(i = 1, 2, \ldots, k\). Hence, \(\bigcup_{i=1}^{k} B_P(z_i, \epsilon) = \mathbb{D}\) and so \((\mathbb{D}, d_P)\) is totally bounded, a contradiction. \(\square\)

In view of Theorems 3.1 and 3.2, the problem of determining whether \((\mathbb{D}, d_T)\) is complete is not immediate. However, this is indeed the case, as shown in the following theorem.

**Theorem 3.3** \((\mathbb{D}, d_T)\) is a complete metric space.

**Proof** Let \((x_n)\) be a Cauchy sequence in \((\mathbb{D}, d_T)\). We first prove that \((x_n)\) is also a Cauchy sequence in \((\mathbb{D}, d_P)\). Let \(\epsilon > 0\). For \(\epsilon' = \tan^{-1} \left( \tanh \frac{\epsilon}{2} \right) > 0\), there is a positive integer \(N\) such that \(d_T(x_m, x_k) < \epsilon'\) for all \(m, k \geq N\). By (3.3), \(d_P(x_m, x_k) < \epsilon\) for all \(m, k \geq N\). Hence, \((x_n)\) is a Cauchy sequence with respect to \(d_P\). As \((\mathbb{D}, d_P)\) is complete, \((x_n)\) converges in \((\mathbb{D}, d_P)\), namely that \(x_n \to x\) and \(x \in \mathbb{D}\).
We claim that \((x_n)\) converges to the same point \(x\) in \((\mathbb{D}, d_T)\). Let \(\epsilon > 0\). Set \(m = \min\{\epsilon, \pi/8\}\) and \(M = \min\{\tan m, 1/2\}\). Then \(M \in (0,1)\) and we can let \(\epsilon' = 2 \tan^{-1} M\). Then \(\epsilon' > 0\) and so there is a positive integer \(N\) such that \(d_T(x_k, x) < \epsilon'\) for all \(k \geq N\). By (3.3),

\[
d_T(x_k, x) < \tan^{-1}\left(\frac{\epsilon'}{2}\right) = \tan^{-1} M \leq \tan^{-1} (\tan m) = m \leq \epsilon
\]

for all \(k \geq N\). This proves the claim.

\[\square\]

3.2. The isometry groups of \((\mathbb{D}, d_P), (\mathbb{D}, \delta_M)\) and \((\mathbb{D}, d_T)\)

The next theorem shows that \((\mathbb{D}, d_T), (\mathbb{D}, \delta_M),\) and \((\mathbb{D}, d_P)\) have the same isometry group.

**Theorem 3.4** Let \(\psi\) be a self-map of \(\mathbb{D}\). Then the following statements are equivalent:

1. \(\psi\) preserves \(d_T\);
2. \(\psi\) preserves \(\delta_M\);
3. \(\psi\) preserves \(d_P\).

Therefore, \(\text{Iso}(\mathbb{D}, d_T) = \text{Iso}(\mathbb{D}, \delta_M) = \text{Iso}(\mathbb{D}, d_P)\), where \(\text{Iso}(\mathbb{D}, d_T), \text{Iso}(\mathbb{D}, \delta_M),\) and \(\text{Iso}(\mathbb{D}, d_P)\) are the isometry groups of \((\mathbb{D}, d_T), (\mathbb{D}, \delta_M),\) and \((\mathbb{D}, d_P)\), respectively.

**Proof** The theorem follows from the fact that

\[
d_T(\psi(w), \psi(z)) = d_T(w, z) \iff |\psi(w) \oplus_M \psi(z)| = |w \oplus_M z| \iff d_P(\psi(w), \psi(z)) = d_P(w, z).
\]

for all \(w, z \in \mathbb{D}\).

Let \(\mathcal{R}\) be the group of rotations of \(\mathbb{D}\); that is, \(\mathcal{R}\) consists precisely of transformations of the form \(z \mapsto \omega z\), \(z \in \mathbb{D}\), where \(\omega\) is a unimodular complex number. Also, define

\[
\mathcal{N} = \{\eta: \eta\ is a bijective self-map of \mathbb{D}\ and\ |\eta(z)| = |z|\ for\ all\ z \in \mathbb{D}\}\}. \quad (3.4)
\]

It is clear that \(\mathcal{N}\) forms a group under composition and that \(\mathcal{R}\) is a subgroup of \(\mathcal{N}\). Furthermore, \(\mathcal{R}\) is proper in \(\mathcal{N}\) as shown in (3.6). Let \(\mathcal{M}(\mathbb{D})\) be the group of Möbius transformations carrying \(\mathbb{D}\) onto itself; that is, \(\mathcal{M}(\mathbb{D})\) consists precisely of holomorphic functions from \(\mathbb{D}\) to itself that are injective and surjective. By Theorem 6.2.3 of [8], \(\tau\) is a Möbius transformation of \(\mathbb{D}\) if and only if \(\tau = \rho \circ \varphi_a\), where \(\rho\) is a rotation in \(\mathcal{R}\), \(a \in \mathbb{D}\), and \(\varphi_a\) is given by

\[
\varphi_a(z) = \frac{z - a}{1 - \overline{a} z}, \quad z \in \mathbb{D}.
\]

Clearly, \(\varphi_a = L_{-a}\). By Equation (55) of [17], \(\rho \circ L_{-a} = L_{\rho(-a)} \circ \rho\). Moreover, if \(w \in \mathbb{D}\) and \(\rho \in \mathcal{R}\), then \(L_w \circ \rho = \rho \circ L_{\rho^{-1}(w)} = \rho \circ \varphi_{-\rho^{-1}(w)}\). This proves that

\[
\mathcal{M}(\mathbb{D}) = \{L_w \circ \rho: w \in \mathbb{D} and \rho \in \mathcal{R}\}. \quad (3.5)
\]

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Lemma 3.5 \textit{If }$\psi$\textit{ is an isometry of }$\mathbb{D},d_T$\textit{ that leaves }$0$\textit{ fixed, then }$\psi$\textit{ preserves the complex modulus.}

Proof \textit{Let }$z \in \mathbb{D}$. Then
\[
|\psi(z)|_T = | - \psi(0) \oplus_M \psi(z)|_T = d_T(\psi(0),\psi(z)) = d_T(0,z) = |z|_T.
\]
Since $\tan^{-1}$ is injective, it follows that $|\psi(z)| = |z|$. \hfill $\Box$

We remark that there is a transformation of $\mathbb{D}$ that preserves the complex modulus but is not additive:

\[
f(z) = \begin{cases} 
    z & \text{if } |z| \neq \frac{1}{2} \\
    -z & \text{if } |z| = \frac{1}{2}
\end{cases}
\]  \hfill (3.6)

for all $z \in \mathbb{D}$. Moreover, $f$ is not a rotation of $\mathbb{D}$. The following theorem shows that the Möbius transformations of $\mathbb{D}$ are indeed isometries of $(\mathbb{D},d_T)$. However, there exists an isometry of $(\mathbb{D},d_T)$ that is not Möbius; see Example 3.9.

Theorem 3.6 \textit{The following inclusions hold:}

$$\mathcal{M}(\mathbb{D}) \subseteq \text{Iso}(\mathbb{D},d_T) \subseteq \{ L_w \circ \eta : w \in \mathbb{D} \text{ and } \eta \in \mathcal{N} \}. $$

Proof By Theorems 2.5 and 2.6, $\mathcal{M}(\mathbb{D}) \subseteq \text{Iso}(\mathbb{D},d_T)$. Let $\psi \in \text{Iso}(\mathbb{D},d_T)$. Then $\psi$ is a bijective self-map of $\mathbb{D}$. By Theorem 11 of [18], $\psi = L_{\psi(0)} \circ \eta$, where $\eta$ is a bijective self-map of $\mathbb{D}$ fixing $0$. Since $\eta = L_{\psi(0)}^{-1} \circ \psi = L_{-\psi(0)} \circ \psi$, it follows that $\eta$ is an isometry of $(\mathbb{D},d_T)$. By Lemma 3.5, $\eta$ preserves the complex modulus and so $\eta \in \mathcal{N}$. Hence, $\psi \in \{ L_w \circ \eta : w \in \mathbb{D} \text{ and } \eta \in \mathcal{N} \}$. This proves the second inclusion. \hfill $\Box$

Corollary 3.7 \textit{If a self-map }$\sigma$\textit{ of }$\mathbb{D}$\textit{ does not preserve the metric }$d_T$\textit{, then }$\sigma$\textit{ is not Möbius.}

Example 3.8 \textit{The function }$f$\textit{ defined by (3.6) preserves the complex modulus but does not define an isometry of }$(\mathbb{D},d_T)$. \textit{In fact, if }$w = \frac{1}{4}$\textit{ and }$z = \frac{1}{2}$\textit{, then }$| - f(w) \oplus_M f(z)| = \frac{6}{7}$\textit{, whereas }$| - w \oplus_M z| = \frac{2}{7}$. \textit{This shows that }$\text{Iso}(\mathbb{D},d_T)$\textit{ is proper in }$\{ L_w \circ \eta : w \in \mathbb{D} \text{ and } \eta \in \mathcal{N} \}$. From Example 3.8, we have seen that a modulus-preserving bijection need not be an isometry of $\mathbb{D}$ with respect to $d_T$. However, a modulus-preserving bijection that respects Möbius addition must be an isometry by Theorem 2.5.

Example 3.9 \textit{The complex conjugation }$\kappa: z \mapsto \bar{z}$\textit{, }$z \in \mathbb{D}$\textit{, is an isometry of }$(\mathbb{D},d_T)$\textit{ but not a Möbius transformation of }$\mathbb{D}$. \textit{In fact, }$\kappa$\textit{ is an automorphism of }$(\mathbb{D},\oplus_M)$\textit{ that preserves the complex modulus and hence becomes an isometry of }$\mathbb{D}$\textit{ with respect to }$d_T$\textit{ and }$d_P$. \textit{However, }$\kappa$\textit{ is not Möbius for it is not holomorphic. This shows that }$\mathcal{M}(\mathbb{D})$\textit{ is proper in }$\text{Iso}(\mathbb{D},d_T)$.\hfill $\Box$

Example 3.9 motivates the following question:

Under what conditions is a $d_T$-preserving self-map of $\mathbb{D}$ a Möbius transformation?
According to the Schwarz–Pick lemma (see, for instance, Theorem 4 on p. 15 of [13]), an obvious answer of the previous question is the condition of “being holomorphic”. We remark that this condition is very strong.

**Theorem 3.10** If \( \psi \) is a \( d_T \)-preserving self-map of \( \mathbb{D} \) that is holomorphic, then \( \psi \) is a Möbius transformation of \( \mathbb{D} \).

**Proof** Let \( w, z \in \mathbb{D} \) with \( w \neq z \). By assumption, \( d_T(\psi(w), \psi(z)) = d_T(w, z) \), which implies

\[
\frac{\psi(z) - \psi(w)}{1 - \overline{\psi(w)} \psi(z)} = \frac{z - w}{1 - wz}
\]

since \( \tan^{-1} \) is injective. By the Schwarz–Pick lemma, \( \psi \) is Möbius. \( \square \)

In order to describe the exact isometry group of \((\mathbb{D}, d_T)\), we need a Euclidean version of Möbius addition. Under the identification \( z = u + v, \) we obtain a complete description of the isometry group of \((\mathbb{D}, d_T)\) using Abe’s result [1] as follows. Define

\[
O(\mathbb{D}) = \{ \Phi|_{\mathbb{D}} : \Phi \text{ is an orthogonal transformation of } \mathbb{R}^2 \},
\]

where \( \Phi|_{\mathbb{D}} \) denotes the restriction of \( \Phi \) to \( \mathbb{D} \).

**Theorem 3.11** The isometry group of \((\mathbb{D}, d_T)\) is given by

\[
\text{Iso}(\mathbb{D}, d_T) = \{ L_w \circ \varphi : w \in \mathbb{D} \text{ and } \varphi \in O(\mathbb{D}) \}.
\]

**Proof** Note that every transformation in \( O(\mathbb{D}) \) is a bijective self-map of \( \mathbb{D} \) since \( \mathbb{D} \) is invariant under the orthogonal transformations of \( \mathbb{R}^2 \). Using (3.7), we obtain that if \( \varphi \in O(\mathbb{D}) \), then \( \varphi \) is an isometry of \((\mathbb{D}, d_T)\). Hence, \( \varphi \) preserves both Möbius addition and the Euclidean inner product. This combined with Theorem 2.6 implies that

\[
\{ L_w \circ \varphi : w \in \mathbb{D} \text{ and } \varphi \in O(\mathbb{D}) \} \subseteq \text{Iso}(\mathbb{D}, d_T).
\]

To prove the reverse inclusion, let \( \tau \in \text{Iso}(\mathbb{D}, d_T) \). By Theorem 11 of [18], \( \tau = L_{\varphi(0)} \circ \varphi \), where \( \varphi \) is a bijective self-map of \( \mathbb{D} \) fixing 0. Thus, \( \varphi = L_{\tau(0)} \circ \tau \) preserves the metric \( d_T \) as well as the Möbius gyrometric \( \delta_M \). By Theorem 3.2 of [1], \( \varphi = \Phi|_{\mathbb{D}} \), where \( \Phi \) is an orthogonal transformation of \( \mathbb{R}^2 \). Hence, \( \varphi \in O(\mathbb{D}) \) and the proof completes. \( \square \)

It is a standard result in linear algebra that any orthogonal transformation of \( \mathbb{R}^2 \) is either a rotation about the origin (which is Möbius) or a reflection about a line passing through the origin (which is not Möbius). Theorem 3.11 shows that any isometry of \((\mathbb{D}, d_T)\) is of the form \( \psi = L_w \circ \varphi \), where \( w \in \mathbb{D} \) and \( \varphi \) is the restriction of an orthogonal transformation of \( \mathbb{R}^2 \) to \( \mathbb{D} \). Furthermore, \( \psi \) is Möbius if and only if \( \varphi \) is a rotation about the origin.
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