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Trigonometric expressions for infinite series involving binomial coefficients

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Abstract: By means of the hypergeometric series approach, we present a new proof of Sun's conjecture on trigonometric series, which is simpler than the original one due to Sun and Meng. Several further infinite series identities are shown as examples.

Key words: Binomial series, hypergeometric series, trigonometric functions

1. Introduction and motivation

For an integer n and an indeterminate x, define the rising and falling factorials, respectively, by the following quotients of Euler's Γ-function:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \quad \text{and} \quad \langle x\rangle_n = \frac{\Gamma(1+x)}{\Gamma(1+x-n)},$$

where for the former we shall utilize the abbreviated multiparameter notation below:

$$\begin{bmatrix} A, & B, & \cdots, & C \\ \alpha, & \beta, & \cdots, & \gamma \end{bmatrix}_n = \frac{(A)_n(B)_n\cdots(C)_n}{(\alpha)_n(\beta)_n\cdots(\gamma)_n}.$$

According to Bailey [1, §2.1], the classical hypergeometric series reads as

$${}_{1+p}F_p\left[\begin{matrix} a_0, a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_p \end{matrix} \,\middle|\, z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k(a_2)_k\cdots(a_p)_k}{k!(b_1)_k(b_2)_k\cdots(b_p)_k} z^k.$$

By introducing the integer sequence

$$S_n = \frac{\binom{6n}{3n}\binom{3n}{n}}{2(2n+1)\binom{2n}{n}},$$

Sun [6] proposed the following conjecture.

Conjecture 1 *There are positive integers $T_1, T_2, T_3...$ such that*

$$\frac{1}{24} - \sum_{k=1}^{\infty} T_k x^{2k} + \sum_{k=0}^{\infty} S_k x^{2k+1} = \frac{\cos\left(\frac{2}{3}\arccos\left(6\sqrt{3}x\right)\right)}{12}$$

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for all real x with $|x| \leq 1/6\sqrt{3}$. Also, $T_p \equiv -2 \pmod{p}$ for any prime p.

Sun and Meng [5] gave an analytic proof for this conjecture. In this paper, we shall present a simpler proof by employing the following well-known hypergeometric series formulae (see, for example, Chu [3, Eqs. 8 and 10] and Chu and Zheng [4, Eqs. 1.3a and 1.3b]):

$$ {}_2F_1\left[\begin{matrix}\frac{x}{2}, -\frac{x}{2}\\ \frac{1}{2}\end{matrix}\,\middle|\, y^2\right] = \cos\left(x \arcsin y\right), \tag{1.1}$$

$$ {}_2F_1\left[\begin{matrix}\frac{1+x}{2}, \frac{1-x}{2}\\ \frac{3}{2}\end{matrix}\,\middle|\, y^2\right] = \frac{\sin\left(x \arcsin y\right)}{xy}. \tag{1.2}$$

The rest of this short paper will be organized as follows. In the next section, a new proof of Conjecture 1 will be given. Then we shall derive, in the third section, further similar results by the hypergeometric series approach.

2. Proof of the conjecture

As a warm-up, we first give a simpler proof of Sun's conjecture 1 by means of the hypergeometric series approach. According to the relations

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ and } (a)_{kn} = k^{kn}\left(\frac{a}{k}\right)_n\left(\frac{a+1}{k}\right)_n \cdots \left(\frac{a+k-1}{k}\right)_n$$

the sequence S_n can be rewritten in the following form:

$$S_n = \frac{108^n}{2}\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}\\ 1, \frac{3}{2}\end{matrix}\right]_n.$$

Then we can evaluate, by (1.2), the following series:

$$\sum_{k=0}^{\infty} S_k x^{2k+1} = \frac{x}{2} \cdot {}_2F_1\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}\\ \frac{3}{2}\end{matrix}\,\middle|\, (6\sqrt{3}x)^2\right] = \frac{\sin\left(\frac{2}{3}\arcsin 6\sqrt{3}x\right)}{8\sqrt{3}}.$$

For $x \in [-1, 1]$, recall the trigonometric identity

$$\arcsin x + \arccos x = \frac{\pi}{2}. \tag{2.1}$$

We can reformulate the difference

$$\begin{aligned}
&\frac{\cos\left(\frac{2}{3}\arccos 6\sqrt{3}x\right)}{12} - \sum_{k=0}^{\infty} S_k x^{2k+1}\\
=\ &\frac{\cos\left(\frac{\pi}{3} - \frac{2}{3}\arcsin 6\sqrt{3}x\right)}{12} - \frac{\sin\left(\frac{2}{3}\arcsin 6\sqrt{3}x\right)}{8\sqrt{3}}\\
=\ &\frac{1}{12}\left\{\cos\left(\frac{\pi}{3} - \frac{2}{3}\arcsin 6\sqrt{3}x\right) - \sin\frac{\pi}{3}\sin\left(\frac{2}{3}\arcsin 6\sqrt{3}x\right)\right\}\\
=\ &\frac{1}{12}\cos\frac{\pi}{3}\cos\left(\frac{2}{3}\arcsin 6\sqrt{3}x\right) = \frac{1}{24}\cos\left(\frac{2}{3}\arcsin 6\sqrt{3}x\right),
\end{aligned}$$

where we have utilized the trigonometric identity

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$$

In view of (1.1), we can expand the last trigonometric function into the Maclaurin series

$$\begin{aligned}\frac{1}{24}\cos\left(\frac{2}{3}\arcsin 6\sqrt{3}x\right)&=\frac{1}{24}{}_2F_1\left[\begin{matrix}\frac{1}{3},-\frac{1}{3}\\ \frac{1}{2}\end{matrix}\,\middle|\,(6\sqrt{3}x)^2\right]\\ &=\sum_{k=0}^{\infty}\frac{16^k}{24(1-3k)}\binom{3k}{k}x^{2k}.\end{aligned}$$

Therefore, we find that the $\{T_k\}$ sequence in Conjecture 1 is explicitly given by

$$T_k=\frac{16^k}{24(3k-1)}\binom{3k}{k}\quad\text{for}\quad k=1,2,\cdots.$$

Observing that

$$T_k=\frac{2}{2+3(k-1)}\binom{2+3(k-1)}{k-1}16^{k-1}\quad\text{for}\quad k=1,2,\cdots,$$

we confirm that $\{T_k\}$ are integers because for $\lambda=1,2,\cdots$ and $n=0,1,2,\cdots$, the ternary Catalan numbers $\left\{\frac{\lambda}{\lambda+3n}\binom{\lambda+3n}{n}\right\}$ are integers (cf. Chu [2]). In addition, we have, for any prime p, the following congruence:

$$\begin{aligned}T_p=&\frac{2\times16^{p-1}}{3p-1}\binom{3p-1}{p-1}=2\times16^{p-1}\prod_{k=2}^{p-1}\left(\frac{3p}{k}-1\right)\\ =&2\times16^{p-1}(-1)^p\pmod p=-2\pmod p\end{aligned}$$

, thanks to Fermat's little theorem. This completes the proof of Conjecture 1, which is simpler than that due to Sun and Meng [5].

Analogously, by examining the trigonometric relation

$$\frac{\cos(\frac{2}{3}\arcsin 6\sqrt{3}x)}{24}-\frac{\sin(\frac{2}{3}\arcsin 6\sqrt{3}x)}{8\sqrt{3}}=\frac{\cos(\frac{\pi}{3}+\frac{2}{3}\arcsin 6\sqrt{3}x)}{12},$$

we can show the following counterpart result.

Theorem 2 *Under the same condition of Conjecture 1, the following identity holds:*

$$\frac{1}{24}-\sum_{k=1}^{\infty}T_kx^{2k}-\sum_{k=0}^{\infty}S_kx^{2k+1}=\frac{\cos(\frac{\pi}{3}+\frac{2}{3}\arcsin(6\sqrt{3}x))}{12}.$$

3. Further infinite series identities

Based on the hypergeometric series (1.1) and (1.2), we can derive further identities similar to those displayed in Conjecture 1 and Theorem 2. Six examples will be illustrated in this section.

Letting $x = \frac{1}{3}$ in (1.1) and (1.2), we have

$$ {}_2F_1\left[\begin{matrix} \frac{2}{3}, \frac{1}{3} \\ \frac{3}{2} \end{matrix} \,\Big|\, y^2\right] = \frac{3}{y}\sin\left(\frac{1}{3}\arcsin y\right), $$

$$ {}_2F_1\left[\begin{matrix} \frac{1}{6}, -\frac{1}{6} \\ \frac{1}{2} \end{matrix} \,\Big|\, y^2\right] = \cos\left(\frac{1}{3}\arcsin y\right). $$

Then for the two sequences defined by

$$ S_n^{(1)} = \left[\begin{matrix} \frac{2}{3}, \frac{1}{3} \\ 1, \frac{3}{2} \end{matrix}\right]_n = \left(\frac{4}{27}\right)^n \frac{\binom{3n}{n}}{1+2n}, $$

$$ T_n^{(1)} = \left[\begin{matrix} \frac{1}{6}, -\frac{1}{6} \\ 1, \frac{1}{2} \end{matrix}\right]_n = \frac{1}{108^n(1-6n)} \frac{\binom{6n}{3n}\binom{3n}{n}}{\binom{2n}{n}}, $$

we get the following two infinite series identities.

Example 1 *For $|y| \leq 1$, there hold the identities*

$$ \frac{1}{3}\sum_{n=0}^{\infty} S_n^{(1)} y^{2n+1} + \sum_{n=0}^{\infty} T_n^{(1)} y^{2n} = \sqrt{2}\sin\left(\frac{1}{3}\arcsin y + \frac{\pi}{4}\right), $$

$$ \frac{1}{3}\sum_{n=0}^{\infty} S_n^{(1)} y^{2n+1} - \sum_{n=0}^{\infty} T_n^{(1)} y^{2n} = \sqrt{2}\sin\left(\frac{1}{3}\arcsin y - \frac{\pi}{4}\right). $$

Analogously, letting $x = \frac{1}{2}$ in (1.1) and (1.2), we get

$$ {}_2F_1\left[\begin{matrix} \frac{3}{4}, \frac{1}{4} \\ \frac{3}{2} \end{matrix} \,\Big|\, y^2\right] = \frac{2}{y}\sin\left(\frac{1}{2}\arcsin y\right), $$

$$ {}_2F_1\left[\begin{matrix} \frac{1}{4}, -\frac{1}{4} \\ \frac{1}{2} \end{matrix} \,\Big|\, y^2\right] = \cos\left(\frac{1}{2}\arcsin y\right). $$

Then for the two sequences defined by

$$ S_n^{(2)} = \left[\begin{matrix} \frac{3}{4}, \frac{1}{4} \\ 1, \frac{3}{2} \end{matrix}\right]_n = \left(\frac{1}{16}\right)^n \frac{\binom{4n}{2n}}{1+2n}, $$

$$ T_n^{(2)} = \left[\begin{matrix} \frac{1}{4}, -\frac{1}{4} \\ 1, \frac{1}{2} \end{matrix}\right]_n = \left(\frac{1}{16}\right)^n \frac{\binom{4n}{2n}}{1-4n}, $$

we find the following two infinite series identities.

Example 2 *For $|y| \leq 1$, there hold the identities*

$$\frac{1}{2}\sum_{n=0}^{\infty} S_n^{(2)} y^{2n+1} + \sum_{n=0}^{\infty} T_n^{(2)} y^{2n} = \sqrt{2}\cos\big(\frac{1}{2}\arccos y\big),$$

$$\frac{1}{2}\sum_{n=0}^{\infty} S_n^{(2)} y^{2n+1} - \sum_{n=0}^{\infty} T_n^{(2)} y^{2n} = -\sqrt{2}\sin\big(\frac{1}{2}\arccos y\big).$$

There exist two companion series (cf. Chu [3, Eqs. 9 and 11] and Chu and Zheng [4, Eqs. 2.1 and 3.2]) similar to (1.1) and (1.2):

$${}_2F_1\left[\begin{matrix}1+x,\ 1-x\\ \frac{3}{2}\end{matrix}\ \Big|\ y^2\right] = \frac{\sin(2x\arcsin y)}{2xy\sqrt{1-y^2}}, \tag{3.1}$$

$${}_2F_1\left[\begin{matrix}\frac{1}{2}+x,\ \frac{1}{2}-x\\ \frac{1}{2}\end{matrix}\ \Big|\ y^2\right] = \frac{\cos(2x\arcsin y)}{\sqrt{1-y^2}}. \tag{3.2}$$

They are utilized below to establish four pairs of infinite series identities.

First, by letting $x = \frac{1}{3}$ in (3.1) and (3.2),

$${}_2F_1\left[\begin{matrix}\frac{4}{3},\ \frac{2}{3}\\ \frac{3}{2}\end{matrix}\ \Big|\ y^2\right] = \frac{3\sin(\frac{2}{3}\arcsin y)}{2y\sqrt{1-y^2}},$$

$${}_2F_1\left[\begin{matrix}\frac{5}{6},\ \frac{1}{6}\\ \frac{1}{2}\end{matrix}\ \Big|\ y^2\right] = \frac{\cos(\frac{2}{3}\arcsin y)}{\sqrt{1-y^2}},$$

and then defining the sequences

$$S_n^{(3)} = \left[\begin{matrix}\frac{4}{3},\ \frac{2}{3}\\ 1,\ \frac{3}{2}\end{matrix}\right]_n = \big(\frac{4}{27}\big)^n\binom{3n+1}{n},$$

$$T_n^{(3)} = \left[\begin{matrix}\frac{5}{6},\ \frac{1}{6}\\ 1,\ \frac{1}{2}\end{matrix}\right]_n = \frac{1}{108^n}\frac{\binom{6n}{3n}\binom{3n}{n}}{\binom{2n}{n}},$$

we derive the following two infinite series identities.

Example 3 *For $|y| \leq 1$, there hold the identities*

$$\frac{2}{3}\sum_{n=0}^{\infty} S_n^{(3)} y^{2n+1} + \sum_{n=0}^{\infty} T_n^{(3)} y^{2n} = \sqrt{\frac{2}{1-y^2}}\sin\big(\frac{2}{3}\arcsin y + \frac{\pi}{4}\big),$$

$$\frac{2}{3}\sum_{n=0}^{\infty} S_n^{(3)} y^{2n+1} - \sum_{n=0}^{\infty} T_n^{(3)} y^{2n} = \sqrt{\frac{2}{1-y^2}}\sin\big(\frac{2}{3}\arcsin y - \frac{\pi}{4}\big).$$

Secondly, by letting $x = \frac{2}{3}$ in (3.1) and (3.2),

$$ {}_2F_1\left[\begin{matrix} \frac{5}{3}, \frac{1}{3} \\ \frac{3}{2} \end{matrix} \middle| y^2\right] = \frac{3\sin(\frac{4}{3}\arcsin y)}{4y\sqrt{1-y^2}}, $$

$$ {}_2F_1\left[\begin{matrix} \frac{7}{6}, -\frac{1}{6} \\ \frac{1}{2} \end{matrix} \middle| y^2\right] = \frac{\cos(\frac{4}{3}\arcsin y)}{\sqrt{1-y^2}}, $$

and then defining the sequences

$$ S_n^{(4)} = \left[\begin{matrix} \frac{5}{3}, \frac{1}{3} \\ 1, \frac{3}{2} \end{matrix}\right]_n = (\frac{4}{27})^n \frac{3n+2}{4n+1}\binom{3n}{n}, $$

$$ T_n^{(4)} = \left[\begin{matrix} \frac{7}{6}, -\frac{1}{6} \\ 1, \frac{1}{2} \end{matrix}\right]_n = \frac{1}{108^n}\frac{6n+1}{1-6n}\frac{\binom{6n}{3n}\binom{3n}{n}}{\binom{2n}{n}}, $$

we get the following two infinite series identities.

Example 4 *For* $|y| \le 1$, *there hold the identities*

$$ \frac{4}{3}\sum_{n=0}^{\infty} S_n^{(4)} y^{2n+1} + \sum_{n=0}^{\infty} T_n^{(4)} y^{2n} = \sqrt{\frac{2}{1-y^2}}\sin(\frac{4}{3}\arcsin y + \frac{\pi}{4}), $$

$$ \frac{4}{3}\sum_{n=0}^{\infty} S_n^{(4)} y^{2n+1} - \sum_{n=0}^{\infty} T_n^{(4)} y^{2n} = \sqrt{\frac{2}{1-y^2}}\sin(\frac{4}{3}\arcsin y - \frac{\pi}{4}). $$

Thirdly, by letting $x = \frac{1}{4}$ in (3.1) and (3.2),

$$ {}_2F_1\left[\begin{matrix} \frac{5}{4}, \frac{3}{4} \\ \frac{3}{2} \end{matrix} \middle| y^2\right] = \frac{2\sin(\frac{1}{2}\arcsin y)}{y\sqrt{1-y^2}}, $$

$$ {}_2F_1\left[\begin{matrix} \frac{3}{4}, \frac{1}{4} \\ \frac{1}{2} \end{matrix} \middle| y^2\right] = \frac{\cos(\frac{1}{2}\arcsin y)}{\sqrt{1-y^2}}, $$

and then defining the sequences

$$ S_n^{(5)} = \left[\begin{matrix} \frac{5}{4}, \frac{3}{4} \\ 1, \frac{3}{2} \end{matrix}\right]_n = \frac{1}{16^n}\binom{4n+1}{2n}, $$

$$ T_n^{(5)} = \left[\begin{matrix} \frac{3}{4}, \frac{1}{4} \\ 1, \frac{1}{2} \end{matrix}\right]_n = \frac{1}{16^n}\binom{4n}{2n}, $$

we have the following two infinite series identities.

Example 5 *For $|y| \leq 1$, there hold the identities*

$$\frac{1}{2}\sum_{n=0}^{\infty} S_n^{(5)} y^{2n+1} + \sum_{n=0}^{\infty} T_n^{(5)} y^{2n} = \sqrt{\frac{2}{1-y^2}}\cos\big(\frac{1}{2}\arccos y\big),$$

$$\frac{1}{2}\sum_{n=0}^{\infty} S_n^{(5)} y^{2n+1} - \sum_{n=0}^{\infty} T_n^{(5)} y^{2n} = -\sqrt{\frac{2}{1-y^2}}\sin\big(\frac{1}{2}\arccos y\big).$$

Finally, by letting $x = \frac{3}{4}$ in (3.1) and (3.2),

$${}_2F_1\left[\begin{matrix}\frac{7}{4}, \frac{1}{4} \\ \frac{3}{2}\end{matrix} \,\Big|\, y^2\right] = \frac{2\sin\big(\frac{3}{2}\arcsin y\big)}{3y\sqrt{1-y^2}},$$

$${}_2F_1\left[\begin{matrix}\frac{5}{4}, -\frac{1}{4} \\ \frac{1}{2}\end{matrix} \,\Big|\, y^2\right] = \frac{\cos\big(\frac{3}{2}\arcsin y\big)}{\sqrt{1-y^2}},$$

and then defining the sequences

$$S_n^{(6)} = \left[\begin{matrix}\frac{7}{4}, \frac{1}{4} \\ 1, \frac{3}{2}\end{matrix}\right]_n = \big(\frac{1}{16}\big)^n \frac{n+1}{3(4n+1)}\binom{4n+3}{2n+1},$$

$$T_n^{(6)} = \left[\begin{matrix}\frac{5}{4}, -\frac{1}{4} \\ 1, \frac{1}{2}\end{matrix}\right]_n = \frac{1}{16^n}\frac{2n+1}{1-4n}\binom{4n+1}{2n},$$

we find the following two infinite series identities.

Example 6 *For $|y| \leq 1$, there hold the identities*

$$\frac{3}{2}\sum_{n=0}^{\infty} S_n^{(6)} y^{2n+1} + \sum_{n=0}^{\infty} T_n^{(6)} y^{2n} = \sqrt{\frac{2}{1-y^2}}\sin\big(\frac{3}{2}\arccos y\big),$$

$$\frac{3}{2}\sum_{n=0}^{\infty} S_n^{(6)} y^{2n+1} - \sum_{n=0}^{\infty} T_n^{(6)} y^{2n} = \sqrt{\frac{2}{1-y^2}}\cos\big(\frac{3}{2}\arccos y\big).$$

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