Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 2891 – 2896
© TÜBİTAK
doi:10.3906/mat-1507-100

On a new subclass of bi-univalent functions defined by using Salagean operator

Bilal ŞEKER*
Department of Mathematics, Faculty of Science, Dicle University, Diyarbakır, Turkey

Received: 28.07.2015 • **Accepted/Published Online:** 23.03.2017 • **Final Version:** 27.11.2018

Abstract: In this manuscript, by using the Salagean operator, new subclasses of bi-univalent functions in the open unit disk are defined. Moreover, for functions belonging to these new subclasses, upper bounds for the second and third coefficients are found.

Key words: Univalent functions, bi-univalent functions, coefficient bounds and coefficient estimates, Salagean operator

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$, which are univalent in $\mathbb{U}$ (for details, see [5]).

In 1983, Salagean [10] introduced the following differential operator:

$D^n : \mathcal{A} \to \mathcal{A}$

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = Df(z) = zf'(z),$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

For the functions given by (1.1), we can easily find that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0).$$

It is known that every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

*Correspondence: bilal.seker@dicle.edu.tr
2000 *AMS Mathematics Subject Classification:* 30C45

This work is licensed under a Creative Commons Attribution 4.0 International License.

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$ given by the Taylor–Maclaurin series expansion (1.1).

For more information about functions in the class Σ, see [11] (see also [3, 8, 9, 13]).

In recent years, the aforementioned study of Srivastava et al. [11] essentially revived the investigation of various subclasses of the bi-univalent function class Σ; it was followed by such studies as those by Ali et al. [2], Srivastava et al. [12], and Jahangiri and Hamidi [7] (see also [1, 4, 6], and the references cited in each of them).

The aim of the this paper is to introduce two new subclasses of the function class Σ related to the Salagean differential operator and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. We have to remember here the following lemma so as to derive our basic results:

Lemma 1.1 *[5] If $p \in \mathcal{P}$ then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of functions p analytic in $\mathbb{U}$ for which $Re\{p(z)\} > 0, p(z) = 1 + c_1z + c_2z^2 + ...$ for $z \in \mathbb{U}$.*

2. Coefficient bounds for the function class $H_{\Sigma}^{m,n}(\alpha)$

By introducing the function class $H_{\Sigma}^{m,n}(\alpha)$, we start by means of the following definition.

Definition 2.1 *A function $f(z)$ given by (1.1) is said to be in the class $H_{\Sigma}^{m,n}(\alpha)$ $(0 < \alpha \leq 1,\ m,n \in \mathbb{N}_0\ , m > n)$ if the following conditions are satisfied:*

$$f \in \Sigma \ \ and \ \ \left|arg\left(\frac{D^m f(z)}{D^n f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}) \tag{2.1}$$

and

$$\left|arg\left(\frac{D^m g(w)}{D^n g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}), \tag{2.2}$$

where the function $g(w)$ is given by (1.2).

For functions in the class $H_{\Sigma}^{m,n}(\alpha)$, we start by finding the estimates on the coefficients $|a_2|$ and $|a_3|$.

Theorem 2.2 *Let the function $f(z)$ given by (1.1) be in the class $H_{\Sigma}^{m,n}(\alpha)$ $(0 < \alpha \leq 1,\ m,n \in \mathbb{N}_0\ , m > n)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha(3^m - 3^n) + (2^m - 2^n)^2 - \alpha(2^{2m} - 2^{2n})}} \tag{2.3}$$

and

$$|a_3| \leq \frac{2\alpha}{3^m - 3^n} + \frac{4\alpha^2}{(2^m - 2^n)^2}. \tag{2.4}$$

Proof It can be written that the inequalities (2.1) and (2.2) are equivalent to

$$\frac{D^m f(z)}{D^n f(z)} = [p(z)]^\alpha \tag{2.5}$$

and

$$\frac{D^m g(w)}{D^n g(w)} = [q(w)]^\alpha \tag{2.6}$$

where $p(z)$ and $q(w)$ are in $\mathcal{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.7}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots . \tag{2.8}$$

Now, equating the coefficients in (2.5) and (2.6), we obtain

$$(2^m - 2^n)a_2 = \alpha p_1 \tag{2.9}$$

$$(3^m - 3^n)a_3 - 2^n(2^m - 2^n)a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2 \tag{2.10}$$

$$-(2^m - 2^n)a_2 = \alpha q_1 \tag{2.11}$$

and

$$(3^m - 3^n)(2a_2^2 - a_3) - 2^n(2^m - 2^n)a_2^2 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$p_1 = -q_1 \tag{2.13}$$

and

$$2(2^m - 2^n)^2 a_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{2.14}$$

Also, from (2.10), (2.12), and (2.14), we find that

$$[2(3^m - 3^n) - 2^n(2^m - 2^n)]a_2^2 = \alpha(p_2 + q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2 + q_1^2) \quad = \alpha(p_2 + q_2) + \frac{\alpha(\alpha-1)}{2}\frac{2(2^m - 2^n)^2 a_2^2}{\alpha^2}.$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2(p_2 + q_2)}{2\alpha(3^m - 3^n) + (2^m - 2^n)^2 - \alpha(2^{2m} - 2^{2n})}. \tag{2.15}$$

If we apply Lemma 1.1 for the coefficients p_2 and q_2, we have

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha(3^m - 3^n) + (2^m - 2^n)^2 - \alpha(2^{2m} - 2^{2n})}}.$$

This gives the desired estimate for $|a_2|$ as asserted in (2.3).

Next, in order to find the bound on $|a_3|$, by subtracting (2.12) from (2.10), we get

$$2(3^m - 3^n)(a_3 - a_2^2) = \alpha(p_2 - q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2 - q_1^2)$$

$$a_3 = \frac{\alpha(p_2 - q_2)}{2(3^m - 3^n)} + \frac{\alpha^2(p_1^2 + q_1^2)}{2(2^m - 2^n)^2}. \tag{2.16}$$

We apply Lemma 1.1 one more time for the coefficients p_2, p_2, q_1, and q_2, obtaining

$$|a_3| \leq \frac{2\alpha}{(3^m - 3^n)} + \frac{4\alpha^2}{(2^m - 2^n)^2}.$$

This completes the proof of Theorem 2.1. □

3. Coefficient bounds for the function class $H_\Sigma^{m,n}(\beta)$

Definition 3.1 *A function $f(z)$ given by (1.1) is said to be in the class $H_\Sigma^{m,n}(\beta)$ $(0 \leq \beta < 1,\ m,n \in \mathbb{N}_0\ , m > n)$ if the following conditions are satisfied:*

$$f \in \Sigma \ \ and \ \ Re\left(\frac{D^m f(z)}{D^n f(z)}\right) > \beta \qquad (z \in \mathbb{U}) \tag{3.1}$$

and

$$Re\left(\frac{D^m g(w)}{D^n g(w)}\right) > \beta \qquad (w \in \mathbb{U}), \tag{3.2}$$

where the function $g(w)$ is given by (1.2).

Theorem 3.2 *Let the function $f(z)$ given by (1.1) be in the class $H_\Sigma^{m,n}(\beta)$ $(0 \leq \beta < 1,\ m,n \in \mathbb{N}_0\ , m > n)$. Then*

$$|a_2| \leq \left(\frac{2(1-\beta)}{(3^m - 3^n) - 2^n(2^m - 2^n)}\right)^{\frac{1}{2}} \tag{3.3}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(2^m - 2^n)^2} + \frac{2(1-\beta)}{(3^m - 3^n)}. \tag{3.4}$$

Proof It follows from (3.1) and (3.2) that there exists $p(z) \in \mathcal{P}$ and $q(z) \in \mathcal{P}$ such that

$$\frac{D^m f(z)}{D^n f(z)} = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$\frac{D^m g(w)}{D^n g(w)} = \beta + (1-\beta)q(w), \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.7) and (2.8), respectively. Equating coefficients in (3.5) and (3.6) yields

$$(2^m - 2^n)a_2 = (1-\beta)p_1, \tag{3.7}$$

$$(3^m - 3^n)a_3 - 2^n(2^m - 2^n)a_2^2 = (1-\beta)p_2, \tag{3.8}$$

$$-(2^m - 2^n)a_2 = (1-\beta)q_1, \tag{3.9}$$

and

$$(3^m - 3^n)(2a_2^2 - a_3) - 2^n(2^m - 2^n)a_2^2 = (1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1 = -q_1, \tag{3.11}$$

$$2(2^m - 2^n)^2 a_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \tag{3.12}$$

Also, from (3.8) and (3.10), we find that

$$2[(3^m - 3^n) - 2^n(2^m - 2^n)]a_2^2 = (1-\beta)(p_2 + q_2). \tag{3.13}$$

Thus, we have

$$|a_2^2| \leq \frac{(1-\beta)(|p_2| + |q_2|)}{2[(3^m - 3^n) - 2^n(2^m - 2^n)]}$$

$$|a_2^2| \leq \frac{2(1-\beta)}{(3^m - 3^n) - 2^n(2^m - 2^n)},$$

which is the bound on $|a_2|$ as given in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$2(3^m - 3^n)(a_3 - a_2^2) = (1-\beta)(p_2 - q_2)$$

or equivalently

$$a_3 = a_2^2 + \frac{(1-\beta)(p_2 - q_2)}{2(3^m - 3^n)}.$$

Upon substituting the value of a_2^2 from (3.12), we have

$$a_3 = \frac{(1-\beta)^2(p_1^2 + q_1^2)}{2(2^m - 2^n)^2} + \frac{(1-\beta)(p_2 - q_2)}{2(3^m - 3^n)}.$$

Applying Lemma 1.1, once again for the coefficients p_2, p_2, q_1, and q_2, we obtain

$$|a_3| \leq \frac{4(1-\beta)^2}{(2^m - 2^n)^2} + \frac{2(1-\beta)}{(3^m - 3^n)},$$

which is the bound on $|a_3|$ as asserted in (3.4). □

References

[1] Akın G, Sümer Eker S. Coefficient estimates for a certain class of analytic and bi-univalent functions defined by fractional derivative. CR Acad Scı I-Math 2014; 352: 1005-1010.

[2] Ali RM, Lee SK, Ravichandran V, Supramaniam S. Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions. Appl Math Lett 2012; 25: 344-351.

[3] Brannan DA, Taha TS. On some classes of bi-univalent functions. Studia Univ. Babeş-Bolyai Math 1986; 31: 70-77.

[4] Deniz E. Certain subclasses of bi-univalent functions satisfying subordinate conditions. J Class Anal 2013; 2: 49-60.

[5] Duren PL. Univalent Functions. Grundlehren der Mathematischen Wissenschaften, Vol. 259. Berlin, Germany: Springer, 1983.

[6] Frasin BA, Aouf MK. New subclasses of bi-univalent functions. Appl Math Lett 2011; 24: 1569-1573.

[7] Jahangiri JM, Hamidi SG. Coefficient estimates for certain classes of bi-univalent functions. Int J Math Math Sci 2013; 2013: 190560.

[8] Lewin M. On a coefficient problem for bi-univalent functions. P Am Math Soc 1967; 18: 63-68.

[9] Netanyahu ME. The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. Arch Rational Mech Anal 1969; 32: 100-112.

[10] Salagean GS. Subclasses of univalent functions. Lect Notes Math 1983; 1013: 362-372.

[11] Srivastava HM, Mishra AK, Gochhayat P. Certain subclasses of analytic and bi-univalent functions. Appl Math Letters 2010; 23: 1188-1192.

[12] Srivastava HM, Süumer Eker S, Ali RM. Coefficient bounds for a certain class of analytic and bi-univalent functions. Filomat 2015; 29: 1839-1845.

[13] Taha TS. Topics in univalent function theory. PhD, University of London, London, UK, 1981.