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# On spanning sets and generators of near-vector spaces

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**Abstract:** In this paper we study the quasi-kernel of certain constructions of near-vector spaces and the span of a vector. We characterize those vectors whose span is one-dimensional and those that generate the whole space.

**Key words:** Field, nearfield, vector space, near-vector space

## 1. Introduction

The near-vector spaces we study in this paper were first introduced by André in 1974 [1]. His near-vector spaces have less linearity than normal vector spaces. They have been studied in several papers, including [2–6]. More recently, since André did a lot of work in geometry, their geometric structure has come under investigation. In order to construct some incidence structures a good understanding of the span of a vector is necessary. It very quickly became clear that near-vector spaces exhibit some strange behavior, where the span of a vector need not be one-dimensional and it is possible for a single vector to generate the entire space.

In this paper we begin by giving the preliminary material of near-vector spaces. In Section 3 we take a closer look at the class of near-vector spaces of the form $(F^n, F)$, where $F$ is a nearfield and $n$ is a natural number, constructed using van der Walt's important construction theorem in [9] for finite dimensional near-vector spaces. We give conditions for when the quasi-kernel will be the whole space. In the last section we prove that when for a near-vector space $(V, A)$, $v \in V$, span $v$ will equal $vA$. We introduce the dimension of a vector and prove that in the case of a field, it is always less than or equal to the number of maximal regular subspaces in the decomposition of $V$. We define a generator for $V$ and give a condition for when $v$ will be a generator for $V$. Finally, we characterize the near-vector spaces that have generators.

## 2. Preliminary material

**Definition 2.1** *A (right) nearfield is a set $F$ together with two binary operations $+$ and $\cdot$ such that*

1. $(F, +)$ *is a group;*
2. $(F \backslash \{0\}, \cdot)$ *is a group;*
3. $(a+b) \cdot c = a \cdot c + b \cdot c$ *for all* $a, b, c \in F$.

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Left nearfields are defined analogously and satisfy the left distributive law. We will use right nearfields throughout this paper. We also have the following definition.

**Definition 2.2** *Let $F$ be a nearfield. We define the kernel of $F$ to be the set of all distributive elements of $F$, i.e.*

$$F_d := \{a \in F | a \cdot (b + c) = a \cdot b + a \cdot c \text{ for every } b, c \in F\}.$$

If $F$ is a nearfield, $F_d$ is a subfield of it [8]; moreover, $F$ is a vector space over $F_d$. We refer the reader to [7] and [8] for more on nearfields.

**Definition 2.3** *([1]) A near-vector space is a pair $(V, A)$ that satisfies the following conditions:*

1. *$(V, +)$ is a group and $A$ is a set of endomorphisms of $V$;*
2. *$A$ contains the endomorphisms $0$, $id$, and $-id$;*
3. *$A^* = A \backslash \{0\}$ is a subgroup of the group $Aut(V)$;*
4. *If $x\alpha = x\beta$ with $x \in V$ and $\alpha, \beta \in A$, then $\alpha = \beta$ or $x = 0$, i.e. $A$ acts fixed point free on $V$;*
5. *The quasi-kernel $Q(V)$ of $V$ generates $V$ as a group. Here, $Q(V) = \{x \in V | \forall \alpha, \beta \in A, \exists \gamma \in A$ such that $x\alpha + x\beta = x\gamma\}$.*

We will write $Q(V)^*$ for $Q(V) \backslash \{0\}$ throughout this paper. The *dimension* of the near-vector space, $\dim(V)$, is uniquely determined by the cardinality of an independent generating set for $Q(V)$, called a *basis* of $V$ (see [1]).

**Definition 2.4** *([6]) We say that two near-vector spaces $(V_1, A_1)$ and $(V_2, A_2)$ are isomorphic (written $(V_1, A_1) \cong (V_2, A_2)$) if there are group isomorphisms $\theta : (V_1, +) \to (V_2, +)$ and $\eta : (A_1^*, \cdot) \to (A_2^*, \cdot)$ such that $\theta(x\alpha) = \theta(x)\eta(\alpha)$ for all $x \in V_1$ and $\alpha \in A_1^*$.*

We will write a near-vector space isomorphism as a pair $(\theta, \eta)$.

**Example 2.5** *([5]) Consider the field ($GF(3^2)$, $+$, $\cdot$) with*

$$GF(3^2) := \{0, 1, 2, \gamma, 1 + \gamma, 2 + \gamma, 2\gamma, 1 + 2\gamma, 2 + 2\gamma\},$$

*where $\gamma$ is a zero of $x^2 + 1 \in \mathbb{Z}_3[x]$. In [8], p. 257, it was observed that ($GF(3^2)$, $+$, $\circ$), with*

$$x \circ y := \begin{cases} x \cdot y & \text{if } y \text{ is a square in } (GF(3^2), +, \cdot) \\ x^3 \cdot y & \text{otherwise} \end{cases}$$

*and*

$$+ : (a + b\gamma) + (c + d\gamma) = (a + c) mod3 \, + ((b + d) mod3 \,)\gamma$$

*is a (right) nearfield, but not a field.*

| $\circ$ | $0$ | $1$ | $2$ | $\gamma$ | $1+\gamma$ | $2+\gamma$ | $2\gamma$ | $1+2\gamma$ | $2+2\gamma$ |
|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $0$ | $1$ | $2$ | $\gamma$ | $1+\gamma$ | $2+\gamma$ | $2\gamma$ | $1+2\gamma$ | $2+2\gamma$ |
| $2$ | $0$ | $2$ | $1$ | $2\gamma$ | $2+2\gamma$ | $1+2\gamma$ | $\gamma$ | $2+\gamma$ | $1+\gamma$ |
| $\gamma$ | $0$ | $\gamma$ | $2\gamma$ | $2$ | $1+2\gamma$ | $1+\gamma$ | $1$ | $2+2\gamma$ | $2+\gamma$ |
| $1+\gamma$ | $0$ | $1+\gamma$ | $2+2\gamma$ | $2+\gamma$ | $2$ | $2\gamma$ | $1+2\gamma$ | $\gamma$ | $1$ |
| $2+\gamma$ | $0$ | $2+\gamma$ | $1+2\gamma$ | $2+2\gamma$ | $\gamma$ | $2$ | $1+\gamma$ | $1$ | $2\gamma$ |
| $2\gamma$ | $0$ | $2\gamma$ | $\gamma$ | $1$ | $2+\gamma$ | $2+2\gamma$ | $2$ | $1+\gamma$ | $1+2\gamma$ |
| $1+2\gamma$ | $0$ | $1+2\gamma$ | $2+\gamma$ | $1+\gamma$ | $2\gamma$ | $1$ | $2+2\gamma$ | $2$ | $\gamma$ |
| $2+2\gamma$ | $0$ | $2+2\gamma$ | $1+\gamma$ | $1+2\gamma$ | $1$ | $\gamma$ | $2+\gamma$ | $2\gamma$ | $2$ |

*The distributive elements of $(GF(3^2),\ +,\ \circ)$, denoted by $(GF(3^2),\ +,\ \circ)_d$, are the elements $0,1,2$. From now on when there is no room for confusion, we will write $x\circ y$ as $xy$. Now let $F=(GF(3^2),+,\circ)$, with $\alpha\in F$ acting as an endomorphism of $V=F^3$ by defining $(x_1,x_2,x_3)\alpha=(x_1\alpha,x_2\alpha,x_3\alpha)$. Thus, $Q(V)=\mathcal{V}_1\cup\mathcal{V}_2\cup\mathcal{V}_3$, with $\mathcal{V}_1=(1,d_1,d_2)F$, $\mathcal{V}_2=(d_1,1,d_2)F$ and $\mathcal{V}_3=(d_1,d_2,1)F$, with $d_1,d_2\in F_d$. We will refer back to this example later in the paper.*

In [9] it was proved that finite-dimensional near-vector spaces can be characterized in the following way:

**Theorem 2.6** *([9]) Let $(G,+)$ be a group and let $A=D\cup\{0\}$, where $D$ is a fixed point free group of automorphism of $G$. Then $(G,A)$ is a finite-dimensional near-vector space if and only if there exist a finite number of nearfields $F_1,\ldots,F_m$, semigroup isomorphisms $\psi_i:(A,\circ)\to(F_i,\cdot)$, and an additive group isomorphism $\Phi: G\to F_1\oplus\ldots\oplus F_m$ such that if $\Phi(g)=(x_1,\ldots,x_m)$, then $\Phi(g\alpha)=(x_1\psi_1(\alpha),\ldots,x_m\psi_m(\alpha))$ for all $g\in G,\ \alpha\in A$.*

Using this theorem we can specify a finite-dimensional near-vector space by taking $n$ copies of a nearfield $F$ for which there are semigroup isomorphisms $\psi_i:(F,\cdot)\to(F,\cdot)$, $i\in\{1,\ldots,n\}$. We then take $V:=F^n$, $n$ a positive integer, as the additive group of the near-vector space and define the scalar multiplication by:

$$(x_1,\ldots,x_n)\alpha:=(x_1\psi_1(\alpha),\ldots,x_n\psi_n(\alpha)),$$

for all $\alpha\in F$ and $i\in\{1,\ldots,n\}$. This is the type of construction we will use throughout this paper and we will use $(F^n,F)$ to denote an instance of a near-vector space of this form.

The concept of regularity is a central notion in the study of near-vector spaces.

**Definition 2.7** *([1]) A near-vector space is regular if any two vectors of $Q(V)^*$ are compatible, i.e. if for any two vectors $u$ and $v$ of $Q(V)^*$ there exists a $\lambda\in A\backslash\{0\}$ such that $u+v\lambda\in Q(V)$.*

**Theorem 2.8** *([1]) Let $F$ be a (right) nearfield and let $I$ be a nonempty index set. Then the set*

$$F^{(I)}:=\{(n_i)_{i\in I}|n_i\in F,n_i\neq 0\ \textit{for at most a finite number of}\ i\in I\}$$

*with the scalar multiplication defined by*

$$(n_i)\lambda:=(n_i\lambda)$$

*gives that $(F^{(I)},F)$ is a near-vector space.*

We describe the quasi-kernel of $F^{(I)}$:

**Theorem 2.9** *([1]) We have*

$$Q(F^{(I)}) = \{(d_i)\lambda | \lambda \in F, d_i \in F_d \text{ for all } i \in I\}.$$

We can also show that the quasi-kernel is not the entire space.

**Theorem 2.10** *Letting $F$ be a proper (right) nearfield and let $I$ be a nonempty index set, then the near-vector space $(F^{(I)}, F)$ has $Q(F^{(I)}) \neq F^{(I)}$.*

**Proof** Consider the element $v = (a_1, 1, \ldots, 0) \in V$, where $a_1 \notin F_d$. We show that $v$ is in $V \backslash Q(V)$. Suppose that $v \in Q(V)$, and then $(a_1, 1, \ldots, 0) = (d_1\lambda, d_2\lambda, \ldots, 0)$. Thus, we get that $a_1 = d_1\lambda$, $1 = d_2\lambda$ and since $F$ is a nearfield, we can solve this and get that $\lambda = d_2^{-1}$. Substituting this in the first equation we get that $a_1 = d_1 d_2^{-1}$, and since $F_d$ is a field, this gives that $a_1 \in F_d$, a contradiction. □

The following theorem gives a characterization of regularity in terms of the near-vector space $(F^{(I)}, F)$.

**Theorem 2.11** *([1]) A near-vector space $(V, F)$, with $F$ a nearfield and $V \neq 0$, is a regular near-vector space if and only if $V$ is isomorphic to $F^{(I)}$ for some index set $I$.*

The following theorem is central in the theory of near-vector spaces.

**Theorem 2.12** *([1]) (The Decomposition Theorem) Every near-vector space $V$ is the direct sum of regular near-vector spaces $V_j$ $(j \in J)$ such that each $u \in Q(V)^*$ lies in precisely one direct summand $V_j$. The subspaces $V_j$ are maximal regular near-vector spaces.*

## 3. Spanning sets and generators

In [5] a study of the subspaces of near-vector spaces was initiated. In this section we add to these results. We begin with some basic definitions.

**Definition 3.1** *([5]) If $(V, A)$ is a near-vector space and $\emptyset \neq V' \subseteq V$ is such that $V'$ is the subgroup of $(V, +)$ generated additively by $XA = \{xa \,|\, x \in X, a \in A\}$, where $X$ is an independent subset of $Q(V)$, then we say that $(V', A)$ is a subspace of $(V, A)$, or simply $V'$ is a subspace of $V$ if $A$ is clear from the context.*

From the definition, since $X$ is a basis for $V'$, the dimension of $V'$ is $|X|$. It is clear that $V$ is a subspace of itself since it is generated by $XA$ where $X$ denotes a basis of $Q(V)$ and we define the trivial subspace, $\{0\}$, to be the space generated by the empty subset of $Q(V)$.

**Definition 3.2** *Letting $(V, A)$ be a near-vector space, then the span of a set $S$ of vectors is defined to be the intersection $W$ of all subspaces of $V$ that contain $S$, denoted span $S$.*

It is straightforward to verify that $W$ is a subspace, called the subspace spanned by $S$, or conversely, $S$ is called a spanning set of $W$ and we say that $S$ spans $W$. Moreover, if we define span $\emptyset = \{0\}$, then it is not difficult to check that span $S$ is the set of all possible linear combinations of $S$.

For a vector space $(V, F)$ the span of a single vector $v$ is always of the form $vF$, but in general this is not true for near-vector spaces. The following two results were recently proved:

{

**Lemma 3.3** *Let $(V, A)$ be a near-vector space. Then for all $v \in V$, $span\{v\} = vA$ if and only if $Q(V) = V$.*

One might wonder if it is possible for a nonzero $w \in V \backslash Q(V)$ to have span$\{w\} = vA$ for some $v \in Q(V)$.

**Lemma 3.4** *Let $(V, A)$ be a near-vector space. Then for all nonzero $w \in V \backslash Q(V)$, $span\{w\} \neq vA$ for some $v \in Q(V)$.*

}

We are interested in what the span of a vector outside of $Q(V)$ looks like.

Let $(V, A)$ be a near-vector space, not necessarily finite-dimensional. By definition, the quasi-kernel $Q(V)$ generates $V$, so for any $v \in V$, there is $u_1, \ldots, u_m \in Q(V) \setminus \{0\}$ and $\alpha_1, \ldots, \alpha_m \in A \setminus \{0\}$, such that $v = u_1\alpha_1 + \cdots + u_m\alpha_m$. This expression is not unique. We can also have $u'_1, \ldots, u'_l \in Q(V) \setminus \{0\}$ and $\alpha'_1, \ldots, \alpha'_l \in A \setminus \{0\}$ such that $v = u'_1\alpha'_1 + \cdots + u'_l\alpha'_l$ with $m \neq l$.

For $v \in V \setminus \{0\}$, we consider

$$n = \min\left\{m \in \mathbb{N} \mid v = \sum_{i=1}^{m} u_i\alpha_i, \text{ with } u_i \in Q(V) \setminus \{0\}, \alpha_i \in A \setminus \{0\}, i = 1, \ldots, m\right\}.$$

**Definition 3.5** *For $v \in V \setminus \{0\}$ we define the dimension of $v$ to be*

$$n = \min\left\{m \in \mathbb{N} \mid v = \sum_{i=1}^{m} u_i\alpha_i, \textit{ with } u_i \in Q(V) \setminus \{0\}, \alpha_i \in A \setminus \{0\}, i = 1, \ldots, m\right\},$$

*and we denote it by $dim(v) = n$ and $dim(v) = 0$ if $v$ is the zero vector.*

**Theorem 3.6** *We have that $dim\,(span\{v\}) = dim(v)$.*

**Proof** Let $n = dim(v)$ and $\{u_1, \ldots, u_n\} \subset Q(V)$, such that $v = \sum_{i=1}^{n} u_i\alpha_i$ for some $\alpha_i \in A \setminus \{0\}$. Then $span\{v\} \subset span\{u_1, \ldots, u_n\} =: W$, since $span\{v\}$ is the smallest subset of $V$ that contains $v$. Since $n$ is minimal, $\{u_1, \ldots, u_n\}$ is a linearly independent subset of $Q(V)$. Hence, $dim(W) = n$ and $dim(span\{v\}) \leqslant n$.

Let us assume that $dim(span\{v\}) < n$. Since $v \in span\{v\}$, there are $u_1, \ldots, u_m \in Q(V) \setminus \{0\}$ and $\beta_1, \ldots, \beta_m \in A \setminus \{0\}$ such that $v = \sum_{i=1}^{m} v_i\beta_i$, with $m < n$. This a contradiction since $n$ is the smallest integer that satisfies this condition. Hence, $dim(span\{v\}) = dim(v)$. □

We know that any subspace of $W$ of $V$ is generated by $XA$, with $X$ a linearly independent subset of $Q(V)$. For $span\{v\}$, $v$ a vector in $V \setminus \{0\}$, the subset $X$ is given by any linearly independent set $\{u_1, \ldots, u_n\} \subset Q(V)$, such that $n = dim(v)$ and $v = \sum_{i=1}^{n} u_i\alpha_i$ for some $\alpha_i \in A \setminus \{0\}$.

By Lemma 3.3, we have that:

**Proposition 3.7** *For any* $v \in V$, $dim(v) = 1$ *if and only of* $v \in Q(V) \setminus \{0\}$.

Also, if $V$ is finite-dimensional, of dimension $n$, then $dim(v) \leqslant n$, and if $dim(v) = n$, then $span\{v\} = V$. Thus, we define:

**Definition 3.8** *Let* $(V, A)$ *be a near-vector space. If* $v \in V$ *such that* $span\{v\} = V$, *then* $v$ *is called a generator of* $V$.

Isomorphisms preserve generators:

**Theorem 3.9** *Let* $(V_1, A_1)$ *and* $(V_2, A_2)$ *be isomorphic near-vector spaces and* $v \in V_1$. *Then* $dim(v) = dim(\theta(v))$, *where* $(\theta, \eta)$ *is the isomorphism.*

**Proof** Let $dim(v) = k$ and $dim(\theta(v)) = k'$. Then there exist $u_1, \ldots, u_k \in Q(V_1)\setminus\{0\}$ and $\alpha_1, \ldots, \alpha_k \in A_1\setminus\{0\}$ such that $v = \sum_{i=1}^{k} u_i\alpha_i$. We have

$$\theta(v) = \theta\left(\sum_{i=1}^{k} u_i\alpha_i\right) = \sum_{i=1}^{k} \theta\left(u_i\alpha_i\right) = \sum_{i=1}^{k} \theta\left(u_i\right)\eta\left(\alpha_i\right).$$

It follows that $dim(\theta(v)) \leq k$.

Assume that $k' = dim(\theta(v)) < k$. There are $v_1, \ldots, v_{k'} \in Q(V_2) \setminus \{0\}$ and $\beta_1, \ldots, \beta_{k'} \in A_2 \setminus \{0\}$ such that $\theta(v) = \sum_{i=1}^{k} v_i\beta_i$. Since $(\theta, \eta)$ is an isomorphism, we have

$$\theta(v) = \sum_{i=1}^{k'} \theta\left(v_i^{'}\right)\eta\left(\beta_i^{'}\right) = \sum_{i=1}^{k'} \theta\left(v_i^{'}\beta_i^{'}\right) = \theta\left(\sum_{i=1}^{k'} v_i^{'}\beta_i^{'}\right).$$

It follows that $v = \sum_{i=1}^{k'} v_i^{'}\beta_i^{'}$ and $dim(v) \leq k' < k$, which is a contradiction. □

**Corollary 3.10** *Let* $(V_1, A_1)$ *and* $(V_2, A_2)$ *be isomorphic near-vector spaces.* $v$ *is a generator of* $V_1$ *if and only if* $\theta(v)$ *is a generator of* $V_2$, *where* $(\theta, \eta)$ *is the isomorphism.*

For $F$ a field, using the following recently proved result, we can show more.

{

**Theorem 3.11** *Let* $F = GF(p^r)$ *and* $V = F^n$ *be a near-vector space with scalar multiplication defined for all* $\alpha \in F$ *by*

$$(x_1, \ldots, x_n)\alpha := (x_1\psi_1(\alpha), \ldots, x_n\psi_n(\alpha)),$$

*where the* $\psi_i^{\prime}s$ *are automorphisms of* $(F, \cdot)$. *If* $Q(V) \neq V$ *and* $V = V_1 \oplus \cdots \oplus V_k$ *is the canonical decomposition of* $V$, *then* $Q(V) = Q_1 \cup \cdots \cup Q_k$ *where* $Q_i = V_i$ *for each* $i \in \{1, \ldots, k\}$.

}

**Theorem 3.12** *Let $F$ be a field and $V = F^n$ be a near-vector space over $F$ with scalar multiplication defined for all $(x_1, \ldots, x_n) \in F$ and $\alpha \in F$ by*

$$(x_1, \ldots, x_n)\alpha := (x_1\psi_1(\alpha), \ldots, x_n\psi_n(\alpha)),$$

*where the $\psi_i's$ are automorphisms of $(F, \cdot)$ for $i \in \{1, \ldots, n\}$ and they can be equal. If $V_1 \oplus \cdots \oplus V_k$ is the canonical decomposition of $V$, then for all $v \in V$, $dim(v) \leq k$.*

**Proof** Let $v \in V$ and suppose that $dim(v) > k$, say $dim(v) = k'$, where $k' > k$. Then $v = \sum_{i=1}^{k'} u_i\lambda_i$, where $u_i \in Q(V)\backslash\{0\}$, $\lambda_i \in F$ for $i \in 1, \ldots, k'$. However, for all $i \in 1, \ldots, k'$, $u_i \in Q_j$ for some $j$ with $1 \leq j \leq k$, since by Theorem 3.11, $Q(V) = Q_1 \cup \cdots \cup Q_k$ and $k' > k$. Suppose, without loss of generality, that $u_s$ and $u_{s'}$ are in $Q_j$, and then $u_s\lambda_s + u_{s'}\lambda_{s'} \in Q_j$, since $Q_j = V_j$ ($F$ is a field). Now we have that $v$ can be written with fewer than $k'$ elements, i.e. $v = u_1\lambda_1 + \cdots + u_k\lambda_k$, a contradiction. □

Thus, in the case where $F$ is a field, unless the dimension of $V$ is less than or equal to 1, or equal to $k$, where $k$ is the number of maximal regular subspaces in the canonical decomposition of the near-vector space, we cannot have any generators. If the dimension of $V$ is exactly $k$ then the maximal regular spaces have dimension 1 and any element of the form $(1, \ldots, 1)$ will be generator of $V$.

### 3.1. Generators for regular near-vector spaces

When $F$ is a proper nearfield, we have the following result:

**Theorem 3.13** *Let $F$ be a proper nearfield and $V' = F^n$ be a near-vector space over $F$ with scalar multiplication defined for all $(x_1, \ldots, x_n) \in V', \alpha \in F$ by*

$$(x_1, \ldots, x_n)\alpha := (x_1\alpha, \ldots, x_n\alpha).$$

*$v = (a_1, \ldots, a_n)$ is a generator of $V'$ if and only for $d_1, \ldots, d_n \in F_d$,*

$$\sum_{i=1}^{n} d_i a_i = 0 \Leftrightarrow d_1 = d_2 = \ldots = d_n = 0.$$

**Proof** Let us assume that there are $d_1, \ldots, d_n \in F_d$ such that $\sum_{i=1}^{n} d_i a_i = 0$ and $d_{i_0} \neq 0$. We show that $dim(v) < n$. Without loss of generality let us assume that $i_0 = 1$. Then $a_1 = \sum_{i=2}^{n} d_1^{-1} d_i a_i$, so we get

$$\begin{aligned}(a_1, \ldots, a_n) =& (\sum_{i=2}^{n} d_1^{-1} d_i a_i, a_2, \ldots, a_n) \\ =& \sum_{i=2}^{n} u_i, \text{ with } u_i = (d_1^{-1} d_i a_i, \ldots, 0, a_i, 0, \ldots, 0).\end{aligned}$$

Since $Q(V') = \{(d_1, \ldots, d_n)\,\alpha | d_1, \ldots, d_n \in F_d, \alpha \in F\}$, $u_i \in Q(V')$ for all $i = 2, \ldots, n$. It follows that $dim(v) < n$. Therefore, $dim(v) = n$ implies that for $d_1, \ldots, d_n \in F_d$,

$$\sum_{i=1}^{n} d_i a_i = 0 \Leftrightarrow d_1 = d_2 = \ldots = d_n = 0.$$

Now let us assume that for $d_1, \ldots, d_n \in F_d$,

$$\sum_{i=1}^{n} d_i a_i = 0 \Leftrightarrow d_1 = d_2 = \ldots = d_n = 0,$$

and that $dim(v) < n$. Thus, $v$ can be written as a linear combination of less than $k$ vectors of the quasi-kernel with $k < n$, so there is

$$(\alpha_i)_{1 \leq i \leq k} \subseteq F \text{ and } (d_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}} \subseteq F_d,$$

such that

$$(a_1, \ldots, a_n) = \sum_{i=1}^{k} (d_{1,i}, \ldots, d_{n,i})\alpha_i.$$

Hence, we get the following system of $n$ equations with $k$ unknowns:

$$\begin{cases} d_{1,1}x_1 + d_{1,2}x_2 + \cdots + d_{1,k}x_k = a_1 \\ d_{2,1}x_1 + d_{2,2}x_2 + \cdots + d_{2,k}x_k = a_2 \\ \vdots \\ d_{n,1}x_1 + d_{n,2}x_2 + \cdots + d_{n,k}x_k = a_n \end{cases}$$

with $(\alpha_1, \ldots, \alpha_k)$ as the solution. Since the equation has a solution, the matrix

$$A = \begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,3} & \ldots & d_{1,k} \\ d_{2,1} & d_{2,2} & d_{2,3} & \ldots & d_{2,k} \\ \vdots & & \ddots & & \vdots \\ & & & & d_{n-1,k} \\ d_{n,1} & d_{n,2} & \ldots & d_{n,k-1} & d_{n,k} \end{pmatrix}$$

has rank $k$ in $F_d$. Therefore, there exist $\delta_1, \ldots, \delta_n \in F_d$ not all zero such that $\sum_{i=1}^{n} \delta_i a_i = 0$. This is a contradiction. □

Let $F$ be a proper nearfield and $V'' = F^n$ be a regular near-vector space over $F$.

**Theorem 3.14** *$v = (a_1, \ldots, a_n)$ is a generator of $V''$ if and only if for $d_1, \ldots, d_n \in F_d$,*

$$\sum_{i=1}^{n} d_i a_i = 0 \Leftrightarrow d_1 = \cdots = d_n = 0.$$

**Proof** It follows from the fact that $(V'', F)$ is isomorphic to $(V', F)$ by Theorem 2.11. □

**Theorem 3.15** *Let $V = F^n$ be a near-vector space with $|F| = |F_d|^m$ and*

$$(x_1, \ldots, x_n)\alpha := (x_1\alpha, \ldots, x_n\alpha),$$

*for all $(x_1, \ldots, x_n) \in V$ and $\alpha \in F$. $v$ is a generator of $V$ if and only if $m \geq n$.*

**Proof** Suppose that there is $v = (a_1, \dots, a_n) \in V$ such that $dim(v) = n$. By Theorem 3.13 we have that for any $d_i \in F_d$, $i = 1, \dots, n$, $\sum_{i=1}^{n} d_i a_i = 0$ implies $d_i = 0$ for all $i$. It follows that $\{a_1, \dots, a_n\}$ is a linearly independent set of vectors in the vector space $F$ over $F_d$. Hence, $m \geq n$.

To show the converse we assume that $m < n$. Then for any $v = (a_1, \dots, a_n) \in V$ there are $d_1, \dots, d_n$ not all zero with $\sum_{i=1}^{n} d_i a_i = 0$. Hence, we cannot have $v \in V$ such that $dim(v) = n$.

□

**Example 3.16** *Let us consider the Dickson nearfield $F = DF(3,2)$ and $V = F^2$ a near-vector space with $(x,y)\alpha := (x\alpha, y\alpha)$. Then the element $v = (1, \gamma)$ has dimension 2. In fact, $v$ is not in any of the subspaces. Suppose that $v \in V_1$, with $V_1$ a one-dimensional subspace of $V$. Let $w$ be a basis of $V'$. It follows that $v = w\lambda$, with $\lambda \in F$, since the quasi-kernel is closed under scalar multiplication $v \in Q(V)$, but $v \notin Q(V)$. Hence, the smallest subspace of $V$ that contains $v$ is $V$ itself. Hence, $v$ is a generator of $V$ and $dim(v) = 2$. Using Theorem 3.15 we can also see that $dim(v) = 2$. For any $d_1, d_2 \in F_d$, $d_1 + d_2\gamma = 0$ implies that $d_1 = d_2 = 0$, since $\{1, \gamma\}$ is a basis of the vector space $F$ over $F_d$.*

*For three copies of $F$, $V = F^3$, it is not possible to have an element that generates $V$.*

### 3.2. Generators for general near-vector spaces

In this subsection we consider the case where $F$ is a proper nearfield and $V = F^n$ is a near-vector space over $F$ with the canonical decomposition $V = \bigoplus_{i=1}^{k} V_i$.

**Lemma 3.17** *If $v_i \in V_i \setminus \{0\}$ and $v_j \in V_j \setminus \{0\}$ with $i \neq j$, then*

$$dim(v_i + v_j) = dim(v_i) + dim(v_j).$$

**Proof** Let $dim(v_i) = l_i, dim(v_j) = l_j$. It is not difficult to check that $dim(v_i + v_j) \leq l_i + l_j$. Suppose that $l = dim(v_i + v_j) < l_i + l_j$. There are $u_1, \dots, u_l \in Q(V_i) \setminus \{0\} \cup Q(V_j) \setminus \{0\}$ and $\alpha_1, \dots, \alpha_l \in F \setminus \{0\}$ such that $v_i + v_j = \sum_{m=1}^{l} u_m \alpha_m$. It follows that we write $v_i$ as $v_i = \sum_{m=1}^{l'} u_m \alpha_m$, with $l' < l_i$ or $v_j = \sum_{m=1}^{l''} u_m \alpha_m$ with $l'' < l_j$, since $V_i \cap V_j = \{0\}$. This is a contradiction since $dim(v_i) = l_i, dim(v_j) = l_j$ and we should have $l_i \geqslant l'$ and $l_j \geqslant l''$. □

**Corollary 3.18** *If $v_i \in V_i \setminus \{0\}$ and $v_j \in V_j \setminus \{0\}$ with $i \neq j$, then*

$$span\{v_i + v_j\} = span\{v_i\} \oplus span\{v_j\}.$$

**Proof** We have $span\{v_i\} \cap span\{v_j\} = \{0\}$, since $span\{v_i\} \subseteq V_i, span\{v_j\} \subseteq V_j$ and $V_i \cap V_j = \{0\}$. We have $span\{v_i + v_j\} \subseteq span\{v_i\} \oplus span\{v_j\}$. Since $dim(v_i + v_j) = dim(v_i) + dim(v_j)$, $span\{v_i + v_j\} = span\{v_i\} \oplus span\{v_j\}$. □

**Corollary 3.19** *Let $v_1, \ldots, v_m \in V$ such that they are all in distinct maximal regular subspaces. We have*

$$dim(v_1 + \cdots + v_m) = dim(v_1) + \cdots + dim(v_m),$$
$$span\{v_1 + \cdots + v_m\} = span\{v_1\} \oplus \cdots \oplus span\{v_m\}.$$

**Theorem 3.20** *A vector $v \in V$ is a generator of $V$ if and only if there are $v_i \in V_i$ generators of $V_i$ for all $i = 1, \ldots, k$, such that $v = v_1 + \cdots + v_k$.*

**Proof** We have $span\{v\} = span\{v_1 + \ldots + v_k\} = span\{v_1\} \oplus \cdots \oplus span\{v_k\}$. If $v$ is a generator of $v$ we have $span\{v\} = V$ and so $span\{v_1\} \oplus \cdots \oplus span\{v_k\} = V$. Hence, $span\{v_i\} = V_i$ for all $i = 1 \ldots, k$. Thus, $v_i$ is a generator of $V_i$ for all $i$. Likewise, if $v_i$ is a generator of $V_i$ for all $i$, then $v$ is a generator of $V$. □

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