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On the Chebyshev coefficients for a general subclass of univalent functions

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Abstract: In this work, considering a general subclass of univalent functions and using the Chebyshev polynomials, we obtain coefficient expansions for functions in this class.

Key words: Chebyshev polynomials, analytic and univalent functions, coefficient bounds, subordination

1. Introduction and definitions

Let D be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and A be the class of functions analytic in D, satisfying the conditions

$$f(0) = 0 \text{ and } \quad f'(0) = 1.$$

Then each function f in A has the following Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Furthermore, by S we shall denote the class of all functions in A that are univalent in D.

If the functions f and g are analytic in D, then f is said to be subordinate to g, written as

$$f(z) \prec g(z), \qquad (z \in D),$$

if there exists a Schwarz function $w(z)$, analytic in D, with

$$w(0) = 0 \ \text{ and } \quad |w(z)| < 1 \qquad (z \in D)$$

such that

$$f(z) = g(w(z)) \qquad (z \in D).$$

Denote by K the subclass of S of convex functions so that $f \in K$ if and only if, for $z \in D$,

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0.$$

The Fekete–Szegö functional $\left|a_3 - \mu a_2^2\right|$ for normalized univalent functions of the form given by (1) is well known for its rich history in geometric function theory. Its origin was in the disproof by Fekete and Szegö

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of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see [5]). The functional has since received great attention, particularly in many subclasses of the family of univalent functions. Nowadays, it seems that this topic had become of interest among researchers (see, for example, [1, 2, 7, 8]).

Chebyshev polynomials have become increasingly important in numerical analysis, from both theoretical and practical points of view. There are four kinds of Chebyshev polynomials. The majority of books and research papers dealing with specific orthogonal polynomials of the Chebyshev family contain mainly results of Chebyshev polynomials of the first and second kinds $T_n(x)$, $U_n(x)$ and their numerous uses in different applications; see, for example, Doha [3] and Mason [6].

The Chebyshev polynomials of the first and second kinds are well known. In the case of a real variable x on $(-1,1)$, they are defined by

$$T_n(x) = \cos n\theta,$$

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},$$

where the subscript n denotes the polynomial degree and where $x = \cos\theta$.

Definition 1 *A function $f \in A$ is said to be in the class $L(\alpha,t)$, $\alpha \geq 0$ and $t \in \left(\frac{1}{2},1\right]$, if the following subordination holds:*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec H(z,t) := \frac{1}{1-2tz+z^2} \qquad (z \in D). \tag{2}$$

We note that if $t = \cos\theta$, $\theta \in \left(-\frac{\pi}{3},\frac{\pi}{3}\right)$, then

$$\begin{aligned} H(z,t) &= \frac{1}{1-2tz+z^2} \\ &= 1+\sum_{n=1}^{\infty}\frac{\sin(n+1)\theta}{\sin\theta}z^n \quad (z \in D). \end{aligned}$$

Thus,

$$H(z,t) = 1 + 2\cos\theta z + (3\cos^2\theta - \sin^2\theta)z^2 + \cdots \quad (z \in D).$$

Following that, we can write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \quad (z \in D, \quad t \in (-1,1)),$$

where $U_{n-1} = \dfrac{\sin(n\arccos t)}{\sqrt{1-t^2}}$ $(n \in \mathbb{N})$ are the Chebyshev polynomials of the second kind. Also, it is known that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$\begin{aligned} &U_1(t) = 2t, \\ &U_2(t) = 4t^2 - 1, \\ &U_3(t) = 8t^3 - 4t, \\ &\vdots \end{aligned} \tag{3}$$

The Chebyshev polynomials $T_n(t),\ t \in [-1,1]$, of the first kind have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t) z^n = \frac{1-tz}{1-2tz+z^2} \quad (z \in D).$$

However, the Chebyshev polynomials of the first kind $T_n(t)$ and the second kind $U_n(t)$ are well connected by the following relationships:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t),$$

$$T_n(t) = U_n(t) - tU_{n-1}(t),$$

$$2T_n(t) = U_n(t) - U_{n-2}(t).$$

In this paper, motivated by the earlier work of Dziok et al. [4], we use the Chebyshev polynomial expansions to provide estimates for the initial coefficients of univalent functions in $L(\alpha, t)$.

2. Coefficient bounds for the function class $L(\alpha, t)$

In this section, we propose to find the estimates on the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the class $L(\alpha, t)$, which we introduced in Definition 1. We first state Theorem 2.

Theorem 2 *Let the function f given by (1) be in the class $L(\alpha, t)$. Then*

$$|a_2| \leq \frac{2t}{2-\alpha}$$

and

$$|a_3| \leq \frac{\left(16 - 13\alpha + \alpha^2\right) t^2}{(3-2\alpha)(2-\alpha)^2} + \frac{t}{3-2\alpha} - \frac{1}{2(3-2\alpha)}.$$

Proof Let $f \in L(\alpha, t)$. From (2), we have

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = 1 + U_1(t)w(z) + U_2(t)w^2(z) + \cdots, \tag{4}$$

for some analytic functions w such that $w(0) = 0$ and $|w(z)| < 1$, for all $z \in D$. From the equality (4), we obtain that

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = 1 + U_1(t)c_1 z + \left[U_1(t)c_2 + U_2(t)c_1^2\right] z^2 + \cdots. \tag{5}$$

It is fairly well known that if $|w(z)| = \left|c_1 z + c_2 z^2 + c_3 z^3 + \cdots\right| < 1,\ z \in D,$ then

$$|c_j| \leq 1, \quad \text{for all} \quad j \in \mathbb{N}, \tag{6}$$

and

$$\left|c_2 - \mu c_1^2\right| \leq \max\{1, |\mu|\}, \quad \text{for } \mu \in \mathbb{R}. \tag{7}$$

It follows from (5) that

$$(2-\alpha)\,a_2 = U_1(t)c_1, \tag{8}$$

$$2\,(3-2\alpha)\,a_3 + \frac{\alpha^2+5\alpha-8}{2}a_2^2 = U_1(t)c_2 + U_2(t)c_1^2. \tag{9}$$

From (3) and (8), we obtain

$$|a_2| \leq \frac{2t}{2-\alpha}. \tag{10}$$

Next, in order to find the bound on $|a_3|$, by using (8) in (9), we obtain

$$2\,(3-2\alpha)\,a_3 = U_1(t)c_2 + \left\{ U_2(t) - \frac{\alpha^2+5\alpha-8}{2\,(2-\alpha)^2}U_1^2(t) \right\} c_1^2. \tag{11}$$

Then, in view of (3) and (6), we have from (11)

$$|a_3| \leq \frac{\left(16-13\alpha+\alpha^2\right)t^2}{(3-2\alpha)\,(2-\alpha)^2} + \frac{t}{3-2\alpha} - \frac{1}{2\,(3-2\alpha)}.$$

□

3. Fekete–Szegö inequalities for the function class $L\,(\alpha, t)$

In this section, we derive the Fekete–Szegö inequalities for functions in the class $L(\alpha, t)$, which is introduced by Definition 1. These inequalities are asserted by Theorem 3.

Theorem 3 *Let f given by (1) be in the class $L\,(\alpha, t)$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{t}{3-2\alpha}; & for\ \mu \in [\mu_1, \mu_2] \\ \dfrac{t}{3-2\alpha}\left|\frac{4t^2-1}{2t} - \frac{\alpha^2+5\alpha-8}{(2-\alpha)^2}t - 4\mu\frac{(3-2\alpha)}{(2-\alpha)^2}t\right|; & for\ \mu \notin [\mu_1, \mu_2] \end{cases},$$

where,

$$\mu_1 = \tfrac{2t^2\left(\alpha^2-13\alpha+16\right)-(2-\alpha)^2(1+2t)}{8(3-2\alpha)t^2}, \quad \mu_2 = \tfrac{2t^2\left(\alpha^2-13\alpha+16\right)-(2-\alpha)^2(1-2t)}{8(3-2\alpha)t^2}.$$

Proof From (8) and (11), we can easily see that

$$a_3 - \mu a_2^2 = \frac{U_1(t)}{2\,(3-2\alpha)}c_2 + \left\{ \frac{U_2(t)}{2\,(3-2\alpha)} - \frac{\alpha^2+5\alpha-8}{4\,(3-2\alpha)\,(2-\alpha)^2}U_1^2(t) - \mu\frac{U_1^2(t)}{(2-\alpha)^2} \right\} c_1^2$$

and

$$\left|a_3 - \mu a_2^2\right| = \frac{U_1(t)}{2\,(3-2\alpha)}\left| c_2 + \left\{ \frac{U_2(t)}{U_1(t)} - \frac{\alpha^2+5\alpha-8}{2\,(2-\alpha)^2}U_1(t) - 2\mu\frac{(3-2\alpha)\,U_1(t)}{(2-\alpha)^2} \right\} c_1^2 \right|.$$

Then, in view of (7), we conclude that

$$\left|a_3-\mu a_2^2\right| \leq \frac{U_1(t)}{2\,(3-2\alpha)}\max\left\{1,\left|\frac{U_2(t)}{U_1(t)}-\frac{\alpha^2+5\alpha-8}{2\,(2-\alpha)^2}U_1(t)-2\mu\frac{(3-2\alpha)\,U_1(t)}{(2-\alpha)^2}\right|\right\}. \tag{12}$$

Finally, by using (3) in (12), we get

$$\left|a_3-\mu a_2^2\right| \leq \frac{t}{3-2\alpha}\max\left\{1,\left|\frac{4t^2-1}{2t}-\frac{\alpha^2+5\alpha-8}{(2-\alpha)^2}t-4\mu\frac{(3-2\alpha)}{(2-\alpha)^2}t\right|\right\}.$$

Because $t>0$, we have

$$\begin{aligned}&\left|\frac{4t^2-1}{2t}-\tfrac{\alpha^2+5\alpha-8}{(2-\alpha)^2}t-4\mu\tfrac{(3-2\alpha)}{(2-\alpha)^2}t\right|\leq 1\\ \Leftrightarrow&\left\{\tfrac{2t^2\left(\alpha^2-13\alpha+16\right)-(2-\alpha)^2(1+2t)}{8(3-2\alpha)t^2}\leq\mu\leq\tfrac{2t^2\left(\alpha^2-13\alpha+16\right)-(2-\alpha)^2(1-2t)}{8(3-2\alpha)t^2}\right\}\\ \Leftrightarrow&\ \mu_1\leq\mu\leq\mu_2.\end{aligned}$$

□

In its special case when $\alpha=0$, Theorem 3 leads us to Corollary 4.

Corollary 4 *If* $f\in L\,(t)$, *then*

$$\begin{aligned}&|a_2|\leq t;\\ &|a_3|\leq\tfrac{4t^2}{3}+\tfrac{t}{3}-\tfrac{1}{6};\\ &\left|a_3-\mu a_2^2\right|\leq\begin{cases}\tfrac{t}{3}; & for\ \mu\in[\mu_1,\mu_2]\\ \left|\tfrac{8t^2-1-6\mu t^2}{6}\right|; & for\ \mu\notin[\mu_1,\mu_2]\end{cases}.\end{aligned}$$

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