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Transversal lightlike submanifolds of metallic semi-Riemannian manifolds

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Abstract: The main purpose of the present paper is to study the geometry of transversal lightlike submanifolds and radical transversal lightlike submanifolds of metallic semi-Riemannian manifolds. We investigate the geometry of distributions and obtain necessary and sufficient conditions for the induced connection on these manifolds to be a metric connection. We also obtain characterization of transversal lightlike submanifolds of metallic semi-Riemannian manifolds. Finally, we give two examples.

Key words: Metallic structure, metallic semi-Riemannian manifold, lightlike submanifolds, transversal lightlike submanifolds, radical transversal submanifolds

1. Introduction

Lightlike submanifolds are one of the most interesting topics in differential geometry. It is well known that a submanifold of a Riemannian manifold is always a Riemannian one. Contrary to that case, in semi-Riemannian manifolds the induced metric by the semi-Riemann metric on the ambient manifold is not necessarily nondegenerate. Since the induced metric is degenerate on lightlike submanifolds, the tools that are used to investigate the geometry of submanifolds in the Riemannian case are not favorable in the semi-Riemannian case and so the classical theory cannot be used to define any induced object on a lightlike submanifold. The main difficulties arise from the fact that the intersection of the normal bundle and the tangent bundle of a lightlike submanifold is nonzero. In 1996, Duggal and Bejancu [14] put forward the general theory of lightlike submanifolds of semi-Riemannian manifolds in their book.

In order to resolve the difficulties that arise while studying lightlike submanifolds, they introduced a nondegenerate distribution called screen distribution to construct a lightlike transversal vector bundle that does not intersect its lightlike tangent bundle. It is well known that a suitable choice of screen distribution gives rises to many substantial results in lightlike geometry. Many authors have studied the geometry of lightlike submanifolds [2–4, 16–18, 29, 33, 34, 37] in different manifolds. For further reading we refer to [14, 15] and the references therein.

Manifolds with various geometric structures are convenient to study submanifold theory [21, 30, 31, 35]. In recent years, one of the most studied manifold types are Riemannian manifolds with metallic structures. Metallic structures on Riemannian manifolds allow many geometric results to be given on a submanifold.

As a generalization of the golden mean, which contains the silver mean, the bronze mean, the copper mean, the nickel mean, etc., the metallic means family was introduced by de Spinadel [12] in 2002. The positive

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solution of the equation given by

$$x^2 - px - q = 0,$$

for some positive integer p and q is called a (p,q)-metallic number [9, 11], which has the form

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

For $p = q = 1$ and $p = 2,\ q = 1$, it is well known that we have the golden mean $\phi = \frac{1+\sqrt{5}}{2}$ and silver mean $\sigma_{2,1} = 1 + \sqrt{2}$, respectively. The metallic mean family plays an important role in establishing a relationship between mathematics and architecture. For example, the golden mean and silver mean can be seen in the sacred art of Egypt, Turkey, India, China, and other ancient civilizations [13].

Goldberg, Yano, and Petridis in [25] and [23] introduced polynomial structures on manifolds. As some particular cases of polynomial structures Hretcanu and Crasmareanu defined the golden structure [6–8, 28] and some generalizations of this called metallic structure [22]. Being inspired by the metallic mean, the notion of metallic manifold $\breve{N}$ was defined in [22] by a $(1,1)$-tensor field $\breve{J}$ on $\breve{N}$, which satisfies $\breve{J}^2 = p\breve{J} + qI$, where I is the identity operator on the Lie algebra $\chi(\breve{N})$ of vector fields on $\breve{N}$ and p, q are fixed positive integer numbers. Moreover, if $(\breve{N}, g)$ is a Riemannian manifold endowed with a metallic structure $\breve{J}$ such that the Riemannian metric $\breve{g}$ is $\breve{J}$-compatible, i.e. $\breve{g}(\breve{J}V, W) = \breve{g}(V, \breve{J}W)$, for any $V, W \in \chi(\breve{N})$, then $(\breve{g}, \breve{J})$ is called a metallic Riemannian structure and $(\breve{N}, \breve{g}, \breve{J})$ is a metallic Riemannian manifold. The metallic structure on the ambient Riemannian manifold provides important geometrical results on the submanifolds, since it is an important tool while investigating the geometry of submanifolds. Invariant, antiinvariant, semiinvariant, slant, and semislant submanifolds of a metallic Riemannian manifold were studied in [5, 26, 27] and the authors obtained important characterizations on submanifolds of metallic Riemannian manifolds.

One of the most important subclasses of metallic Riemannian manifolds is the golden Riemannian manifolds. Many authors have studied golden Riemannian manifolds and their submanifolds in recent years (see [6–8, 19, 20, 32, 36]). Poyraz Önen and Yaşar [34] initiated the study of lightlike geometry in golden semi-Riemannian manifolds by investigating lightlike hypersurfaces of golden semi-Riemannian manifolds. Acet introduced lightlike hypersurfaces in metallic semi-Riemannian manifolds [1].

Motivated by the studies on submanifolds of metallic Riemannian manifolds and lightlike submanifolds of semi-Riemannian manifolds, in the present paper we introduce the transversal lightlike submanifolds of a metallic semi-Riemannian manifold.

Considering the brief background given above, in this paper, we introduce transversal lightlike submanifolds of metallic semi-Riemannian manifolds and study their differential geometry. The paper is organized as follows: Section 2 is devoted to basic definitions needed for the rest of the paper. In Section 3 and Section 4, we introduce a metallic semi-Riemannian manifold along with its subclasses, namely radical transversal and transversal lightlike submanifolds, and obtain some characterizations. We investigate the geometry of distributions and find necessary and sufficient conditions for the induced connection to be a metric connection. Furthermore, we give two examples.

2. Preliminaries

A submanifold $\acute{N}^m$ immersed in a semi-Riemannian manifold $(\breve{N}^{m+k}, \breve{g})$ is called a lightlike submanifold if it admits a degenerate metric g induced from $\breve{g}$, whose radical distribution $Rad\,T\acute{N}$ is of rank r, where $1 \leq r \leq m$. Then $Rad\,T\acute{N} = T\acute{N} \cap T\acute{N}^{\perp}$, where

$$T\acute{N}^{\perp} = \cup_{x\in\acute{N}} \left\{ u \in T_x\breve{N} \mid \breve{g}\,(u,v) = 0, \forall v \in T_x\acute{N} \right\}.$$

Let $S(T\acute{N})$ be a screen distribution that is a semi-Riemannian complementary distribution of $Rad\,T\acute{N}$ in $T\acute{N}$, i.e. $T\acute{N} = Rad\,T\acute{N} \perp S(T\acute{N})$.

We consider a screen transversal vector bundle $S(T\acute{N}^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad\,T\acute{N}$ in $T\acute{N}^{\perp}$ since, for any local basis $\{\xi_i\}$ of $Rad\,T\acute{N}$, there exists a lightlike transversal vector bundle $ltr(T\acute{N})$ locally spanned by $\{N_i\}$ [14]. Let $tr(T\acute{N})$ be complementary (but not orthogonal) vector bundle to $T\acute{N}$ in $T\breve{N}^{\perp}\mid_{\acute{N}}$. Then we have

$$\begin{aligned} tr(T\acute{N}) &= ltr\,T\acute{N} \perp S(T\acute{N}^{\perp}), \\ T\breve{N}\mid_{\acute{N}} &= S(T\acute{N}) \perp [Rad\,T\acute{N} \oplus ltr\,T\acute{N}] \perp S(T\acute{N}^{\perp}). \end{aligned}$$

Although $S(T\acute{N})$ is not unique, it is canonically isomorphic to the factor vector bundle $T\acute{N}/Rad\,T\acute{N}$ [14].

The following result is important for this paper.

Proposition 2.1 *The lightlike second fundamental forms of a lightlike submanifold $\acute{N}$ do not depend on $S(T\acute{N})$, $S(T\acute{N}^{\perp})$, and $ltrT\acute{N}$ [14].*

We say that a submanifold $(\acute{N}, g, S(T\acute{N}),\ S(T\acute{N}^{\perp}))$ of $\breve{N}$ is

Case 1: r-lightlike if $r < \min\{m, k\}$;

Case 2: Co-isotropic if $r = k < m$; $S(T\acute{N}^{\perp}) = \{0\}$;

Case 3: Isotropic if $r = m = k$; $S(T\acute{N}) = \{0\}$;

Case 4: Totally lightlike if $r = k = m$; $S(T\acute{N}) = \{0\} = S(T\acute{N}^{\perp})$.

The Gauss and Weingarten equations are

$$\breve{\nabla}_W U = \nabla_W U + h\,(W,U)\,, \quad \forall W, U \in \Gamma(T\acute{N}), \tag{2.1}$$

$$\breve{\nabla}_W V = -A_V W + \nabla^t_W V, \quad \forall W \in \Gamma(T\acute{N}), V \in \Gamma(tr(T\acute{N})), \tag{2.2}$$

where $\{\nabla_W U, A_V W\}$ and $\left\{h\,(W,U)\,, \nabla^t_W V\right\}$ belong to $\Gamma(T\acute{N})$ and $\Gamma(tr(T\acute{N}))$, respectively. Here, ∇ and ∇^t denote linear connections on $\acute{N}$ and the vector bundle $tr\,(T\acute{N})$, respectively. Moreover, we have

$$\breve{\nabla}_W U = \nabla_W U + h^{\ell}\,(W,U) + h^s\,(W,U)\,, \quad \forall W, U \in \Gamma(T\acute{N}), \tag{2.3}$$

$$\breve{\nabla}_W N = -A_N W + \nabla^{\ell}_W N + D^s\,(W,N)\,, \quad N \in \Gamma(ltr\,T\acute{N}), \tag{2.4}$$

$$\breve{\nabla}_W Z = -A_Z W + \nabla^s_W Z + D^{\ell}\,(W,Z)\,, \quad Z \in \Gamma(S(T\acute{N}^{\perp})). \tag{2.5}$$

Denote the projection of $T\acute{N}$ on $S(T\acute{N})$ by P. Then by using (2.1), (2.3)–(2.5), and the fact that $\breve{\nabla}$ is a metric connection, we obtain

$$\breve{g}(h^s(W,U),Z)+\breve{g}(U,D^\ell(W,Z)) = \breve{g}(A_Z W,U), \tag{2.6}$$

$$\breve{g}(D^s(W,N),Z) = \breve{g}(N,A_Z W). \tag{2.7}$$

From the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_W PU = \nabla^*_W PU + h^*(W,PU), \tag{2.8}$$

$$\nabla_W \xi = -A^*_\xi W + \nabla^{*t}_W \xi, \tag{2.9}$$

for $W,U\in\Gamma(T\acute{N})$ and $\xi\in\Gamma(Rad\,T\acute{N})$. By using above equations, we obtain

$$g\left(h^\ell(W,PU),\xi\right) = g\left(A^*_\xi W,PU\right), \tag{2.10}$$

$$g\left(h^s(W,PU),N\right) = g\left(A_N W,PU\right), \tag{2.11}$$

$$g\left(h^\ell(W,\xi),\xi\right) = 0, \quad A^*_\xi\xi=0. \tag{2.12}$$

In general, the induced connection ∇ on $\acute{N}$ is not a metric connection. Since $\breve{\nabla}$ is a metric connection, by using (2.3) we get

$$(\nabla_W g)(U,V)=\breve{g}\left(h^\ell(W,U),V\right)+\breve{g}\left(h^\ell(W,V),U\right). \tag{2.13}$$

However, we note that ∇^* is a metric connection on $S(T\acute{N})$.

Fix two positive integers p and q. The positive solution of the equation

$$x^2-px-q=0$$

is an entitled member of the metallic means family [9]–[13]. These numbers, denoted by

$$\sigma_{p,q}=\frac{p+\sqrt{p^2+4q}}{2}, \tag{2.14}$$

are called (p,q)-metallic numbers.

Definition 2.1 *A polynomial structure on a manifold $\breve{N}$ is called a metallic structure if it is determined by a $(1,1)$-tensor field $\breve{J}$, which satisfies*

$$\breve{J}^2=p\breve{J}+qI, \tag{2.15}$$

where I is the identity map on $\breve{N}$ and p,q are positive integers. Also, if

$$\breve{g}(\breve{J}W,U)=\breve{g}(W,\breve{J}U) \tag{2.16}$$

holds, then the semi-Riemannian metric $\breve{g}$ is called $\breve{J}$-compatible for every $U,W\in\Gamma(T\breve{N})$. In this case $(\breve{N},\breve{g},\breve{J})$ is called a metallic semi-Riemannian manifold. Also, a metallic semi-Riemannian structure $\breve{J}$ is called a locally metallic structure if $\breve{J}$ is parallel with respect to the Levi-Civita connection $\breve{\nabla}$, that is

$$\breve{\nabla}_W\breve{J}U=\breve{J}\breve{\nabla}_W U \tag{2.17}$$

[6].

If $\breve{J}$ is a metallic structure, then (2.16) is equivalent to

$$\breve{g}(\breve{J}W, \breve{J}U) = p\breve{g}(\breve{J}W, U) + q\breve{g}(W, U), \tag{2.18}$$

for any $W, U \in \Gamma(T\breve{N})$.

3. Radical transversal lightlike submanifolds of metallic semi-Riemannian manifolds

In this section, we introduce radical transversal lightlike submanifolds of a metallic semi-Riemannian manifold.

Definition 3.1 *Let $(\acute{N}, g, S(T\acute{N}), S(T\acute{N}^{\perp}))$ be a lightlike submanifold of a metallic semi-Riemannian manifold $(\breve{N}, \breve{g}, \breve{J})$. If the following conditions are satisfied, then the lightlike submanifold $\acute{N}$ is called a radical transversal lightlike submanifold:*

$$\breve{J}Rad\,T\acute{N} = ltr\,T\acute{N}, \tag{3.1}$$

$$\breve{J}S(T\acute{N}) = S(T\acute{N}). \tag{3.2}$$

Proposition 3.1 *Let $\breve{N}$ be a metallic semi-Riemannian manifold. In this case, there is no 1-radical transversal lightlike submanifold of $\breve{N}$.*

Proof Let $\acute{N}$ be a 1-radical transversal lightlike submanifold. Hence, $RadT\acute{N} = \{\xi\}$ and $ltrT\acute{N} = \{N\}$. From equation (2.18), we have

$$\breve{g}(\breve{J}\xi, \xi) = \breve{g}(\xi, \breve{J}\xi) = 0. \tag{3.3}$$

On the other hand, from (3.1), since $\breve{J}\xi \in \Gamma(T\breve{N})$, we have

$$\breve{g}(\breve{J}\xi, \xi) \neq 0,$$

which contradicts equation (3.3). The proof is completed. □

Theorem 3.1 *Let $\acute{N}$ be a radical transversal lightlike submanifold of a metallic semi-Riemannian manifold $\breve{N}$. In this case, the distribution $S(T\acute{N}^{\perp})$ is invariant with respect to $\breve{J}$.*

Proof For $V \in \Gamma(S(T\acute{N}^{\perp}))$ and $\xi \in \Gamma(Rad\,T\acute{N})$, from (2.18), we find

$$\breve{g}(\breve{J}V, \xi) = \breve{g}(V, \breve{J}\xi) = 0,$$

which implies that there is not a component of $\breve{J}V$ in $ltr\,T\acute{N}$.

Similarly, for $N \in \Gamma(ltr\,T\acute{N})$ from (2.18), we have

$$\breve{g}(\breve{J}V, N) = \breve{g}(V, \breve{J}N) = \frac{1}{p}\breve{g}(\breve{J}V, \breve{J}N). \tag{3.4}$$

From the definition of a radical transversal lightlike submanifold, for $\xi_1 \in \Gamma(Rad\,T\acute{N})$ there exists a $N_1 \in \Gamma(ltr\,T\acute{N})$ such that

$$\breve{J}\xi_1 = N_1.$$

If we apply $\breve{J}$ to the last equation, we can write

$$p\breve{J}\xi_1 + q\xi_1 = \breve{J}N_1,$$

which implies that equation (3.4) equals zero. Namely, we see that there is no component of $\breve{J}V$ in $Rad\,T\acute{N}$.

In a similar way, for $W \in \Gamma(S(T\acute{N}))$, we obtain

$$\breve{g}(\breve{J}V, W) = \breve{g}(V, \breve{J}W) = 0;$$

that is, there is no component of $\breve{J}V$ in $S(T\acute{N})$. Hence, the proof is completed. □

Let $\acute{N}$ be a radical transversal lightlike submanifold of metallic semi-Riemannian manifold $\breve{N}$. Q and T denote projection morphisms in $Rad\,T\acute{N}$ and $S(T\acute{N})$, respectively. For any $W \in \Gamma(T\acute{N})$, we can write

$$W = TW + QW, \tag{3.5}$$

where $TW \in \Gamma(S(T\acute{N}))$ and $QW \in \Gamma(Rad\,T\acute{N})$. By applying $\breve{J}$ to (3.5), we have

$$\breve{J}W = \breve{J}TW + \breve{J}QW. \tag{3.6}$$

Here, if we write $\breve{J}TW = SW$ and $\breve{J}QW = LW$, then (3.6) becomes

$$\breve{J}W = SW + LW, \tag{3.7}$$

where $SW \in \Gamma(S(T\acute{N}))$ and $LW \in \Gamma(ltr\,T\acute{N})$.

Assume $\acute{N}$ to be a radical transversal submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. From (2.17), (2.3), and (2.5), we have

$$\breve{\nabla}_U(SW + LW) = \breve{J}(\nabla_U W + h^l(U, W)) + h^s(U, W)),$$

where $U, W \in \Gamma(T\acute{N})$. If we write $\breve{J}h^l(U, W) = K_1\breve{J}h^l(U, W) + K_2\breve{J}h^l(U, W)$, where K_1 and K_2 are projection morphisms of $\breve{J}ltr\,T\acute{N}$ in $ltr\,T\acute{N}$ and $Rad\,T\acute{N}$, respectively, we find

$$\begin{pmatrix} \nabla_U SW + h^l(U, SW) + h^s(U, SW) \\ -A_{LW}U + \nabla^l_U LW + D^s(U, LW) \end{pmatrix} = \begin{pmatrix} S\nabla_U W + L\nabla_U W + \breve{J}h^s(U, W) \\ +K_1\breve{J}h^l(U, W) + K_2\breve{J}h^l(U, W) \end{pmatrix}.$$

Thus, by equating the tangent, screen transversal, and lightlike transversal parts components, we have

$$\begin{aligned} \nabla_U SW - A_{LW}U &= S\nabla_U W + K_2\breve{J}h^l(U, W), \\ h^s(U, SW) + D^s(U, LW) &= \breve{J}h^s(U, W), \\ h^l(U, SW) + \nabla^l_U LW &= L\nabla_U W + K_1\breve{J}h^l(U, W). \end{aligned}$$

Therefore, we give the following proposition.

Proposition 3.2 *Let $\acute{N}$ be a radical transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. Then we have*

$$(\nabla_U S)W = A_{LW}U + K_2\breve{J}h^l(U,W), \tag{3.8}$$

$$0 = h^S(U,SW) + D^s(U,LW) - \breve{J}h^s(U,W), \tag{3.9}$$

$$0 = h^l(U,SW) + \nabla^l_U LW - L\nabla_U W - K_1\breve{J}h^l(U,W), \tag{3.10}$$

for $W,U \in \Gamma(T\acute{N})$.

Theorem 3.2 *Let $\acute{N}$ be a radical transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. Then the induced connection ∇ on $\acute{N}$ is a metric connection if and only if there is no component of $A_{\breve{J}\xi}W$ in $\Gamma(S(T\acute{N}))$, for $W \in \Gamma(T\acute{N})$ and $\xi \in \Gamma(Rad\,T\acute{N})$.*

Proof Assume that the induced connection ∇ is a metric connection. In this case, for $W \in \Gamma(T\acute{N})$ and $\xi \in \Gamma(Rad\,T\acute{N})$, $\nabla_\xi W \in \Gamma(Rad\,T\acute{N})$. For any $U \in \Gamma(S(T\acute{N}))$, we have

$$g(\nabla_\xi W, U) = \breve{g}(\breve{\nabla}_\xi W, U) = 0.$$

If we use (2.18), we find

$$0 = \breve{g}(\breve{J}\breve{\nabla}_\xi W, \breve{J}U) - p\breve{g}(\breve{\nabla}_\xi W, \breve{J}U),$$

and from (2.4), we have

$$\breve{g}(A_{\breve{J}\xi}W, \breve{J}U) = 0,$$

which implies that there is no component of $A_{\breve{J}\xi}W$ in $\Gamma(S(T\acute{N}))$.

Since the converse is obvious, we omit it. □

Now we shall investigate the conditions for integrability of the distributions involved in the definition of radical transversal lightlike submanifolds.

Theorem 3.3 *Let $\acute{N}$ be a radical transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. In this case, the screen distribution is integrable if and only if*

$$h^l(U,SW) = h^l(W,SU),$$

for $W,U \in \Gamma(S(T\acute{N}))$.

Proof For $W,U \in \Gamma(S(T\acute{N}))$, if we use equation (3.10), and by interchanging the roles of U and W, we find

$$h^l(U,SW) - h^l(W,SU) - K_1\left(\breve{J}h^l(U,W) - \breve{J}h^l(W,U)\right) = L\,[U,W]\,.$$

Since h^l is symmetric, we obtain

$$h^l(U,SW) - h^l(W,SU) = L\,[U,W]\,,$$

which completes the proof. □

Theorem 3.4 *Let $\acute{N}$ be a radical transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. The radical distribution is integrable if and only if*

$$A_{LU}W = A_{LW}U,$$

for $U, W \in \Gamma(Rad\, T\acute{N})$.

Proof For any $U, W \in \Gamma(Rad\, T\acute{N})$, if we use equation (3.8), we have

$$-S\nabla_U W = A_{LW}U + K_2 \breve{J}h^l(U, W),$$

by virtue of $SW = 0$. By changing the roles of U and W, we find

$$S(\nabla_W U - \nabla_U W) = A_{LU}W - A_{LW}U + K_2\left(\breve{J}h^l(W, U) - \breve{J}h^l(U, W)\right).$$

Since h^l is known to be symmetric, we obtain

$$S[W, U] = A_{LU}W - A_{LW}U.$$

Therefore, the proof is completed. □

Theorem 3.5 *Let $\acute{N}$ be a radical transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. Then the radical distribution is defined as a totally geodesic foliation if and only if*

$$h^*(W, \breve{J}Z) = ph^*(W, Z),$$

for $W \in \Gamma(Rad\, T\acute{N})$, $Z \in \Gamma(S(T\acute{N}))$.

Proof By using the definition of a lightlike submanifold, it is known that the radical distribution defines totally geodesic foliation if and only if

$$\breve{g}(\nabla_W U, Z) = 0,$$

for any $W, U \in \Gamma(Rad\, T\acute{N})$ and $Z \in S(T\acute{N})$. Since $\breve{\nabla}$ is a metric connection, if we use (2.3), (2.17), and (2.18), we have

$$\breve{g}(\breve{J}U, \breve{\nabla}_W \breve{J}Z) - p\breve{g}(\breve{J}U, \breve{\nabla}_W Z) = 0.$$

Then, from (2.8), we get

$$\breve{g}\left(\breve{J}U, h^*(W, \breve{J}Z) - ph^*(W, Z)\right) = 0.$$

Hence, the proof is completed. □

Theorem 3.6 *Let $\acute{N}$ be a radical transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. Then the screen distribution defines a totally geodesic foliation if and only if either*

$$h^*(W, \breve{J}U) + K_2 h^l(W, \breve{J}U) = p(h^*(W, U) + K_2 h^l(W, U))$$

or there is no component of $\breve{J}N$ in $ltr\, T\acute{N}$ for $W, U \in \Gamma(S(T\acute{N}))$, $N \in \Gamma(ltr\, T\acute{N})$.

Proof Since the screen distribution defines a totally geodesic foliation if and only if

$$\breve{g}\left(\nabla_W U, N\right) = 0,$$

for any $W, U \in \Gamma(S(T\acute{N}))$, $N \in \Gamma(ltr\, T\acute{N})$. Here, if we use (2.3), then we have

$$\breve{g}(\breve{\nabla}_W U, N) = 0.$$

Also, from (2.18) and (2.17), we have

$$\breve{g}(\breve{\nabla}_W \breve{J}U, \breve{J}N) - p\breve{g}(\breve{\nabla}_W U, \breve{J}N) = 0.$$

By using (2.3) and (2.8) in the last equation, we find

$$\breve{g}(h^*(W, \breve{J}U) + K_2 h^l(W, \breve{J}U), \breve{J}N) - p\breve{g}(h^*(W, U) + K_2 h^l(W, U), \breve{J}N) = 0.$$

Therefore, we conclude. □

Example 3.1 *Let* $(\breve{N} = \mathbb{R}^5_2, \breve{g}, \breve{J})$ *be the* 5*-dimensional semi-Euclidean space with the semi-Euclidean metric of signature* $(-, +, -, +, +)$ *and the structure* $\breve{J}$ *given by*

$$\breve{J}(x_1, x_2, x_3, x_4, x_5) = ((p - \sigma)\, x_1, \sigma x_2, (p - \sigma)\, x_3, \sigma x_4, \sigma x_5),$$

where $(x_1, x_2, x_3, x_4, x_5)$ *is the standard coordinate system of* $\mathbb{R}^5_2$ *. If we take* $\sigma = \frac{p+\sqrt{p^2+4q}}{2}$*, then we have*

$$\breve{J}^2 = p\breve{J} + qI,$$

which implies $\breve{J}$ *is a metallic structure on* $\mathbb{R}^5_2$*. Hence,* $(\breve{N} = \mathbb{R}^5_2, \breve{g}, \breve{J})$ *is a metallic semi-Riemannian manifold. Let* $\acute{N}$ *be a submanifold in* $\breve{N}$ *defined by*

$$x_2 = 0, \quad x_4 = \sigma x_1 + \sigma x_3.$$

Then $T\acute{N} = Sp\{W_1, W_2, W_3\}$*, where*

$$W_1 = \frac{\partial}{\partial x_1} + \sigma \frac{\partial}{\partial x_4}, \quad W_2 = \frac{\partial}{\partial x_3} + \sigma \frac{\partial}{\partial x_4}, \quad W_3 = \frac{\partial}{\partial x_5}.$$

It is easy to check that $\acute{N}$ *is a lightlike submanifold. Therefore,*

$$\begin{aligned} Rad\, T\acute{N} &= Sp\{\xi = \sigma W_1 - \sigma W_2 + \sigma\sqrt{2} W_3\}, \\ ltr\, T\acute{N} &= Sp\left\{N = \frac{1}{2\sigma^2\,(2\sigma - p)}\left((p - \sigma)\, \frac{\partial}{\partial x_1} - (p - \sigma)\, \frac{\partial}{\partial x_3} + \sigma\sqrt{2}\frac{\partial}{\partial x_5}\right)\right\}, \\ S(T\acute{N}) &= Sp\{W_3\}, \end{aligned}$$

and we have

$$N = \frac{1}{4\sigma^3}\breve{J}\xi.$$

That is, $\breve{J}\xi \in \Gamma(ltr\, T\acute{N})$ *and* $\breve{J}W_3 = \sigma W_3 \in S(T\acute{N})$*. Therefore,* $\acute{N}$ *is a radical transversal lightlike submanifold of* $(\breve{N} = \mathbb{R}^5_2, \breve{g}, \breve{J})$*.*

4. Transversal lightlike submanifolds of metallic semi-Riemannian manifolds

In this section, we give a definition of transversal lightlike submanifolds and investigate the geometry of distributions.

Definition 4.1 *Let $(\acute{N}, g, S(T\acute{N}), S(T\acute{N}^{\perp}))$ be a lightlike submanifold of a metallic semi-Riemannian manifold $(\breve{N}, \breve{g}, \breve{J})$. If the following conditions are satisfied, then the lightlike submanifold $\acute{N}$ is called a transversal lightlike submanifold:*

$$\breve{J} Rad\, T\acute{N} = ltr\, T\acute{N},$$
$$\breve{J}(S(T\acute{N})) \subseteq S(T\acute{N}^{\perp}).$$

We shall denote the orthogonal complement subbundle to $\breve{J}(S(T\acute{N}))$ in $S(T\acute{N}^{\perp})$ by μ.

Proposition 4.1 *Let $\acute{N}$ be a transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. In this case, the distribution μ is invariant according to $\breve{J}$.*

Proof For $V \in \Gamma(\mu)$, $\xi \in \Gamma(Rad\, T\acute{N})$, and $N \in \Gamma(ltr\, T\acute{N})$, from (2.15), (2.16), and (2.18), we have

$$\breve{g}(\breve{J}V, \xi) = \breve{g}(V, \breve{J}\xi) = 0, \tag{4.1}$$

and

$$\breve{g}(\breve{J}V, N) = \breve{g}(V, \breve{J}N) = 0. \tag{4.2}$$

Therefore, there is no component of $\breve{J}V$ in $Rad\, T\acute{N}$ and $ltr\, T\acute{N}$.

Similarly, for $W \in \Gamma(S(T\acute{N}))$ and $V_1 \in \Gamma(\breve{J}S(T\acute{N}^{\perp}))$, we have

$$\breve{g}(\breve{J}V, W) = \breve{g}(V, \breve{J}W) = 0 \tag{4.3}$$

and

$$\breve{g}(\breve{J}V, V_1) = \breve{g}(V, \breve{J}V_1) = 0, \tag{4.4}$$

which imply that there is no component of $\breve{J}V$ in $S(T\acute{N})$ and $\breve{J}(S(T\acute{N}))$. From (4.1), (4.2), (4.3), and (4.4), we conclude. □

Proposition 4.2 *There does not exist a 1-lightlike transversal lightlike submanifold of a locally metallic semi-Riemannian manifold.*

Proof Assume that $\acute{N}$ is a 1-lightlike transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. In this case, $Rad\, T\acute{N} = Sp\{\xi\}$ and $ltr\, T\acute{N} = Sp\{N\}$. From (2.15) and (2.18), we obtain

$$\breve{g}(\breve{J}\xi, \xi) = \breve{g}(\xi, \breve{J}\xi) = 0. \tag{4.5}$$

On the other hand, from the fact that $\breve{J} Rad\, T\acute{N} = ltr\, T\acute{N}$, we have $\breve{J}\xi \in \Gamma(ltr\, T\acute{N})$ and so we find

$$\breve{g}(\xi, \breve{J}\xi) \neq 0,$$

which contradicts (4.5). The proof is completed. □

From Definition 4.1 and Proposition 4.2, we have:

Corollary 4.1 *Let $\acute{N}$ be a transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. Then:*

(i) $\dim(Rad\,T\acute{N}) \geq 2$,

(ii) *The transversal lightlike submanifold of* 3*-dimensional is* 2*-lightlike.*

Let $\acute{N}$ be a transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. Q and T are projection morphisms in $Rad\,T\acute{N}$ and $S(T\acute{N})$, respectively. For any $W \in \Gamma(T\acute{N})$, we can write

$$W = TW + QW, \tag{4.6}$$

where $TW \in \Gamma(S(T\acute{N}))$ and $QW \in \Gamma(Rad\,T\acute{N})$. If we apply $\breve{J}$ to (4.6), we have

$$\breve{J}W = \breve{J}TW + \breve{J}QW. \tag{4.7}$$

By writing $\breve{J}TW = KW$ and $\breve{J}QW = LW$, expression (4.7) is

$$\breve{J}W = KW + LW. \tag{4.8}$$

Here, $KW \in \Gamma(S(T\acute{N}^{\perp}))$ and $LW \in \Gamma(ltr\,T\acute{N})$. Besides, let D and E be projection morphisms in $\breve{J}S(T\acute{N})$ and μ in $S(T\acute{N}^{\perp})$, respectively. For $V \in \Gamma(S(T\acute{N}^{\perp}))$, we write

$$V = DV + EV. \tag{4.9}$$

By applying $\breve{J}$ to (4.9), we have

$$\breve{J}V = \breve{J}DV + \breve{J}EV. \tag{4.10}$$

If we write $\breve{J}DV = BV$ and $\breve{J}EV = CV$, expression (4.10) becomes

$$\breve{J}V = BV + CV, \tag{4.11}$$

where $BV \in \breve{J}S(T\acute{N}) \oplus S(T\acute{N})$, $CV \in \Gamma(\mu)$. Since $\breve{N}$ is a locally metallic semi-Riemannian manifold, then from (2.3), (2.5), and (4.8), we have

$$\begin{pmatrix} -A_{KW}U + \nabla^s_U KW + D^l(U, KW) \\ -A_{LW}U + \nabla^l_U LW + D^s(U, LW) \end{pmatrix} = \begin{pmatrix} K\nabla_U W + L\nabla_U W + \breve{J}h^l(U, W) \\ +Bh^s(U, W) + Ch^s(U, W) \end{pmatrix}, \tag{4.12}$$

where $U, W \in \Gamma(T\acute{N})$. For projection morphisms K_1 and K_2 of $\breve{J}ltr\,T\acute{N}$ in $ltr\,T\acute{N}$ and $Rad\,T\acute{N}$, respectively, we write

$$\breve{J}h^l(U, W) = K_1\breve{J}h^l(U, W) + K_2\breve{J}h^l(U, W).$$

Also, for projection morphisms S_1 and S_2 of $\breve{J}S(T\acute{N}^{\perp})$ in $\breve{J}S(T\acute{N}) \subseteq S(T\acute{N}^{\perp})$ and $S(T\acute{N})$, we have

$$Bh^s(U, W) = S_1Bh^s(U, W) + S_1Bh^s(U, W).$$

Therefore, (4.12) can be rewritten as

$$\begin{pmatrix} -A_{KW}U + \nabla^s_U KW + D^l(U, KW) \\ -A_{LW}U + \nabla^l_U LW + D^s(U, LW) \end{pmatrix} = \begin{pmatrix} K\nabla_U W + L\nabla_U W + K_1\breve{J}h^l(U, W) \\ +K_2\breve{J}h^l(U, W) + S_1Bh^s(U, W) \\ +S_2Bh^s(U, W) + Ch^s(U, W) \end{pmatrix}.$$

If we equate the tangent and transversal parts of the above equation, we get

$$-A_{KW}U - A_{LW}U = K_2\breve{J}h^l(U,W) + S_2Bh^s(U,W), \quad (4.13)$$

$$\nabla^s_U KW + D^s(U,LW) = \begin{pmatrix} K\nabla_U W + S_1Bh^s(U,W) \\ +Ch^s(U,W) \end{pmatrix}, \quad (4.14)$$

$$D^l(U,KW) + \nabla^l_U LW = L\nabla_U W + K_1\breve{J}h^l(U,W). \quad (4.15)$$

Now we shall investigate the integrable of the distributions on transversal lightlike submanifolds.

Theorem 4.1 *Let $\acute{N}$ be a transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. Then the radical distribution is integrable if and only if*

$$D^s(U,LW) = D^s(W,LU),$$

for $U,W \in \Gamma(Rad\,T\acute{N})$.

Proof For $U,W \in \Gamma(Rad\,T\acute{N})$, from equation (4.14), by interchanging the roles of W and U, we find

$$\nabla^s_U KW - \nabla^s_W KU + D^s(U,LW) - D^s(W,LU) - K(\nabla_U W - \nabla_W U) = 0,$$

since h^s is symmetric. Also, we have $\nabla^s_U KW = \nabla^s_W KU = 0$. Then we get

$$D^s(U,LW) - D^s(W,LU) = K[U,W],$$

which completes the proof . □

Theorem 4.2 *Let $\acute{N}$ be a transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. Then the screen distribution is integrable if and only if*

$$D^l(U,KW) = D^l(W,KU),$$

$W,U \in \Gamma(S(T\acute{N}))$.

Proof From (4.15), the fact that h^l is symmetric, and $LW = LU = 0$, we have

$$D^l(U,KW) - D^l(W,KU) = L[U,W],$$

by interchanging the roles of $W,U \in \Gamma(S(T\acute{N}))$. Thus, the proof is completed. □

Theorem 4.3 *Let $\acute{N}$ be a transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. Then the screen distribution defines a totally geodesic foliation if and only if $D^l(W,\breve{J}U) = -ph^l(W,U)$, $h^*(W,U) = 0$, and there is no component of $A_{\breve{J}U}W$ in $Rad\,T\acute{N}$, for $W,U \in \Gamma(S(T\acute{N}))$, $N \in \Gamma(ltr\,T\acute{N})$.*

Proof By the definition of a lightlike submanifold, it is known that $S(T\acute{N})$ defines a totally geodesic foliation if and only if

$$\breve{g}(\nabla_W U, N) = 0,$$

where $W,U \in \Gamma(S(T\acute{N}))$ and $N \in \Gamma(ltr\,T\acute{N})$. If we use (2.16), (2.17), and (2.18), we find

$$0 = \breve{g}(\breve{\nabla}_W \breve{J}U, \breve{J}N) - p\breve{g}(\breve{\nabla}_W U, \breve{J}N).$$

Since $\breve{J}U \in \Gamma(S(T\acute{N}^{\perp}))$ and from equation (2.3) and (2.5), we have

$$\breve{g}(-A_{\breve{J}U}W + D^l(W, \breve{J}U), \breve{J}N) - p\breve{g}(\nabla_W U + h^l(W, U), \breve{J}N) = 0.$$

Then, by using (2.8), we obtain

$$\breve{g}(-A_{\breve{J}U}W + D^l(W, \breve{J}U) - ph^*(W, U) - ph^l(W, U), \breve{J}N) = 0,$$

which completes the proof. □

Theorem 4.4 *Let $\acute{N}$ be a transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. Then the radical distribution defines a totally geodesic foliation if and only if there is no component in $Rad\,T\acute{N}$ of $A_{\breve{J}Z}W$, that is, either $K_2\breve{J}h^l(W,Z) = 0$ or $-A_{\breve{J}Z}W = S_2Bh^s(W,Z)$, for $W, U \in \Gamma(Rad\,T\acute{N})$, $Z \in \Gamma(S(T\acute{N}))$.*

Proof The radical distribution defines a totally geodesic foliation if and only if

$$\breve{g}\left(\nabla_W U, Z\right) = 0,$$

for $W, U \in \Gamma(Rad\,T\acute{N})$ and $Z \in S(T\acute{N})$. From (2.3), we find

$$\breve{g}\left(\nabla_W U, Z\right) = \breve{g}(\breve{\nabla}_W U, Z) = 0.$$

Since $\breve{\nabla}$ is a metric connection, from (2.16), (2.17), and (2.18), we have

$$0 = -\breve{g}(\breve{J}U, \breve{\nabla}_W \breve{J}Z) + p\breve{g}(\breve{J}U, \breve{\nabla}_W Z).$$

For $\breve{J}Z \in \Gamma(S(T\acute{N}^{\perp}))$, from (2.5) and (2.8), we get

$$0 = \breve{g}(\breve{J}U, A_{\breve{J}Z}W) + p\breve{g}(\breve{J}U, h^*(W, Z)).$$

Here, since $\breve{J}U \in \Gamma(ltr\,T\acute{N})$, we conclude that either there is no component of $A_{\breve{J}Z}W$ in $Rad\,T\acute{N}$ or, by changing the roles of U and W and taking $U = Z$, we have $K_2\breve{J}h^l(W, Z) = 0$, by virtue of

$$-A_{\breve{J}Z}W = S_2Bh^s(W, Z).$$

□

Theorem 4.5 *Let $\acute{N}$ be a transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\breve{N}$. Then the induced connection on $\acute{N}$ is a metric connection if and only if*

$$Q_1\breve{J}D^s(W, \breve{J}\xi) = pM_1\breve{J}h^s(W, \xi),$$

for $W \in \Gamma(T\acute{N}), \xi \in \Gamma(Rad\,T\acute{N})$.

Proof For $W \in \Gamma(T\acute{N})$ and $\xi \in \Gamma(Rad\,T\acute{N})$, we have

$$\breve{\nabla}_W \breve{J}\xi = \breve{J}\breve{\nabla}_W \xi.$$

From equations (2.3) and (2.4), we write

$$-A_{\breve{J}\xi}W + \nabla^l_W \breve{J}\xi + D^s(W, \breve{J}\xi) = \breve{J}\left(\nabla_W \xi + h^l(W, \xi) + h^s(W, \xi)\right).$$

If we apply $\breve{J}$ to the above equation and use (2.15), (4.8), and (4.11), we obtain

$$\left(\begin{array}{c} -KA_{\breve{J}\xi}W - LA_{\breve{J}\xi}W \\ +T_1\breve{J}\nabla^l_W\breve{J}\xi + T_2\breve{J}\nabla^l_W\breve{J}\xi \\ +Q_1\breve{J}D^s\left(W,\breve{J}\xi\right) \\ +Q_2\breve{J}D^s\left(W,\breve{J}\xi\right) \end{array}\right) = \left(\begin{array}{c} p\breve{J}\nabla_W\xi + q\nabla_W\xi + \\ p\breve{J}h^l(W,\xi) + qh^l(W,\xi) \\ +p\breve{J}h^s(W,\xi) + qh^s(W,\xi) \end{array}\right), \tag{4.16}$$

for $W \in \Gamma(S(T\acute{N}^{\perp}))$, where T_1 and T_2 are projection morphisms of $\breve{J}\nabla^l_W\breve{J}\xi$ in $Rad\,T\acute{N}$ and $ltr\,T\acute{N}$, respectively. Then we have

$$\breve{J}\nabla^l_W\breve{J}\xi = T_1\breve{J}\nabla^l_W\breve{J}\xi + T_2\breve{J}\nabla^l_W\breve{J}\xi.$$

Also, for projection morphisms M_1 and M_2 of $\breve{J}h^s(W,\xi)$ in $S(T\acute{N})$ and $\breve{J}S(T\acute{N})$, respectively, then we have

$$\breve{J}h^s(W,\xi) = M_1\breve{J}h^s(W,\xi) + M_2\breve{J}h^s(W,\xi).$$

Additionally, we get

$$\breve{J}D^s(W,\breve{J}\xi) = Q_1\breve{J}D^s(W,\breve{J}\xi) + Q_2\breve{J}D^s(W,\breve{J}\xi),$$

where Q_1 and Q_2 are projection morphisms of $\breve{J}D^s(W,\breve{J}\xi)$ in $S(T\acute{N})$ and $S(T\acute{N}^{\perp})$, respectively. By equating the tangent parts in equation (4.16), we find

$$\frac{1}{q}\left(T_1\breve{J}\nabla^l_W\breve{J}\xi + Q_1\breve{J}D^s(W,\breve{J}\xi) - pM_1\breve{J}h^s(W,\xi) - pK_2\breve{J}h^l(W,\xi)\right) = \nabla_W\xi.$$

Therefore, $\nabla_W\xi$ belongs to $Rad\,T\acute{N}$ if and only if

$$Q_1\breve{J}D^s(W,\breve{J}\xi) = pM_1\breve{J}h^s(W,\xi).$$

This completes the proof. □

Example 4.1 *Let* $(\breve{N} = R^6_2, \breve{g}, \breve{J})$ *be the* 6*-dimensional semi-Euclidean space with the semi-Euclidean metric of signature* $(-,+,+,-,+,+)$ *and the structure*

$$\breve{J}(x_1,x_2,x_3,y_1,y_2,y_3) = ((p-\sigma)x_1, \sigma x_2, \sigma x_3, (p-\sigma)y_1, \sigma y_2, \sigma y_3),$$

where $(x_1,x_2,x_3,y_1,y_2,y_3)$ *is the standard coordinate system of* R^6_2*. If we take* $\sigma = \frac{p+\sqrt{p^2+4q}}{2}$*, then we have*

$$\breve{J}^2 = p\breve{J} + qI,$$

which implies that $\breve{J}$ *is a metallic structure on* R^6_2*. Thus,* $(\breve{N} = R^6_2, \breve{g}, \breve{J})$ *is a metallic semi-Riemannian manifold. Let* $\acute{N}$ *be a submanifold in* $\breve{N}$ *defined by*

$$x^1 = y^2, \quad x^2 = y^1, \quad x^3 = y^3.$$

Then $T\acute{N} = Span\{W_1, W_2, W_3\}$*, where*

$$W_1 = \partial x_1 + \partial y_2, \qquad W_2 = \partial x_2 + \partial y_1, \qquad W_3 = \partial x_3 + \partial y_3.$$

It is easy to check that $\acute{N}$ *is a lightlike submanifold. Therefore,*

$$\begin{aligned} RadT\acute{N} &= Span\{\xi = W_1 + W_2\}, \\ ltrT\acute{N} &= Span\{N = \frac{1}{2p(2\sigma - p)}\left(\begin{array}{c} ((p-\sigma)^2 - q)\partial x_1 + (\sigma^2 - q)\partial x_2 \\ +((p-\sigma)^2 - q)\partial y_{1+}(\sigma^2 - q)\partial y_2 \end{array}\right)\}, \\ S(T\acute{N}) &= Span\{W = \frac{1}{\sqrt{2}}W_3\}, \end{aligned}$$

and we have $N = \breve{J}\xi$, *for* $2\sigma^2 - 2p\sigma + p^2 - 2q = 0$, *where* $\breve{J}^2\xi = p\breve{J}\xi + q\xi$. *Also, screen transversal bundle* $S(T\acute{N}^{\perp})$ *is spanned by* $V = \frac{1}{\sqrt{2}}(\sigma\partial x_3 - \sigma\partial y_3)$ *and* $\breve{J}W \subseteq S(T\acute{N}^{\perp})$. *Therefore,* $\acute{N}$ *is a transversal lightlike submanifold of* $(\breve{N} = R^6_2, \breve{g}, \breve{J})$.

References

[1] Acet BE. Lightlike hypersurfaces of metallic semi-Rieamnnian Manifolds. Int J Geom Meth M (in press).

[2] Acet BE, Perktaş SY. Screen slant radical transversal null submanifolds of para-Sasakian manifolds. Facta Universitatis Series Mathematics and Informatics 2016; 31: 543-557.

[3] Acet BE, Perktaş SY, Kılıç E. Symmetric lightlike hypersurfaces of a para-Sasakian space form. An Sitiint Univ Al I Cuza Iaşi Mat (N.S.) Tomul LXII 2016; 3: 915-926

[4] Acet BE, Perktaş SY. On geodesic paracontact CR-lightlike submanifolds. British Journal of Mathematics & Computer Science 2016; 14: 1-11.

[5] Blaga AM, Hretcanu CE. Invariant, anti-invariant and slant submanifolds of a metallic Riemannian manifold. arXiv: 1803.01415.

[6] Crasmareanu M, Hretcanu CE. Golden differential geometry. Chaos Soliton Fract 2008; 38: 1229-1238.

[7] Crasmareanu M, Hretcanu CE. Applications of the Golden Ratio on Riemannian Manifolds. Turk J Math 2009; 33: 179-191.

[8] Crasmareanu M, Hretcanu CE. On some invariant submanifolds in a Riemannian manifold with golden structure. An Stiins Univ Al I Cuza Iasi Mat (N.S.) 2007; 53: 199-211.

[9] De Spinadel VW. On characterization of the onset to chaos. Chaos Soliton Fract 1997; 8: 1631-1643.

[10] De Spinadel VW. The metallic means family and multifractal spectra. Nonlinear Anal-Real 1999; 36: 721-745.

[11] De Spinadel VW. The Family of Metallic Means. 1999. http://vismath1.tripod.com/spinadel/.

[12] De Spinadel VW. The metallic means family and forbidden symmetries. Int Math J 2002; 2: 279-288.

[13] De Spinadel VW. The metallic means family and renormalization group techniques. P Steklov Inst Math 2000; 1: 194-209.

[14] Duggal KL, Bejancu A. Lightlike Submanifolds of Semi-Riemannian Manifolds and Its Applications. Dordrecht, the Netherlands: Kluwer, 1996.

[15] Duggal KL, Şahin B. Differential Geometry of Lightlike Submanifolds. Basel, Switzerland: Birkhauser, 2010.

[16] Erdoğan FE, Güneş R, Şahin B. Half-lightlike submanifold with planar normal sections in R^4_2. Turk J Math 2014; 38: 764-777.

[17] Erdoğan FE, Şahin B, Güneş R. Lightlike surfaces with planar normal sections in Minkowski 3-spaces. International Electronic Journal of Geometry 2015; 7: 133-142.

[18] Erdoğan FE, Yıldırım C. Lightlike submanifolds with planar normal section in semi-Riemannian product manifolds. International Electronic Journal of Geometry 2016; 9: 70-77.

[19] Erdoğan FE, Yıldırım C. Semi-invariant submanifolds of Golden Riemannian manifolds. In: AIP Conference Proceedings; 2017.

[20] Erdoğan FE, Yıldırım C. On a study of the totally umbilical semi-invariant submanifolds of golden Riemannian manifolds. Politeknik Dergisi 2018; 21: 967-970.

[21] Gezer AKÇ. On metallic Riemannian structures. Turk J Math 2015; 39: 954-962.

[22] Gezer A, Cengiz N, Salimov A. On integrability of Golden Riemannian structures. Turk J Math 2013; 37: 693-703.

[23] Goldberg SI, Petridis NC. Differentiable solutions of algebraic equations on manifolds. Kodai Math Sem Rep 1973; 25: 111-128.

[24] Goldberg SI, Pedridis NC. Differentiable solutions of algebraic on manifolds. Kodai Math Sem Rep 1973; 25: 111-128.

[25] Goldberg SI, Yano K. Polynomial structures on manifolds. Kodai Math Sem Rep 1970; 22: 199-218.

[26] Hretcanu CE, Blaga AM. Submanifolds in metallic Riemannian manifolds. arXiv:1803.02184, 2018.

[27] Hretcanu CE, Blaga AM. Slant and semi-slant submanifolds in metallic Riemannian manifolds. arXiv:1803.03034, 2018.

[28] Hretcanu CE, Crasmareanu M. Metallic structure on Riemannian manifolds. Rev Un Mat Argentina 2013; 54: 15-27.

[29] Kılıç E, Şahin. Radical anti-invariant lightlike submanifolds of Semi-Riemannian Product Manifolds. Turk J Math 2008; 32: 429-449.

[30] Özgür C, Özgür NY. Classification of metallic shaped hypersurfaces in real space forms. Turk J Math 2015; 39: 784-794.

[31] Özgür C, Özgür NY. Metallic shaped hypersurfaces in Lorentzian space forms. Rev Un Mat Argentina 2017; 58: 215-226.

[32] Özkan M. Prolongations of golden structures to tangent bundles. Differ Geom Dyn Syst 2014; 16: 227-238.

[33] Perktaş SY, Kılıç E, Acet BE. Lightlike hypersurfaces of para-Sasakian space form. Gulf J Math 2014; 2: 333-342.

[34] Poyraz NÖ, Yaşar E. Lightlike hypersurfaces of a golden semi-Riemannian manifold. Mediterr J Math 2017; 14: 204.

[35] Şahin B. Almost poly-Norden manifolds. International Journal of Maps in Mathematics 2015; 1: 68-79.

[36] Şahin B, Akyol MA. Golden maps between Golden Riemannian manifolds and constancy of certain maps. Math Commun 2014; 19: 333-342.

[37] Yıldırım C, Şahin B. Screen transversal lightlike submanifolds of indefinite Sasakian manifolds. An Şt Univ Ovidius Constanta 2010; 18: 315-336.