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Curves whose pseudo spherical indicatrices are elastic

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Abstract: The pseudo spherical indicatrix of a curve in Minkowski 3-space emerges as three types: the pseudo spherical tangent indicatrix, principal normal indicatrix, and binormal indicatrix of the curve. The pseudo spherical tangent, principal normal, and binormal indicatrix of a regular curve may be positioned on De Sitter 2-space (pseudo sphere), pseudo hyperbolic 2-space, and two-dimensional null cone in terms of causal character of the curve. In this paper, we separately derive Euler–Lagrange equations of all pseudo spherical indicatrix elastic curves in terms of the causal character of a curve in Minkowski 3-space. Then we give some results of solutions of these equations.

Key words: Elastic curve, Euler–Lagrange equation, pseudo spherical indicatrix

1. Introduction

An elastic curve γ minimizes the bending energy

$$\mathcal{F}(\gamma) = \int_{\gamma} \kappa^2(s)ds \tag{1}$$

with fixed length and boundary conditions. Euler–Lagrange equations corresponding to critical points of the functional (1) are given by

$$\begin{aligned} 2\kappa'' + \kappa^3 - 2\kappa\tau^2 - \lambda\kappa &= 0, \\ \kappa\tau' + 2\kappa'\tau &= 0, \end{aligned}$$

where κ and τ are curvature and torsion of γ, respectively. These equations can be solved by using Jacobi elliptic functions in Euclidean 3-space [5, 12]. Elastic curves (spherical elastic curves) have been characterized on the sphere in Euclidean 3-space. Brunnett and Crouch classified the forms of spherical elastic curves based on the differential equation

$$2\kappa_g'' + \kappa_g^3 + \left(\frac{2}{r^2} - \rho\right)\kappa_g = 0,$$

where κ_g is the geodesic curvature, ρ is the tension parameter, and r is the radius of the sphere [1]. Elastic curves have also been studied in Minkowski 3-space and are characterized as follows:

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i) A timelike elastic curve satisfies the following Euler–Lagrange equations:

$$\begin{aligned} 2\kappa'' - \kappa^3 - 2\kappa\tau^2 + \lambda\kappa &= 0, \\ \kappa\tau' + 2\kappa'\tau &= 0. \end{aligned}$$

ii) A spacelike elastic curve is characterized by:

$$\begin{aligned} 2\kappa'' - \kappa^3 + 2\kappa\tau^2 + \lambda\kappa &= 0, \\ \kappa\tau' + 2\kappa'\tau &= 0. \end{aligned}$$

iii) A null elastic curve satisfies the following equation:

$$\tau = 0$$

[11]. On the other hand, nonnull elastic curves are investigated on De Sitter 2-space (the pseudo sphere) $\mathbb{S}_1^2$ and the pseudo hyperbolic 2-space $\mathbb{H}_0^2$ known as hyperquadrics in Minkowski 3-space. Characterization equations for nonnull elastic curves on the De Sitter 2-space $\mathbb{S}_1^2$ are derived as follows:
i) For the timelike curves,

$$2\kappa_g'' - \kappa_g^3 + (\rho - \frac{2}{r^2})\kappa_g = 0;$$

ii) For the spacelike curves,

$$2\kappa_g'' - \kappa_g^3 + (\rho + \frac{2}{r^2})\kappa_g = 0$$

[14]. In [8], Euler–Lagrange equations were derived for elastic curves on the pseudo hyperbolic 2-space $\mathbb{H}_0^2$ in the following way:

$$2\kappa_g'' + \kappa_g^3 - (\rho + \frac{2}{r^2})\kappa_g = 0.$$

Özkan Tükel and Yücesan also gave the following theorem, which determines elastic curves in the 2-dimensional null cone $\mathbb{Q}^2$.

Theorem 1 *γ is an elastic curve on a two-dimensional null cone $\mathbb{Q}^2$ if and only if the cone curvature of spacelike curve γ is a constant that is equal to the negative half of the tension parameter [10].*

The authors in [9] characterize the tangent, principal normal, and binormal indicatrix elastic curve in Euclidean 3-space as follows:

i) For spherical tangent indicatrix of a unit speed curve:

Theorem 2 *If the spherical tangent indicatrix of a unit speed curve in Euclidean 3-space is an elastic curve, the equation $\frac{\tau}{\kappa} = const.$ must be satisfied; that is, the curve must be a general helix.*

ii) For spherical binormal indicatrix of a unit speed curve:

Theorem 3 *If the spherical binormal indicatrix of a unit speed curve in Euclidean 3-space is an elastic curve, the equation $\frac{\tau}{\kappa} = const.$ must be satisfied; that is, the curve must be a general helix.*

iii) For spherical principal normal indicatrix of a unit speed curve:

Theorem 4 *Any curve whose spherical principal normal indicatrix is elastic can be determined by the Euler–Lagrange equation:*

$$
\begin{aligned}
&\frac{1}{\sqrt{\kappa^2+\tau^2}}\Bigg(-\bigg(\Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'\frac{1}{\sqrt{\kappa^2+\tau^2}}\bigg)'\frac{1}{\sqrt{\kappa^2+\tau^2}}+\frac{1}{\sqrt{\kappa^2+\tau^2}}\Bigg)'-\bigg(\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'\frac{\kappa}{(\kappa^2+\tau^2)\tau}-\Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'\frac{1}{\kappa^2+\tau^2}\bigg)\\
&\times\bigg(\Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'\frac{\kappa}{\kappa^2+\tau^2}+\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'\frac{\tau}{\kappa^2+\tau^2}\bigg)'-\frac{1}{\kappa^2+\tau^2}\frac{\kappa}{\tau}\Bigg(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}-\bigg(\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'\frac{1}{\sqrt{\kappa^2+\tau^2}}\bigg)'\frac{1}{\sqrt{\kappa^2+\tau^2}}\Bigg)'\\
&+\bigg(-\Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'\frac{\kappa}{\kappa^2+\tau^2}+\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'\frac{\kappa^2}{(\kappa^2+\tau^2)\tau}\bigg)\times\Bigg(\bigg(-\Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'\frac{1}{\kappa^2+\tau^2}\bigg)'\frac{1}{\sqrt{\kappa^2+\tau^2}}+\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Bigg)\\
&-\bigg(-\frac{\kappa}{\kappa^2+\tau^2}\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'+\Big(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\Big)'\frac{\tau}{\kappa^2+\tau^2}\bigg)\times\Bigg(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}-\bigg(\Big(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big)'\frac{1}{\sqrt{\kappa^2+\tau^2}}\bigg)'\frac{1}{\sqrt{\kappa^2+\tau^2}}\Bigg)=0.
\end{aligned}
$$

The organization of the paper: We obtain some characterizations for elastic curves, which are pseudo spherical indicatrices of a timelike, a spacelike, and a null curve. We separately derive Euler–Lagrange equations corresponding to the pseudo spherical indicatrix elastic curve of a timelike, a spacelike, and a null curve in three subsections. We solve these problems for pseudo spherical tangent and binormal indicatrix elastic curves and give special solutions for the pseudo spherical principal normal indicatrix.

2. Pseudo spherical indicatrices of regular curves in Minkowski 3-space

Pseudo spherical indicatrices of a regular curve in Minkowski 3-space lie on De Sitter 2-space $\mathbb{S}_1^2$, pseudo hyperbolic 2-space $\mathbb{H}_0^2$, and two-dimensional null cone $\mathbb{Q}^2$ in terms of the causal character of the curve. We next investigate all of the indicatrices types, but we first give the Frenet frame field.

Let $\gamma = \gamma(s) : I \subset \mathbb{R} \to \mathbb{R}_1^3$ be a nonnull unit-speed curve in Minkowski 3-space $\mathbb{R}_1^3$. At a point $\gamma(s)$ of γ, let $T = \gamma^{'}(s)$ denote the unit tangent vector to γ, and $N(s)$ denote the unit principal normal and $\varepsilon_2 B(s) = T(s) \times N(s)$ is the unit binormal vector. Then $\{T, N, B\}$ is an orthonormal basis known as the Frenet frame along γ for all vectors at $\gamma(s)$ on γ. The derivative equations of Frenet frame $\{T, N, B\}$ are given by

$$
\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_1\kappa & 0 \\ -\varepsilon_0\kappa & 0 & \varepsilon_2\tau \\ 0 & -\varepsilon_1\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2}
$$

where $\varepsilon_0 =< T, T >$, $\varepsilon_1 =< N, N >$, and $\varepsilon_2 =< B, B >$. Also, $\kappa > 0$ and τ are the curvature and torsion of γ, respectively [6, 7].

2.1. The pseudo spherical indicatrix of a timelike curve

The pseudo spherical tangent indicatrix of a timelike curve γ is a curve $\sigma(s_t) = T(s)$, so $\sigma(s_t)$ lies on the pseudo hyperbolic 2-space $\mathbb{H}_0^2$. In this subsection, we investigate extremals of the functional (1) defined on pseudo spherical indicatrices curves under some boundary conditions. We first study the pseudo spherical tangent indicatrix elastic curve and later we give some results of the other types of indicatrix elastic curves.

Now we calculate some derivatives of $\sigma(s_t)$. The first derivative of σ is obtained as

$$
\frac{d\sigma(s_t)}{ds} = \frac{dT(s)}{ds}, \tag{3}
$$

$$\frac{d\sigma(s_t)}{ds_t}\frac{ds_t}{ds} = \kappa(s)N(s).$$

Then we get

$$\frac{ds_t}{ds} = \kappa(s).$$

σ is a spacelike curve, since N is a spacelike vector field. In the following, we calculate other derivatives of σ:

$$\frac{d^2\sigma}{ds_t^2} = T(s) + (\frac{\tau(s)}{\kappa(s)})B(s),$$

$$\frac{d^3\sigma}{ds_t^3} = (1 - (\frac{\tau(s)}{\kappa(s)})^2)N(s) + (\frac{\tau(s)}{\kappa(s)})' \frac{1}{\kappa(s)}B(s), \tag{4}$$

and

$$\begin{aligned}\frac{d^4\sigma}{ds_t^4} =& (1 - (\frac{\tau(s)}{\kappa(s)})^2)T - 3(\frac{\tau(s)}{\kappa(s)})'(\frac{\tau(s)}{\kappa^2(s)})N(s) \\ &+ [(1 - (\frac{\tau(s)}{\kappa(s)})^2)\frac{\tau(s)}{\kappa(s)} + ((\frac{\tau(s)}{\kappa(s)})'\frac{1}{\kappa(s)})'\frac{1}{\kappa(s)}]B(s)\end{aligned} \tag{5}$$

On the other hand, we define a set Ω of all smooth pseudo tangent spherical indicatrix curves with the fixed initial and end points and direction, respectively. Then $\mathcal{F}^\Lambda : \Omega \to \mathbb{R}$ is defined by

$$\mathcal{F}^\Lambda(\sigma) = \frac{1}{2}\int_\sigma \left(\left\| \frac{d^2\sigma}{ds_t^2} \right\|^2 + \Lambda(\left\| \frac{d\sigma}{ds_t} \right\|^2 - 1) \right) ds_t.$$

According to the Lagrange multiplier principle, a critical point of $\mathcal{F}$ on the subset Ω_u, defined as all unit speed curves on Ω, is a stationary point for $\mathcal{F}^\Lambda$ for some Λ, which is a Lagrange multiplier.

Now, we suppose that σ is a critical point of the functional $\mathcal{F}^\Lambda$. If we denote by a variation σ with a variational vector field W, then we have

$$\partial\mathcal{F}^\Lambda(W) = \frac{\partial}{\partial\delta}\mathcal{F}^\Lambda(\sigma + \delta W)|_{\delta=0} = 0,$$

[13]. By using a standard argument involving some integrations by parts, we obtain the first variation formula of the functional as follows:

$$\int_0^\ell < E(\sigma), W > ds_t + (< \frac{d^2\sigma}{ds_t^2}, \frac{dW}{ds_t} > + < \Lambda\frac{d\sigma}{ds_t} - \frac{d^3\sigma}{ds_t^3}, W >)\Big|_0^\ell = 0,$$

where

$$E(\sigma) = \frac{d^4\sigma}{ds_t^4} - \frac{d}{ds_t}(\Lambda\frac{d\sigma}{ds_t}).$$

Then, the pseudo spherical tangent indicatrix elastic curve must satisfy the equation

$$\frac{d^4\sigma}{ds_t^4} - \frac{d}{ds_t}(\Lambda\frac{d\sigma}{ds_t}) = 0, \tag{6}$$

for some Λ. By using equations (2), (3), (4), (5), and (6), we have following Euler–Lagrange equations:

$$1-(\frac{\tau}{\kappa})^2-\Lambda=0, \tag{7}$$

$$-3(\frac{\tau}{\kappa})'(\frac{\tau}{\kappa^2})-\Lambda'\frac{1}{\kappa}=0, \tag{8}$$

and

$$(1-(\frac{\tau}{\kappa})^2)\frac{\tau}{\kappa}+((\frac{\tau}{\kappa})'\frac{1}{\kappa})'\frac{1}{\kappa}-\Lambda\frac{\tau}{\kappa}=0. \tag{9}$$

If we combine (7) and (9), then we obtain

$$((\frac{\tau}{\kappa})'\frac{1}{\kappa})'\frac{1}{\kappa}=0. \tag{10}$$

From (7) and (8), we find

$$(\frac{\tau}{\kappa})'(\frac{\tau}{\kappa^2})=0. \tag{11}$$

Thus, the pseudo spherical tangent indicatrix elastic curves are characterized by Euler–Lagrange equations (10) and (11). The solution of these equations is obtained as follows:

$$\frac{\tau}{\kappa}=const.$$

Then we can give the following theorem.

Theorem 5 *If the pseudo spherical tangent indicatrix of a timelike curve in Minkowski 3-space is an elastic curve on pseudo hyperbolic 2-space, then the equation $\frac{\tau}{\kappa}=const.$ must be satisfied. It means that this timelike curve is a general helix.*

When we apply similar calculations for the other types of pseudo spherical indicatrices, we have the following theorems:

Theorem 6 *If the pseudo spherical binormal indicatrix of a timelike curve in Minkowski 3-space is an elastic curve on De Sitter 2-space, then the curve satisfies $\frac{\tau}{\kappa}=const.$, such that $\tau\neq 0$. It means that the timelike curve is a general helix.*

Theorem 7 *If the pseudo spherical principal normal indicatrix of a timelike curve is an elastic curve, then the following equations are provided depending on the conditions:*

i) The case $\tau>\kappa$:

$$\begin{aligned}&\Big(((\tfrac{\kappa}{\sqrt{\tau^2-\kappa^2}})'\tfrac{1}{\sqrt{\tau^2-\kappa^2}})'-\kappa\Big)'-(\tfrac{\kappa}{\tau})\Big(((\tfrac{\tau}{\sqrt{\tau^2-\kappa^2}})'\tfrac{1}{\tau^2-\kappa^2})'\tfrac{1}{\sqrt{\tau^2-\kappa^2}}-\tfrac{\tau}{\sqrt{\tau^2-\kappa^2}}\Big)'\\&+\tfrac{1}{\sqrt{\tau^2-\kappa^2}}[((\tfrac{\kappa}{\sqrt{\tau^2-\kappa^2}})'\tfrac{1}{\sqrt{\tau^2-\kappa^2}})'\tfrac{\kappa}{\sqrt{\tau^2-\kappa^2}}-\tfrac{\kappa^2}{\sqrt{\tau^2-\kappa^2}}+(\tfrac{\kappa}{\sqrt{\tau^2-\kappa^2}})'\tfrac{\kappa^2}{\sqrt{\tau^2-\kappa^2}}\\&-(\tfrac{\tau}{\sqrt{\tau^2-\kappa^2}})'\tfrac{\tau\kappa}{\sqrt{\tau^2-\kappa^2}}+\tfrac{\tau^2}{\sqrt{\tau^2-\kappa^2}}-((\tfrac{\tau}{\sqrt{\tau^2-\kappa^2}})'\tfrac{1}{\sqrt{\tau^2-\kappa^2}})'\tfrac{1}{\sqrt{\tau^2-\kappa^2}}]((\tfrac{\tau}{\tau^2-\kappa^2})'\tfrac{\kappa}{\tau}\\&-(\tfrac{\kappa}{\sqrt{\tau^2-\kappa^2}})'\big)=0,\end{aligned}$$

ii) The case $\kappa > \tau$:

$$
\begin{aligned}
&\left(((\tfrac{\kappa}{\sqrt{\kappa^2-\tau^2}})'\tfrac{1}{\sqrt{\kappa^2-\tau^2}})'+\kappa\right)'-(\tfrac{\kappa}{\tau})\left(((\tfrac{\tau}{\sqrt{\kappa^2-\tau^2}})'\tfrac{1}{\kappa^2-\tau^2})'\tfrac{1}{\sqrt{\kappa^2-\tau^2}}+\tfrac{\tau}{\sqrt{\kappa^2-\tau^2}}\right)' \\
&+\tfrac{1}{\sqrt{\kappa^2-\tau^2}}[((\tfrac{\kappa}{\sqrt{\kappa^2-\tau^2}})'\tfrac{1}{\sqrt{\kappa^2-\tau^2}})'\tfrac{\kappa}{\sqrt{\kappa^2-\tau^2}}-\tfrac{\kappa^2}{\sqrt{\kappa^2-\tau^2}}+(\tfrac{\kappa}{\sqrt{\kappa^2-\tau^2}})'\tfrac{\kappa^2}{\sqrt{\kappa^2-\tau^2}} \\
&-(\tfrac{\tau}{\sqrt{\kappa^2-\tau^2}})'\tfrac{\tau\kappa}{\sqrt{\kappa^2-\tau^2}}+\tfrac{\tau^2}{\sqrt{\kappa^2-\tau^2}}-((\tfrac{\tau}{\sqrt{\kappa^2-\tau^2}})'\tfrac{1}{\sqrt{\kappa^2-\tau^2}})'\tfrac{1}{\sqrt{\kappa^2-\tau^2}}]((\tfrac{\tau}{\kappa^2-\tau^2})'\tfrac{\kappa}{\tau} \\
&-(\tfrac{\kappa}{\sqrt{\kappa^2-\tau^2}})')=0.
\end{aligned}
$$

Corollary 1 *A pseudo principal spherical normal indicatrix of a timelike helix in Minkowski* 3*-space is an elastic curve on De Sitter* 2*-space.*

2.2. The pseudo spherical indicatrix of a spacelike curve

In this subsection, we study spacelike curves whose spherical indicatrices are elastic. The pseudo spherical tangent indicatrix of a spacelike curve γ is a curve $\sigma(s_t) = T(s)$, so $\sigma(s_t)$ lies on the De Sitter space $\mathbb{S}^2_1$. The pseudo spherical principal normal indicatrix of a spacelike curve γ is a curve $\sigma(s_t) = N(s)$ and $\sigma(s_t)$ can lie on the pseudo hyperbolic 2-space $\mathbb{H}^2_0$, De Sitter 2-space $\mathbb{S}^2_1$, and null cone $\mathbb{Q}^2$. Previously, we investigate extremals of the functional (1) defined on pseudo spherical principal normal indicatrix curves under some boundary conditions and later we give some results of the other types indicatrix elastic curves. We do not dwell on variational and differential calculations of the problem of finding a spacelike curve whose pseudo spherical indicatrix is an elastic curve since the same procedure would repeat; however, we give some necessary notices different from the timelike one.

The first derivative of $\sigma(s_t)$ is calculated as

$$\frac{d\sigma(s_t)}{ds} = \frac{dN(s)}{ds}. \tag{12}$$

From (12), we obtain

$$\frac{ds_t}{ds} = \sqrt{|\kappa^2 + \varepsilon_2\tau^2|} = A.$$

Then some derivatives of σ are calculated as

$$\frac{d\sigma}{ds_t} = -\frac{\kappa}{A}T + \frac{\tau}{A}B,$$

$$\frac{d^2\sigma}{ds_t^2} = -\left(\frac{\kappa}{A}\right)'\frac{1}{A}T - \varepsilon_1\left[\frac{\kappa^2+\tau^2}{A^2}\right]N + \left(\frac{\tau}{A}\right)'\frac{1}{A}B,$$

$$
\begin{aligned}
\frac{d^3\sigma}{ds_t^3} \;=\; & -\underbrace{\left[\left(\left(\frac{\kappa}{A}\right)'\frac{1}{A}\right)'\frac{1}{A} + \frac{\varepsilon_1\kappa\left(\kappa^2+\tau^2\right)}{A^3}\right]}_{a}T - \varepsilon_1\underbrace{\left[\left(\frac{\kappa}{A}\right)'\frac{\kappa}{A^2} + \left(\frac{\tau}{A}\right)'\frac{\tau}{A^2} + \left(\frac{\kappa^2+\tau^2}{A^2}\right)'\frac{1}{A}\right]}_{b}N \\
& + \underbrace{\left[\left(\left(\frac{\tau}{A}\right)'\frac{1}{A}\right)'\frac{1}{A} - \varepsilon_1\varepsilon_2\frac{\tau\left(\kappa^2+\tau^2\right)}{A^3}\right]}_{c}B,
\end{aligned}
$$

and

$$\begin{aligned}\frac{d^4\sigma}{ds_t^4} = \left(\frac{\kappa b - a'}{A}\right) T - \left(\frac{\varepsilon_1 a\kappa + b' - \varepsilon_1 c\tau}{A}\right) N \\ + \left(\frac{\varepsilon_2 \tau b}{A} + \frac{c'}{A}\right) B.\end{aligned} \tag{13}$$

We will use a, b, and c in Equation (13) for simplicity. By using the same calculation method in Subsection 2.1, we show that the pseudo spherical principal normal indicatrix elastic curve must satisfy Equation (6), too. Also, by using Frenet equations (2), derivatives of σ (13), (12), and Equation (6), we have the following Euler–Lagrange equations:

$$\frac{\kappa b - a'}{A} + \frac{\kappa}{A^2}\Lambda' + \left(\frac{\kappa}{A}\right)' \frac{1}{A}\Lambda = 0, \tag{14}$$

$$-\frac{\varepsilon_1 (a\kappa - c\tau) + b'}{A} + \varepsilon_1 \Lambda \frac{\kappa^2 + \tau^2}{A^2} = 0, \tag{15}$$

and

$$\frac{\varepsilon_2 \tau b + c'}{A} - \Lambda' \frac{\tau}{A^2} - \left(\frac{\tau}{A}\right)' \frac{1}{A}\Lambda = 0. \tag{16}$$

If we combine (14), (15), and (16), we obtain the following equation:

$$\begin{aligned}
&-\tfrac{1}{\sqrt{|\kappa^2-\tau^2|}} \left(((\tfrac{\kappa}{\sqrt{|\kappa^2-\tau^2|}})' \tfrac{1}{\sqrt{|\kappa^2-\tau^2|}})' \tfrac{1}{\sqrt{|\kappa^2-\tau^2|}} + \tfrac{\varepsilon_1 \kappa \left(\kappa^2+\tau^2\right)}{\left(\sqrt{|\kappa^2-\tau^2|}\right)^3} \right)' \\
&+\tfrac{1}{\kappa^2+\tau^2} \left(\left(\tfrac{\kappa}{\sqrt{|\kappa^2-\tau^2|}} \right)' - \tfrac{\kappa}{\tau} \left(\tfrac{\tau}{\sqrt{|\kappa^2-\tau^2|}} \right)' \right) \\
&\times \left(\left(\tfrac{\kappa}{\sqrt{|\kappa^2-\tau^2|}} \right)' \tfrac{\kappa}{|\kappa^2-\tau^2|} + \left(\tfrac{\tau}{\sqrt{|\kappa^2-\tau^2|}} \right)' \tfrac{\tau}{|\kappa^2-\tau^2|} + \left(\tfrac{\kappa^2+\tau^2}{|\kappa^2-\tau^2|} \right)' \tfrac{1}{|\kappa^2-\tau^2|} \right)' \\
&+\tfrac{1}{\sqrt{|\kappa^2-\tau^2|}} \left(\tfrac{\kappa}{\tau}\right) \left(\left(\left(\tfrac{\tau}{\sqrt{|\kappa^2-\tau^2|}} \right)' \tfrac{1}{|\kappa^2-\tau^2|} \right)' \tfrac{1}{\sqrt{|\kappa^2-\tau^2|}} - \varepsilon_1 \varepsilon_2 \tfrac{\left(\kappa^2+\tau^2\right)\tau}{\left(\sqrt{|\kappa^2-\tau^2|}\right)^3} \right)' \\
&+\tfrac{\kappa}{\kappa^2+\tau^2} \left(\left(\tfrac{\kappa}{\sqrt{|\kappa^2-\tau^2|}} \right)' - \tfrac{\kappa}{\tau} \left(\tfrac{\tau}{\sqrt{|\kappa^2-\tau^2|}} \right)' \right) \\
&\times \left(\left(\left(\tfrac{\kappa}{\sqrt{|\kappa^2-\tau^2|}} \right)' \tfrac{1}{\sqrt{|\kappa^2-\tau^2|}} \right)' \tfrac{1}{\sqrt{|\kappa^2-\tau^2|}} + \varepsilon_1 \tfrac{\kappa\left(\kappa^2+\tau^2\right)}{\left(\sqrt{|\kappa^2-\tau^2|}\right)^3} \right) \\
&+\tfrac{\kappa}{\sqrt{|\kappa^2-\tau^2|}} (1+\varepsilon_2) (\left(\tfrac{\kappa}{\sqrt{|\kappa^2-\tau^2|}} \right)' \tfrac{\kappa}{|\kappa^2-\tau^2|} + \left(\tfrac{\tau}{\sqrt{|\kappa^2-\tau^2|}} \right)' \tfrac{\tau}{|\kappa^2-\tau^2|} \\
&+ \left(\tfrac{\kappa^2+\tau^2}{|\kappa^2-\tau^2|} \right)^3 \tfrac{1}{\sqrt{|\kappa^2-\tau^2|}}) + \tfrac{\tau}{\kappa^2+\tau^2} \left(\tfrac{\kappa}{\tau} \left(\tfrac{\tau}{\sqrt{|\kappa^2-\tau^2|}} \right)' - \left(\tfrac{\kappa}{\sqrt{|\kappa^2-\tau^2|}} \right)' \right) \\
&\times \left(\left(\tfrac{\tau}{\sqrt{|\kappa^2-\tau^2|}} \right)' \left(\tfrac{1}{\sqrt{|\kappa^2-\tau^2|}} \right)' \tfrac{1}{\sqrt{|\kappa^2-\tau^2|}} - \varepsilon_1 \varepsilon_2 \tfrac{\left(\kappa^2+\tau^2\right)\tau}{\left(\sqrt{|\kappa^2-\tau^2|}\right)^3} \right) = 0.
\end{aligned} \tag{17}$$

Then, we can give the following theorem.

Theorem 8 *If the pseudo spherical principal normal indicatrix of a spacelike curve with nonnull principal normal vector is an elastic curve, then the curve satisfies Equation* (17).

Corollary 2 *A pseudo spherical principal normal indicatrix of a spacelike helix in Minkowski* 3*-space is an elastic curve on De Sitter* 2*-space.*

If we make similar calculations for finding pseudo spherical tangent and binormal indicatrices of a spacelike curve, we obtain the following theorems.

Theorem 9 *If the pseudo spherical tangent indicatrix of a spacelike curve with nonnull principal normal vector in Minkowski* 3*-space is an elastic curve on De Sitter* 2*-space, then the curve satisfies* $\frac{\tau}{\kappa} = const.$*; that is, this spacelike curve in Minkowski* 3*-space is a general helix.*

Theorem 10 *If the pseudo spherical binormal indicatrix of a spacelike curve with nonnull principal normal vector in Minkowski* 3*-space is an elastic curve, then the curve satisfies* $\frac{\tau}{\kappa} = const.$ *such that* $\tau \neq 0$*; that is, this spacelike curve in Minkowski* 3*-space is a general helix.*

2.3. The pseudo spherical indicatrix of a null curve

In this subsection, we give the characterization for null curves whose pseudo spherical indicatrix are elastic. We first recall the Cartan frame of a null curve.

Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}_1^3$ be a null curve in Minkowski 3-space. $\{T, V, U\}$ is the Cartan frame along the null curve γ with the Cartan equations given by

$$\begin{pmatrix} T' \\ V' \\ U' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\tau & 0 & -\kappa \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ V \\ U \end{pmatrix},$$

$$< T, U > = < V, V > = 1,$$

$$< T, T > = < U, U > = < T, V > = < U, V > = 0,$$

where $\kappa = < \gamma^{''}(s), \gamma^{''}(s) >$ and $\tau = \frac{1}{2} < \gamma^{'''}(s), \gamma^{'''}(s) >$ denote the Cartan curvature functions of the curve γ [2, 4].

The pseudo spherical tangent indicatrix and pseudo spherical binormal indicatrix of the null curve lie on the 2-dimensional null cone $\mathbb{Q}^2$. The pseudo spherical normal indicatrix of the null curve γ is a curve $\sigma(s_t) = V(s)$ and $\sigma(s_t)$ lies on De Sitter 2-space $\mathbb{S}_1^2$ [3].

Now we will obtain a characterization for pseudo normal spherical indicatrix elastic curves and give results of the other types of elastic curves.

The first derivative of σ is obtained as

$$\frac{d\sigma(s_t)}{ds} = \frac{dV(s)}{ds}, \tag{18}$$

$$\frac{d\sigma(s_t)}{ds_t}\frac{ds_t}{ds} = -\tau(s)T(s) - \kappa(s)U(s).$$

Then we have $\frac{ds_t}{ds} = \sqrt{2\left|\kappa(s)\tau(s)\right|}$, (see [3]). The other derivatives of σ are calculated as

$$\frac{d^2\sigma}{ds_t^2} = (-\frac{1}{\sqrt{2\left|\kappa(s)\tau(s)\right|}})' \frac{1}{\sqrt{2\left|\kappa(s)\tau(s)\right|}}(-\tau'(s)T - 2\kappa(s)\tau(s)V - \kappa^{'}(s)U(s)),$$

$$\frac{d^3\sigma}{ds_t^3} = -\,((\frac{1}{\sqrt{2\left|\kappa(s)\tau(s)\right|}})'\frac{1}{\sqrt{2\left|\kappa(s)\tau(s)\right|}})'\frac{1}{\sqrt{2\left|\kappa(s)\tau(s)\right|}}((-\tau''(s) \tag{19}$$
$$+\,2\kappa(s)\tau^2(s))T(s) - 2(\kappa(s)\tau(s))'V(s) + (-\kappa''(s) + 2\kappa^2(s)\tau(s))U(s)).$$

In order to find critical points of the functional (1) , we use the similar method in Subsection 2.1 and 2.2. If we substitute (18) and (19) in Equation (6) and make similar calculations, then we obtain the following theorem.

Theorem 11 *The pseudo spherical normal indicatrix of a null curve in Minkowski* 3*-space can be determined by the following Euler–Lagrange equation:*

$$\begin{array}{l} 2\kappa^2\tau''\left(\tau'+\kappa\tau\right) - 2\kappa\tau\kappa'\tau'' - 10\kappa^3\tau^2\tau' - 4\kappa^2\tau^3\kappa' + \kappa\kappa'\tau'\left(3\tau'+2\kappa\tau\right) \\ -\kappa'\tau'\left(3\kappa'\tau+4\kappa^2\tau^2\right) + 2\kappa''\tau\left(\kappa\tau'-\kappa'\tau\right) - \kappa\tau\left(\tau'''\kappa-\kappa'''\tau\right) = 0. \end{array}$$

We can find a special solution of the equation as follows.

Corollary 3 *A pseudo spherical normal indicatrix of a null helix is an elastic curve on De Sitter* 2*-space.*

We have the following theorems for pseudo spherical tangent and binormal indicatrices elastic curves.

Theorem 12 *If the pseudo spherical tangent indicatrix of a null curve in Minkowski* 3*-space is an elastic curve, then the curve satisfies* $\frac{\tau}{\kappa} = const.$*; that is, this curve in Minkowski* 3*-space is a null helix.*

Theorem 13 *If the pseudo spherical binormal indicatrix of a null curve in Minkowski* 3*-space is an elastic curve, then the curve satisfies* $\frac{\tau}{\kappa} = const.$ *such that* $\tau \neq 0$*; that is, this null curve in Minkowski* 3*-space is a helix.*

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