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Some characterizations of right c-regularity and (b, c)-inverse

Ruju ZHAO*, **Hua YAO**, **Long WANG**, **Junchao WEI**
School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu, P.R. China

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Abstract: Let R be a ring and $a, b, c \in R$. We give a novel characterization of group inverses (resp. EP elements) by the properties of right (resp. left) c-regular inverses of a and discuss the relation among the strongly left (b, c)-invertibility of a, the right ca-regularity of b, and the (b, c)-invertibility of a. Finally, we investigate the sufficient and necessary condition for a ring to be a strongly left min-Abel ring by means of the (b, c)-inverse of a.

Key words: Right c-regular element, (b, c)-inverse, group inverse, EP element, left min-Abel ring

1. Introduction

Let S be a semigroup and $a, b, c \in S$. Then a is said to be (b, c)-invertible [4] if there exists $y \in bSy \cap ySc$ such that $yab = b$ and $cay = c$. Such an y is called a (b, c)-inverse of a, which is always unique if it exists, denoted by $a^{||(b,c)}$.

In [5], Drazin considered the following problem: in any semigroup S (or any associative ring) with unit element 1, and for any given $a \in S$, the properties $1 \in Sa$ ($1 \in aS$) of left (right) invertibility are often useful as weaker versions of ordinary two-sided invertibility, and it is natural to seek corresponding one-sided versions for at least some types of generalized invertibility. Hence, Drazin in [5] introduced the left (b, c)-inverse as follows: let S be any semigroup and let $a, b, c \in S$. Then a is said to be left (b, c)-invertible if $b \in Scab$, or equivalently if there exists $x \in Sc$ such that $xab = b$, in which case any such x will be called a left (b, c)-inverse of a. The left (b, c)-inverse of a is not unique [5, Example 3.4]. Dually, a is said to be right (b, c)-invertible if $c \in cabS$, or equivalently if there exists $z \in bS$ such that $caz = c$, and any such z will be called a right (b, c)-inverse of a. Related studies of the one-sided (b, c)-inverse can be found in [7] and [12]. The main purpose of this article is to do some further research on the left (right) (b, c)-inverse of a. Therefore, the following concepts need to be introduced.

Let R be a ring and $a, c \in R$. If there exists $b \in R$ such that $a = abca$ ($a = acba$), then we say that a is right (left) c-regular and b is a right (left) c-regular inverse of a. We denote by a_c^- (${}_ca^-$) the set of all right (left) c-regular inverses of a.

In [1], an element a of a ring R is said to be group invertible if there is $a^\# \in R$ such that

$$aa^\#a = a,\ a^\#aa^\# = a^\#,\ aa^\# = a^\#a.$$

*Correspondence: zrj0115@126.com
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Denote by $R^{\#}$ the set of all group invertible elements of R. An element $a \in R$ is group invertible if and only if $a \in a^2R \cap Ra^2$ [3, 6]. Clearly, a ring R is strongly regular if and only if $R = R^{\#}$.

An involution $a \longmapsto a^*$ in a ring R is an antiisomorphism of degree 2; that is,

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^*a^*.$$

A ring R with an involution $*$ is called a $*$-ring. An element $p \in R$ is called a projection if $p^2 = p = p^*$.

An element $a^\dagger$ in a $*$-ring R is called the Moore–Penrose inverse (or MP-inverse) [9] of a, if

$$aa^\dagger a = a,\ a^\dagger aa^\dagger = a^\dagger,\ aa^\dagger = (aa^\dagger)^*,\ a^\dagger a = (a^\dagger a)^*.$$

In this case, we say a is MP-invertible in R. The set of all MP-invertible elements of R is denoted by $R^\dagger$.

In [2], an element a of a $*$-ring R is said to be EP if $a \in R^\dagger$ and $a^\dagger a = aa^\dagger$, which is equivalent to $a \in R^{\#} \cap R^\dagger$ and $a^{\#} = a^\dagger$. Denote by R^{EP} the set of all EP-invertible elements of R.

An idempotent $e \in R$ is called a left minimal idempotent if Re is a minimal left ideal of R. We denote by $ME_l(R)$ the set of all left minimal idempotents of R, and e is said to be left (right) semicentral if $ae = eae$ ($ea = eae$) for each $a \in R$. A ring R is said to be (strongly) left min-Abel [10] if either $ME_l(R) = \emptyset$ or every element e of $ME_l(R)$ is (right) left semicentral.

In this paper, we first study the right (left) c-regular elements by means of left and right (b,c)-inverses of a. Next, with the help of right (left) c-regular elements, we characterize group invertible elements, MP-invertible elements, and EP elements. Finally, we give some new characterizations of directly finite rings, left min-Abel rings, and strongly left min-Abel rings.

2. c-Regular inverses

Recall that an element a of a ring R is said to be regular if there exists $b \in R$ such that $a = aba$. Such a b is called an inner inverse of a. Clearly, if b is an inner inverse of a, then so is bab. We denote by a^- the set of all inner inverses of a.

Let R be a ring. For any $a, c \in R$, if there exists $b \in R$ such that $a = abca$ ($a = acba$), then we say that a is right (left) c-regular and b is right (left) c-regular inverse of a. Obviously, if a is right c-regular, then a is regular, but the converse is not true from the following example.

Example 2.1 *Let* $R = T_2(\mathbb{Z}_2) = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| x, y, z \in \mathbb{Z}_2 \right\}$. *It is easy to check that* $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ *is regular. Take* $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. *Then* $CA = 0$. *Consequently, we obtain that* $ABCA \neq A$, *for any* $B \in R$. *That is,* A *is not right* C*-regular.*

In order to study the (b,c)-inverse of a in the next section, we first discuss right (left) c-regular inverses of a in this section.

Remark 2.2 *Let* R *be a ring. For each* $a, b, c \in R$, *if* b *is a right* c*-regular inverse of* a, *so is* $bcab$. *In fact,* $a(bcab)ca = (abca)bca = abca = a$. *If* a *is right (left)* c*-regular, then we denote by* a_c^- $({}_ca^-)$ *the set of all right (left)* c*-regular inverses of* a.

Example 2.3 *Let a be a regular element of a ring R. If $d \in a^-$, then a is right ad-regular and left da-regular. In fact, $a = ada = ad(ad)a = a(da)da$, which implies $d \in a^-_{ad}$ and $d \in_{da} a^-$.*

If a is regular and $b \in a^-$, then $b \in a^-_{ab} \cap {}_{ba}a^-$. Conversely, if a is regular and $b \in R$ satisfying $b \in a^-_{ab} \cap {}_{ba}a^-$, then $b \in a^-$?

From the following example, we know that the above question is not true.

Example 2.4 *Let $R = T_2(\mathbb{Z}_3) = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| x, y, z \in \mathbb{Z}_3 \right\}$. Write $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \in R$. It is easy to check that $ABA = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \neq A$ and $ABABA = A$. Therefore, $B \in A^-_{AB} \cap {}_{BA}A^-$, but $B \notin A^-$.*

Proposition 2.5 *Let R be a ring and $a, b, c \in R$. Then the following conditions are equivalent:*

(1) ab is right c-regular and $Rb = Rab$;
(2) ab is right c-regular and $Rb = Rcab$;
(3) cab is regular and $Rb = Rcab$.

Proof (1) $\Rightarrow$ (2) Since ab is right c-regular, we get $ab = ab(ab)^-_c cab$. This clearly forces $Rb = Rab = Rab(ab)^-_c cab \subseteq Rcab \subseteq Rab$. That is, $Rb = Rcab$.

(2) $\Rightarrow$ (3) Since ab is right c-regular, we have $ab = ab(ab)^-_c cab$. Premultiplying by c, we have $cab = cab(ab)^-_c cab$. Hence, cab is regular.

(3) $\Rightarrow$ (1) Since $Rb = Rcab$, $b = vcab$ for some $v \in R$. From the hypothesis that cab is regular, we have $b = vcab(cab)^- cab = b(cab)^- cab$. Premultiplying by a, we get $ab = ab(cab)^- cab$. Therefore, ab is right c-regular, and $(cab)^- \subseteq (ab)^-_c$. □

Corollary 2.6 *Let R be a ring and $a, b, c \in R$. Then the following conditions are equivalent:*

(1) ab is right c-regular, and $Rb = Rab$;
(2) $b \in bRcab$;
(3) b is right ca-regular.

Proof (1) $\Rightarrow$ (2) Write $b = vab$. We deduce that

$$b = vab = vab(ab)^-_c cab = b(ab)^-_c cab \in bRcab.$$

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) Since $b = bb^-_{ca}cab$, we obtain that $ab = abb^-_{ca}cab$. Hence, ab is right c-regular and $b^-_{ca} \subseteq (ab)^-_c$. Moreover, we have $Rb = Rbb^-_{ca}cab \subseteq Rab \subseteq Rb$. That is, $Rb = Rab$. □

Proposition 2.7 *Let R be a ring and $a, b, c \in R$. Then the following conditions are equivalent:*

(1) $b \in bRcab$;
(2) $r(ca) \cap bR = 0$, and b is right ca-regular;
(3) $r(ab) = r(b)$, and ab is right c-regular.

Proof (1) $\Rightarrow$ (2) Set $b = bvcab$. Then b is right ca-regular. Assume that $t \in r(ca) \cap bR$. Writing $t = bs$, we get $cabs = cat = 0$. Moreover, we get $bs = bvcabs = 0$. This means that $t = 0$.

(2) $\Rightarrow$ (3) For any $y \in r(ab)$, we have $aby = 0$. Premultiplying by c, we get $caby = 0$. It follows that $by \in r(ca) \cap bR = 0$. Thus, $y \in r(b)$. This gives $r(b) \supseteq r(ab)$. However, $r(b) \subseteq r(ab)$ is clear. Hence, $r(b) = r(ab)$. Moreover, we get that ab is right c-regular, because $b = bb^-_{ca}cab$.

(3) $\Rightarrow$ (1) Since $ab = ab(ab)^-_c cab$, we obtain that $1-(ab)^-_c cab \in r(ab) = r(b)$. Therefore, $b = b(ab)^-_c cab \in bRcab$. □

Next, we give some characterizations of group invertible elements, MP-invertible elements, and EP-elements with c-regular inverses.

Proposition 2.8 *Let R be a ring and $a \in R^\#$. Then $a^-_{a^\#} = \{x \in R | a^\# a = axa^\#\}$.*

Proof Since $a \in R^\#$, $a^\#$ exists and $a = a(a^\# a)a^\# a$. It follows that a is right $a^\#$-regular and $a^\# a \in a^-_{a^\#}$. Thus, $a^-_{a^\#}$ is not empty. For any $x \in a^-_{a^\#}$, we have $a = axa^\# a$. This gives $aa^\# = axa^\# aa^\# = axa^\#$. That is, $x \in \{x \in R | a^\# a = axa^\#\}$. Conversely, if $x \in \{x \in R | a^\# a = axa^\#\}$, then $a = a^\# a^2 = axa^\# a$. Therefore, $x \in a^-_{a^\#}$. □

Proposition 2.9 *Let R be a ring and a be a regular element of R. Then $a \in R^\#$ if and only if there exists $b \in R$ such that $b \in a^-_{ba} \cap {}_{ab}a^-$.*

Proof Assume that $a \in R^\#$. Then $a^\#$ exists. Write $b = a^\# \in R$. Then we have

$$ab(ba)a = aa^\#(a^\# a^2) = aa^\# a = a,$$
$$a(ab)ba = a^2 a^\# a^\# a = aa^\# a = a,$$

which imply $b \in a^-_{ba} \cap {}_{ab}a^-$.

Conversely, since $b \in a^-_{ba} \cap {}_{ab}a^-$, we get $ab(ba)a = a = a(ab)ba$, which yields $a \in a^2R \cap Ra^2$. Therefore, $a \in R^\#$. □

Proposition 2.10 *Let R be a ring and $a \in R$. Then the following conditions are equivalent:*

(1) $a \in R^\#$;
(2) there exist $x \in R$ and $d \in {}_x a^-$, such that ${}_x a^- = a^-_x$ is not empty and $dxa = axd$.

Proof (1) $\Rightarrow$ (2) Assume that $a \in R^\#$. Then $a^\#$ exists and $a^\# a \in a^-_{a^\#} \cap {}_{a^\#}a^-$. Thus, $a^-_{a^\#}$ and ${}_{a^\#}a^-$ are not empty. Set $y \in {}_{a^\#}a^-$. We get $a = aa^\# ya$. Premultiplying by a, we have $a^2 = a^2 a^\# ya = aya$. We conclude from the above equality that $a^\# a = aa^\# = a^2(a^\#)^2 = aya(a^\#)^2 = aya^\#$, which gives $y \in a^-_{a^\#}$, and hence that ${}_{a^\#}a^- \subseteq a^-_{a^\#}$. In the same manner we can see that $a^-_{a^\#} \subseteq {}_{a^\#}a^-$, and so ${}_{a^\#}a^- = a^-_{a^\#}$. Since $a^\# a \in {}_{a^\#}a^-$, we have $(a^\# a)a^\# a = a^\# a = aa^\# = aa^\#(aa^\#) = aa^\#(a^\# a)$. Thus, the conclusion is proved by writing $x = a^\#$ and $d = a^\# a$.

(2) $\Rightarrow$ (1) Let $x \in R$ satisfy ${}_x a^- = a^-_x$, which is not empty, and let $d \in {}_x a^-$ satisfy $dxa = axd$. Then $a = axda = adxa$. Write $y = dxaxd$. We get

$$aya = adxaxda = axda = a\,,$$
$$yay = dxaxdadxaxd = dxadxaxd = dxaxd = y\,,$$
$$ya = dxaxda = dxa = axd = adxaxd = ay\,.$$

Consequently, $a \in R^{\#}$ and $a^{\#} = y = dxaxd$. □

Proposition 2.11 *Let R be a ring and $a \in R$. Then the following conditions are equivalent:*

(1) $a \in R^{\dagger}$;

(2) there exists $x \in a^{-}_{ax}$ such that ax and xa are projections.

Proof (1) ⇒ (2) From the hypothesis that $a \in R^{\dagger}$, $a^{\dagger}$ exists. Write $x = a^{\dagger}$. It is easy to check that the element x satisfies condition (2).

(2) ⇒ (1) Assume that there exists $x \in a^{-}_{ax}$ such that ax and xa are projections. Then we get $ax(ax)a = a$, $ax = axax = (ax)^{*}$, and $xa = xaxa = (xa)^{*}$. Thus, $axa = (axax)a = a$. Take $b = xax$. Then we obtain

$$ab = axax = ax = (ax)^{*} = (ab)^{*}\,,$$
$$ba = xaxa = xa = (xa)^{*} = (ba)^{*}\,,$$
$$aba = axa = a\,,\; bab = (xax)(ax) = xax = b\,.$$

Consequently, $a \in R^{\dagger}$ and $a^{\dagger} = b = xax$. □

Proposition 2.12 *Let R be a ring and $a \in R$. Then the following conditions are equivalent:*

(1) $a \in R^{EP}$;

(2) $a \in R^{\dagger}$, ${}_{a^{\dagger}}a^{-} = a^{-}_{a^{\dagger}}$, and there exists $d \in {}_{a^{\dagger}}a^{-}$, such that $da^{\dagger}a = aa^{\dagger}d = aa^{\dagger}$.

Proof (1) ⇒ (2) Suppose that $a \in R^{EP}$. Then $a \in R^{\#} \cap R^{\dagger}$. From the proof of Proposition 2.10, we know that ${}_{a^{\#}}a^{-} = a^{-}_{a^{\#}}$ and there exists $d \in {}_{a^{\#}}a^{-}$ such that $da^{\#}a = aa^{\#}d = aa^{\#}$. Accordingly, we have $d \in {}_{a^{\dagger}}a^{-} = a^{-}_{a^{\dagger}}$, which satisfies $da^{\dagger}a = aa^{\dagger}d = aa^{\dagger}$.

(2) ⇒ (1) Let $d \in {}_{a^{\dagger}}a^{-}$ satisfy $da^{\dagger}a = aa^{\dagger}d = aa^{\dagger}$. Then $a = aa^{\dagger}da = ada^{\dagger}a$ follows from $d \in {}_{a^{\dagger}}a^{-} = a^{-}_{a^{\dagger}}$. Write $x = da^{\dagger}d$. Then we get

$$axa = ada^{\dagger}da = ada^{\dagger}aa^{\dagger}da = aa^{\dagger}da = a\,,$$
$$xax = da^{\dagger}dada^{\dagger}d = d(a^{\dagger}aa^{\dagger})dada^{\dagger}d = da^{\dagger}(aa^{\dagger}da)da^{\dagger}d = da^{\dagger}ada^{\dagger}d = da^{\dagger}ad(a^{\dagger}aa^{\dagger})d = da^{\dagger}(ada^{\dagger}a)a^{\dagger}d =$$
$$da^{\dagger}aa^{\dagger}d = da^{\dagger}d = x\,,$$
$$ax = ada^{\dagger}d = ad(a^{\dagger}aa^{\dagger})d = aa^{\dagger}d = da^{\dagger}a = da^{\dagger}(aa^{\dagger}da) = d(a^{\dagger}aa^{\dagger})da = da^{\dagger}da = xa\,.$$

Thus, we deduce that $a \in R^{\#}$ and $a^{\#} = x = da^{\dagger}d$. Premultiplying by a, we obtain that $aa^{\#} = ada^{\dagger}d = aa^{\dagger}d = aa^{\dagger}$. That is, $a \in R^{EP}$ by [8, Theorem 7.3]. □

Recall that a ring R is quasinormal [11] if $eR(1-e)Re = 0$ for each $e^2 = e \in R$. The following theorem gives a new characterization of quasinormal rings. At the end of this section, we study the quasinormal rings and the directly finite rings by means of c-regular inverses.

Theorem 2.13 *Let R be a ring and e be an idempotent of R. Then the following conditions are equivalent:*

(1) R is a quasinormal ring;

(2) if there exists an idempotent $g \in R$ such that $e_{eg}^{-} \neq \emptyset$, then $e_{eg}^{-} = e_{ge}^{-}$.

Proof $\Rightarrow$ Assume that R is quasinormal and $e^2 = e, g^2 = g \in R$ with $e_{eg}^{-} \neq \emptyset$. Choose $x \in e_{eg}^{-}$. Then $e = exege$. Note that R is quasinormal. Then $ex(1-e)ge \in eR(1-e)Re = 0$, and it follows that $exge = exege$. Hence, $e = exge = ex(ge)e$, which implies that $x \in e_{ge}^{-}$, so $e_{eg}^{-} \subseteq e_{ge}^{-}$. Conversely, assume that $y \in e_{ge}^{-}$, and then $e = ey(ge)e = eyge$. Since R is quasinormal, $eyge = eyege = ey(eg)e$, one obtains that $y \in e_{eg}^{-}$. Hence, $e_{ge}^{-} \subseteq e_{eg}^{-}$.

$\Leftarrow$ Assume that $e^2 = e \in R$. For any $a, b \in R$, write $g = e + (1-e)ae, f = e + eb(1-e)$. Then $eg = e = fe, ge = g, ef = f, g^2 = g$, and $f^2 = f$. Note that $e = ef(eg)e$. Then $f \in e_{eg}^{-}$, by hypothesis, and we have $e_{eg}^{-} = e_{ge}^{-}$. Hence, $f \in e_{ge}^{-}$; that is, $e = ef(ge)e = fg = e + eb(1-e)ae$, and we have $eb(1-e)ae = 0$ for any $a, b \in R$. Therefore, $eR(1-e)Re = 0$, and so R is quasinormal. □

Proposition 2.14 *Let R be a ring. Then the following conditions are equivalent:*

(1) R is a directly finite ring;

(2) if $ab = 1$ for $a, b \in R$, then $a_b^{-} = \{1\}$.

Proof (1) $\Rightarrow$ (2) Assume that $ab = 1$. Then we get $a = a(ba)ba$. That is, $ba \in a_b^{-}$. Since R is a directly finite ring, we see that $ba = 1$. It follows that a and b are invertible and $1 \in a_b^{-}$. For any $x \in a_b^{-}$, we conclude that $a = axba = ax$. Thus, $x = 1$. Hence, $a_b^{-} = \{1\}$.

(2) $\Rightarrow$ (1) Let $a, b \in R$ satisfy $ab = 1$. By the hypothesis, we know $a_b^{-} = \{1\}$. As $ba \in a_b^{-}$, we have $ba = 1$. Consequently, R is a directly finite ring. □

Proposition 2.15 *Let R be a ring. Then the following conditions are equivalent:*

(1) R is a directly finite ring;

(2) if $ab = 1$ for $a, b \in R$, then $a_b^{-} = b_a^{-}$.

Proof (1) $\Rightarrow$ (2) Suppose that R is a directly finite ring and $ab = 1$. Then we could find $a_b^{-} = \{1\}$ by Proposition 2.14. Since $ba = 1$, we have $b_a^{-} = \{1\}$ by Proposition 2.14. Hence, $a_b^{-} = b_a^{-}$.

(2) $\Rightarrow$ (1) Let $a, b \in R$ satisfy $ab = 1$. Then $a_b^{-} = b_a^{-}$ follows from the hypothesis. We have $ba \in a_b^{-} = b_a^{-}$, because $a = a(ba)ba$. That is, $b = b(ba)ab = b^2a$. This clearly forces $1 = ab = ab^2a = (ab)(ba) = ba$. Therefore, R is a directly finite ring. □

3. Characterizations of the (b, c)-inverse of a

Let R be a ring. For each $a, b, c \in R$, a is said to be strongly left (b, c)-invertible if there exists $x \in bRc$ such that $b = xab$. Such an x is called a strongly left (b, c)-inverse of a. Clearly, if x is a strongly left (b, c)-inverse of a, then so is xax. Denote by $a_l^{s\|(b,c)}$ the set of all strongly left (b, c)-inverses of a.

In this section, we will consider the relation among the strongly left (b,c)-invertibility of a, the right ca-regularity of b, and the (b,c)-invertibility of a.

In the following, we give an example in which the strongly left (b,c)-inverse of a is not unique.

Example 3.1 *Let $R = M_2(\mathbb{Z}_2)$. Write $a = x_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $b = x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $v = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It is obvious that $x_1 = buc \in bRc$, $x_2 = bvc \in bRc$, and $x_1ab = b = x_2ab$. This gives $x_1, x_2 \in a_l^{s\|(b,c)}$, but $x_1 \neq x_2$.*

Proposition 3.2 *Let R be a ring and $a,b,c \in R$. If a is strongly left (b,c)-invertible and $x \in a_l^{s\|(b,c)}$, then we have:*

(1) $x \in bRx \cap xRc$;
(2) $xax = x$;
(3) cax is left ab-regular;
(4) $xR = bR$;
(5) $r(c) \subseteq r(x)$.

Proof It follows from $x \in a_l^{s\|(b,c)}$ that $x \in bRc$ and $b = xab$. Write $x = bvc$. Then we get $xax = xabvc = bvc = x$. This gives $bvcax = xax = x = bvc = xabvc$. Thus, $x \in bRx \cap xRc$. Furthermore, we have

$$cax = caxax = cabvcax = ca(xab)vcax = cax(ab)vcax.$$

Hence, cax is left ab-regular. We have $xR = bR$, because $xR = bvcR \subseteq bR = xabR \subseteq xR$. Finally, for any $d \in r(c)$, we have $cd = 0$. Premultiplying by bv, we get $xd = bvcd = 0$. That is, $d \in r(x)$. □

We first give some equivalent conditions for an element to be strongly left (b,c)-invertible.

Corollary 3.3 *Let R be a ring and $a,b,c \in R$. Then the following conditions are equivalent:*

(1) a is strongly left (b,c)-invertible;
(2) there exists $x \in R$, such that $xax = x$, $l(x) = l(b)$, $Rx \subseteq Rc$, and $xR \subseteq bR$. In this case, $x \in a_l^{s\|(b,c)}$.

Proof (1) $\Rightarrow$ (2) Fix $x \in a_l^{s\|(b,c)}$. It follows from Proposition 3.2 that

$$xax = x,\ xR = bR,\ Rx \subseteq Rc, \text{ and } l(x) = l(b).$$

(2) $\Rightarrow$ (1) Since $1 - xa \in l(x) = l(b)$, it follows that $b = xab$. Write $x = vc = bs$. Then we obtain $x = xax = (bs)a(vc) \in bRc$. Hence, a is strongly left (b,c)-invertible. This means that $x \in a_l^{s\|(b,c)}$. □

Corollary 3.4 *Let R be a ring and $a,b,c \in R$. Then the following conditions are equivalent:*

(1) a is strongly left (b,c)-invertible;
(2) there exists $x \in R$ such that $xax = x$, $xR = bR$, and $Rx \subseteq Rc$.

Proof (1) $\Rightarrow$ (2) Let $x \in a_l^{s\|(b,c)}$. Then $b = xab$ and $x \in bRc$. This gives that $bR = xR$ and $Rx \subseteq Rc$. Again, by Proposition 3.2, we have that $x = xax$.

(2) $\Rightarrow$ (1) Since $xR = bR$ and $Rx \subseteq Rc$, one has that $x = xax \in bRc$. By $1 - xa \in l(x) = l(b)$, we get that $b = xab$. Thus, a is strongly left (b,c)-invertible, and $x \in a_l^{s\|(b,c)}$. □

Corollary 3.5 *Let R be a ring and $a, b, c \in R$. Then the following conditions are equivalent:*

(1) a is strongly left (b,c)-invertible;

(2) $b \in bRcab$.

Proof (1) $\Rightarrow$ (2) It is clear from the definition of strongly left (b,c)-invertibility.

(2) $\Rightarrow$ (1) Set $b = bvcab$ and $x = bvc$. Then we get $x \in bRc$ and $b = xab$. That is, a is strongly left (b,c)-invertible. □

Next, we discuss when a strongly left (b,c)-invertible element actually becomes a (b,c)-invertible element.

Proposition 3.6 *Let R be a ring and $a, b, c \in R$. Then the following conditions are equivalent:*

(1) a is (b,c)-invertible;

(2) a is strongly left (b,c)-invertible and $caa_l^{s\|(b,c)} = c$. In this case, $a^{\|(b,c)} \in a_l^{s\|(b,c)}$.

Proof (1) $\Rightarrow$ (2) Set $y = a^{\|(b,c)}$. It is straightforward that

$$y \in bRy \cap yRc,\ y = yay,\ yab = b, \text{ and } cay = c.$$

Thus, $y = yay \in (bRy)a(yRc) \subseteq bRc$. Therefore, a is strongly left (b,c)-invertible, $y \in a_l^{s\|(b,c)}$, and $cay = c$. Now, for each $x \in a_l^{s\|(b,c)}$, we get $l(x) = l(b) = l(y)$ by Corollary 3.3. We conclude from $1 - xa \in l(x) = l(y)$ that $y = xay$, and hence that $c = cay = caxay$ and finally that $1 - axay \in r(c) \subseteq r(x)$ by Corollary 3.3. We thus get $x = xaxay = xay = y$. Hence, $cax = cay = c$.

(2) $\Rightarrow$ (1) Since a is strongly left (b,c)-invertible, there exists $x \in R$ such that

$$x = xax,\ l(x) = l(b),\ Rx \subseteq Rc,\ xR \subseteq bR, \text{ and } x \in a_l^{s\|(b,c)}.$$

It follows that $cax = c$. Write $x = dc = bt$. We have

$$x = xax = btax \in bRx, \text{ and } x = xax = xadc \in xRc.$$

Namely, $b = xab$ because $1 - xa \in l(x) = l(b)$. Thus, a is (b,c)-invertible and $a^{\|(b,c)} = x$. It is obvious that $a^{\|(b,c)} = x \in a_l^{s\|(b,c)}$. □

Proposition 3.7 *Let R be a ring and $a, b, c \in R$. Then the following conditions are equivalent:*

(1) a is (b,c)-invertible;

(2) a is strongly left (b,c)-invertible and $Rc \cap l(ab) = 0$.

Proof (1) $\Rightarrow$ (2) It follows from Proposition 3.6 that a is strongly left (b,c)-invertible. Now let $a^{\|(b,c)} = y$. Then $y = yay$, $yab = b$, and $cay = c$. Assume that $z \in Rc \cap l(ab)$. Then we have $z = dc$ and $zab = 0$, where $d \in R$. Thus, $dcab = 0$. Set $y = bs$. Then $z = dc = dcay = zay = zabs = 0$.

(2) $\Rightarrow$ (1) Let $x \in a_l^{s\|(b,c)}$. Then by Proposition 3.2, we get $xax = x$, $x = bvc$, $l(x) = l(b)$, and $cax = caxaxax = caxa(bvc)ax$. Hence, $ca - caxabvca \in l(x) = l(b)$. This gives $cab = caxabvcab$. We thus get $c - caxabvc \in l(ab) \cap Rc = 0$. This yields that $c = caxabvc = caxax = cax$. By Proposition 3.6, we have that a is (b,c)-invertible. □

Corollary 3.8 *Let R be a ring and $a, b, c \in R$. Then the following conditions are equivalent:*

(1) a is (b,c)-invertible;
(2) a is strongly left (b,c)-invertible and $l(c) = l(cab)$.

Proof (1) $\Rightarrow$ (2) Take any $x \in l(cab)$. We have $xcab = 0$. Thus, $xc \in Rc \cap l(ab) = 0$ by Proposition 3.7. That is, $x \in l(c)$.

(2) $\Rightarrow$ (1) For any $y \in Rc \cap l(ab)$, we know that $y = dc$ and $yab = 0$, where $d \in R$. Thus, $dcab = 0$. This means that $d \in l(cab) = l(c)$. Therefore, $y = dc = 0$. It follows from Proposition 3.7 that a is (b,c)-invertible. □

Corollary 3.9 *Let R be a ring and $a, b, c \in R$. Then the following conditions are equivalent:*

(1) a is (b,c)-invertible;
(2) a is strongly left (b,c)-invertible and $R = Rc \oplus l(ab)$.

Proof Assume that a is (b,c)-invertible. By Proposition 3.7, we know that $Rc \cap l(ab) = 0$. Write $a^{\|(b,c)} = y$. Then we have $y \in yRc$ and $b = yab$. Hence, $ab = ayab$. It follows that $1 - ay \in l(ab)$. We thus get $1 \in Ry + l(ab) \subseteq Rc + l(ab)$. Then $R = Rc + l(ab)$. That $R = Rc \oplus l(ab)$ follows from Proposition 3.7. The converse is obvious. □

Corollary 3.10 *Let R be a ring and $a, b, c \in R$. If a is (b,c)-invertible, then $R = Ra_l^{s\|(b,c)} \oplus l(ab)$.*

Proof Since a is (b,c)-invertible, $c = caa_l^{s\|(b,c)}$ by Proposition 3.6 and $R = Rc \oplus l(ab)$ by Corollary 3.9. Hence, $R = Ra_l^{s\|(b,c)} + l(ab)$. For any $z \in Ra_l^{s\|(b,c)} \cap l(ab)$, we have $z = ya_l^{s\|(b,c)}$ and $zab = 0$, where $y \in R$. This gives $ya_l^{s\|(b,c)}ab = 0$. Write $a_l^{s\|(b,c)} = btc$ for $t \in R$. Since $b = a_l^{s\|(b,c)}ab$, we have $z = ya_l^{s\|(b,c)} = ybtc = ya_l^{s\|(b,c)}abtc = 0$. The result is $Ra_l^{s\|(b,c)} \cap l(ab) = 0$. Therefore, $R = Ra_l^{s\|(b,c)} \oplus l(ab)$. □

Naturally, is the converse of the Corollary 3.10 true? The problem has not yet been solved.

Question 3.11 *If a is strongly left invertible and $Ra_l^{s\|(b,c)} \oplus l(ab) = R$, then is a (b,c)-invertible?*

4. Left min-Abel ring and (b,c)-inverse of a

This section is devoted to the study of left (resp. strongly left) min-Abel ring.

Let R be a ring and $e^2 = e \in R$. We denote by $E(R)$ the set of all idempotents of R. If Re is a left minimal ideal of R, then e is called a left minimal idempotent of R. Denote by $ME_l(R)$ the set of all left minimal idempotents of R. If either $ME_l(R)$ is an empty set or every element of $ME_l(R)$ is left (resp. right) semicentral in R, then R is called a left (resp. strongly left) min-Abel ring.

We first give some conditions to ensure that a ring R is a left min-Abel ring, by means of left semicentral elements and left (b, c)-invertible elements in R.

Lemma 4.1 *Let R be a ring and $e \in ME_l(R)$ a left semicentral idempotent. If $e = abe$ for $a, b \in R$, then $e = bae$.*

Proof Since e is left semicentral and $e = abe$, we have $e = aebe$. Thus, $ae \neq 0$. This gives $Re = Rae$. Writing $e = cae$ for $c \in R$, we can assert that $ce = c(aebe) = (cae)be = ebe = be$. It is obvious that $bae = beae = ceae = cae = e$. □

Proposition 4.2 *Let R be a ring. Then the following conditions are equivalent:*

(1) R is a left min-Abel ring;
(2) $e_a^- \subseteq {}_ae^-$ for any $e \in ME_l(R)$ and $a \in R$.

Proof (1) $\Rightarrow$ (2) Assume that R is a left min-Abel ring, $e \in ME_l(R)$, and $a \in R$. Fix $x \in e_a^-$. Then we have $e = (ex)ae$. Since R is a left min-Abel ring, we deduce that e is left semicentral. That $e = aexe = axe$ follows from Lemma 4.1. Thus, $e = eaxe$. That is, $x \in {}_ae^-$.

(2) $\Rightarrow$ (1) For any $e \in ME_l(R)$ and $a \in R$, writing $h = (1 - e)ae$, we can assert that $he = h$, $eh = 0$, and $h^2 = 0$. If $h \neq 0$, then $Rh = Re$. Taking $e = ch$ for $c \in R$, we get $e = eche$. That is, $c \in e_h^-$. From the hypothesis, we obtain that $e_h^- \subseteq {}_he^-$. It follows that $c \in {}_he^-$. We thus get $e = ehce = 0$. This contradicts our assumption. From this, we see that $h = 0$. It follows that $(1 - e)ae = h = 0$ for any $a \in R$. This gives $(1 - e)Re = 0$. Consequently, R is a left min-Abel ring.

□

Proposition 4.3 *Let R be a ring and $k \in E(R)$. Then the following conditions are equivalent:*

(1) k is a left minimal idempotent of R;
(2) if $ak \neq 0$ for $a \in R$, then a is left $(k, 1)$-invertible.

Proof (1) $\Rightarrow$ (2) Suppose that k is a left minimal idempotent of R and $ak \neq 0$. Then we get $Rk = Rak$. It follows that a is left $(k, 1)$-invertible.

(2) $\Rightarrow$ (1) Let $0 \neq L$ be any left ideal of R contained in Rk. Then we get $0 \neq y \in L \subseteq Rk$. Write $y = ak$. It follows that $ak \neq 0$. From the assumption, we know that a is left $(k, 1)$-invertible and $k \neq 0$. Then it is easy to see that $0 \neq Rk \subseteq R1ak = Ry \subseteq L$. That is, $Rk = L$. Hence, Rk is a left minimal ideal of R. □

Proposition 4.4 *Let R be a ring. Then the following conditions are equivalent:*

(1) R is a left min-Abel ring;
(2) if $ae \neq 0$ for $e \in ME_l(R)$ and $a \in R$, then there exists $c \in Re$ such that $e = cae$.

Proof (1) $\Rightarrow$ (2) Suppose that $ae \neq 0$. It follows from Proposition 4.3 that a is left $(e,1)$-invertible. For each $x \in a_l^{\|(e,1)}$, we get $e = xae$. Since R is a left min-Abel ring, we know that e is a left semicentral idempotent, i.e. $e = xeae$. Taking $c = xe \in Re$, the result holds.

(2) $\Rightarrow$ (1) For any $e \in ME_l(R)$, if $(1-e)Re \neq 0$, then there exists $a \in R$ such that $h = (1-e)ae \neq 0$. By assumption, there exists $c \in Re$ such that $e = che$ for $he = h \neq 0$. Write $c = te$. It is easy to show that $e = tehe = te(1-e)ae = 0$. It is a contradiction, so we have $(1-e)Re = 0$. Hence, R is a left min-Abel ring. □

Motivated by Propositions 4.2–4.4, in the following, we give the main result for this section.

Theorem 4.5 *Let R be a ring. Then the following conditions are equivalent:*

(1) R is a strongly left min-Abel ring;
(2) if $ea \neq 0$ for $e \in ME_l(R)$ and $a \in R$, then a is right (e,e)-invertible.

Proof (1) $\Rightarrow$ (2) We first show that eR is a minimal right ideal of R. Assume that $0 \neq K$ is an arbitrary right ideal of R contained in eR. For every $0 \neq x \in K$, we know $x = ex$. Since R is a strongly left min-Abel ring, e is a right semicentral idempotent. It follows that $x = xe$ and $0 \neq Rx = Rxe = Re$. Write $e = yx$ and $g = xy$, where $y \in R$. It is clear that

$$g^2 = xyxy = xey = xy = g,\ g = xy = exy = eg \text{ and } e = (yx)(yx) = ygx.$$

Moreover, $ge = ege = eg = g$. It follows that $0 \neq Rg = Rge \subseteq Re$. That is, $Rg = Re$. Thus, $g \in ME_l(R)$. This means that g is also a right semicentral idempotent. Furthermore, we get

$$e = ygx = ygxg = eg = g, \text{ and } eR = gR = xyR \subseteq xR \subseteq K \subseteq eR.$$

Thus, eR is a minimal right ideal of R.

Now we assume that $ea \neq 0$. Then we get $eaR = eR$ and write $e = eac$ for some $c \in R$. Since e is central, we have $e = eaec$, which means that a is right (e,e)-invertible.

(2) $\Rightarrow$ (1) Suppose that $e \in ME_l(R)$. If $eR(1-e) \neq 0$, then there exists some $a \in R$ such that $h = ea(1-e) \neq 0$. Since $eh = h$, we have that h is right (e,e)-invertible by (2). This clearly forces $e \in eheR$, so $e = 0$, which is a contradiction. It follows that $eR(1-e) = 0$. Hence, R is a strongly left min-Abel ring. □

Corollary 4.6 *Let R be a ring. Then the following conditions are equivalent:*

(1) R is a strongly left min-Abel ring;
(2) for each $e \in ME_l(R)$ and $x, y \in R$, $e = xy$ implies that $e = yx$.

Proof (1) $\Rightarrow$ (2) The proof is straightforward from Theorem 4.5.

(2) $\Rightarrow$ (1) For any $a \in R$, we denote $g = e + ea(1-e)$. It follows that $eg = g$ and $ge = e$. By assumption, we get $e = ge = eg = g$. It is obvious that $eR(1-e) = 0$. □

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