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Construction of the second Hankel determinant for a new subclass of bi-univalent functions

Şahsene ALTINKAYA*, **Sibel YALÇIN**
Department of Mathematics, Faculty of Arts and Science, Uludağ University, Bursa, Turkey

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Abstract: In this paper, we will discuss a newly constructed subclass of bi-starlike functions. Furthermore, we establish bounds for the coefficients and get the second Hankel determinant for the class $S_\Sigma(\alpha,\beta)$.

Key words: Analytic functions, bi-starlike functions, Hankel determinant

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$, and let S be the subclass of A consisting of the form (1), which are also univalent in U.

Among the most famous subclasses of univalent functions are the class $S^\star(\beta)$ of starlike functions of order β. By definition, we have

$$S^\star(\beta) = \left\{ f \in S : \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \right\} \qquad (0 \le \beta < 1, \quad z \in U).$$

The Koebe one-quarter theorem [9] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus, every such univalent function has an inverse f^{-1} that satisfies

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f)\ ,\ r_0(f) \ge \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

*Correspondence: sahsene@uludag.edu.tr
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A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions defined in the unit disk U.

For a brief history and interesting examples in the class Σ, see [26]. Examples of functions in the class Σ are

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in S such as

$$z-\frac{z^2}{2} \text{ and } \frac{z}{1-z^2}$$

are also not members of Σ (see [26]).

Lewin [16] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Later, Netanyahu [19] showed that $max\,|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Taha [6] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses, $S^{\star}(\beta)$ and $K(\beta)$ of starlike and convex function of order β $(0 \leq \beta < 1)$, respectively (see [19]). The classes $S_{\Sigma}^{\star}(\beta)$ and $K_{\Sigma}(\beta)$ of bi-starlike functions of order α and bi-convex functions of order β, corresponding to the function classes $S^{\star}(\beta)$ and $K(\beta)$, were also introduced analogously. For each of the function classes $S_{\Sigma}^{\star}(\beta)$ and $K_{\Sigma}(\beta)$, they found nonsharp estimates on the initial coefficients. Recently, many authors investigated bounds for various subclasses of bi-univalent functions [1, 4, 11, 17, 25–27]. Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions [2, 7, 13, 15]. The coefficient estimate problem for each of $|a_n|$ $(n \in \mathbb{N}\backslash\{1,2\};\ \ \mathbb{N}=\{1,2,3,...\})$ is still an open problem.

The Fekete–Szegö functional $\left|a_3-\mu a_2^2\right|$ for normalized univalent functions

$$f(z)=z+a_2z^2+\cdots$$

is well known for its rich history in the theory of geometric functions. Its origin was in the disproof by Fekete and Szegö of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see [10]). The functional has since received great attention, particularly in many subclasses of the family of univalent functions. Nowadays, it seems that this topic had become a point of interest among researchers (see, for example, [3, 18, 22, 28]).

In 1976, Noonan and Thomas [20] defined the q^{th} Hankel determinant of f for $n \geq 0$ and $q \geq 1$, defined by

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1=1).$$

This determinant has also been considered by several authors. For example, Noor [21] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained by the authors of articles [14, 21] for different classes of functions.

Note that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

The Hankel determinant $H_2(1) = a_3 - a_2^2$ is well known as the Fekete-Szegö functional. Very recently, the upper bounds of $H_2(2)$ for some classes were discussed by Deniz et al. [8] (see also [23]).

Definition 1 *A function $f \in \Sigma$ is said to be in the class $S_\Sigma(\alpha,\beta)$ if the following conditions holds:*

$$\Re\left\{\frac{1}{2}\left(\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}}\right)\right\} > \beta, \qquad (0 \le \beta < 1, \quad 0 < \alpha \le 1, \quad z \in U)$$

and

$$\Re\left\{\frac{1}{2}\left(\frac{wg'(w)}{g(w)} + \left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\alpha}}\right)\right\} > \beta, \qquad (0 \le \beta < 1, \quad 0 < \alpha \le 1, \quad w \in U),$$

where $g = f^{-1}$.

In this paper, we get the upper bound for the functional $H_2(2) = a_2 a_4 - a_3^2$ for functions f belonging to the class $S_\Sigma(\alpha,\beta)$.

2. Preliminary results

Let P be the class of functions with positive real part consisting of all analytic functions $p : U \to C$ satisfying $p(0) = 1$ and $Rep(z) > 0$.

Lemma 2 *[24] If the function $p \in P$,*

$$|p_n| \le 2 \qquad (n \in \mathbb{N} = \{1, 2, \ldots\})$$

and

$$\left|p_2 - \frac{p_1^2}{2}\right| \le 2 - \frac{|p_1|^2}{2}.$$

Lemma 3 *[12] If the function $p \in P$, then*

$$\begin{aligned} 2p_2 &= p_1^2 + x(4 - p_1^2) \\ 4p_3 &= p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z \end{aligned}$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

3. Main results

Theorem 4 *Let f given by (1) be in the class $S_\Sigma(\alpha,\beta)$, $0<\alpha\le 1$ and $0\le\beta<1$. Then*

$$\left|a_2a_4-a_3^2\right|\le\begin{cases}\frac{16\alpha^2}{3(\alpha+1)^2}(1-\beta)^2\left[\frac{8\left(6\alpha^3+2\alpha^2+3\alpha+1\right)}{3(\alpha+1)^3}(1-\beta)^2+1\right], \\ \qquad \beta\in\left[0,1-\frac{9\alpha(\alpha+1)^2+(\alpha+1)\sqrt{81\alpha^2(\alpha+1)^2+192(\alpha+1)(6\alpha^3+2\alpha^2+3\alpha+1)}}{32(6\alpha^3+2\alpha^2+3\alpha+1)}\right] \\ \frac{4\alpha^2}{(\alpha+1)^2}(1-\beta)^2\left\{1-\frac{9\{2\alpha(1-\beta)+(\alpha+1)\}^2(\alpha+1)}{32(6\alpha^3+2\alpha^2+3\alpha+1)(1-\beta)^2-36\alpha(\alpha+1)^2(1-\beta)-15(\alpha+1)^3}\right\} \\ \qquad \beta\in\left[1-\frac{9\alpha(\alpha+1)^2+(\alpha+1)\sqrt{81\alpha^2(\alpha+1)^2+192(\alpha+1)(6\alpha^3+2\alpha^2+3\alpha+1)}}{32(6\alpha^3+2\alpha^2+3\alpha+1)},1\right).\end{cases}.$$

Proof Let $f\in B(\alpha,\beta)$. Then

$$\frac{1}{2}\left(\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}}\right)=\beta+(1-\beta)p(z) \tag{2}$$

$$\frac{1}{2}\left(\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\alpha}}\right)=\beta+(1-\beta)q(z) \tag{3}$$

where $p,q\in P$ and $g=f^{-1}$.

It follows from (2) and (3) that

$$\frac{\alpha+1}{2\alpha}a_2=(1-\beta)\,p_1, \tag{4}$$

$$\frac{\alpha+1}{2\alpha}\left(2a_3-a_2^2\right)+\frac{1-\alpha}{4\alpha^2}a_2^2=(1-\beta)\,p_2, \tag{5}$$

$$\frac{\alpha+1}{2\alpha}\left(3a_4+a_2^3-3a_2a_3\right)+\frac{1-\alpha}{2\alpha^2}\left(2a_2a_3-a_2^3\right)+\frac{(1-\alpha)(1-2\alpha)}{12\alpha^3}a_2^3=(1-\beta)\,p_3 \tag{6}$$

$$-\frac{\alpha+1}{2\alpha}a_2=(1-\beta)\,q_1, \tag{7}$$

$$\frac{\alpha+1}{2\alpha}\left(3a_2^2-2a_3\right)+\frac{1-\alpha}{4\alpha^2}a_2^2=(1-\beta)\,q_2 \tag{8}$$

$$-\frac{\alpha+1}{2\alpha}\left(3a_4+10a_2^3-12a_2a_3\right)-\frac{1-\alpha}{2\alpha^2}(3a_2^3-2a_2a_3)-\frac{(1-\alpha)(1-2\alpha)}{12\alpha^3}a_2^3=(1-\beta)q_3. \tag{9}$$

From (4) and (7) we obtain

$$p_1=-q_1 \tag{10}$$

and

$$a_2=\frac{2\alpha}{\alpha+1}(1-\beta)\,p_1. \tag{11}$$

Subtracting (5) from (8), we have

$$a_3 = \frac{4\alpha^2}{(\alpha+1)^2}(1-\beta)^2 p_1^2 + \frac{\alpha}{2(\alpha+1)}(1-\beta)(p_2-q_2). \tag{12}$$

Also, subtracting (6) from (9), we have

$$a_4 = \frac{12\alpha^3+16\alpha^2-3\alpha-1}{18\alpha^2(\alpha+1)}\frac{8\alpha^3}{(\alpha+1)^3}(1-\beta)^3 p_1^3 + \frac{5\alpha^2}{2(\alpha+1)^2}(1-\beta)^2 p_1(p_2-q_2) + \frac{\alpha}{3(\alpha+1)}(1-\beta)(p_3-q_3). \tag{13}$$

Then we can establish that

$$\begin{aligned}\left|a_2a_4-a_3^2\right| = \Big| & -\frac{(6\alpha^3+2\alpha^2+3\alpha+1)}{18\alpha^2(\alpha+1)}\frac{16\alpha^4}{(\alpha+1)^4}(1-\beta)^4 p_1^4 + \frac{\alpha^3}{(\alpha+1)^3}(1-\beta)^3 p_1^2(p_2-q_2) \\ & + \frac{2\alpha^2}{3(\alpha+1)^2}(1-\beta)^2 p_1(p_3-q_3) - \frac{\alpha^2}{4(\alpha+1)^2}(1-\beta)^2(p_2-q_2)^2\Big|.\end{aligned} \tag{14}$$

According to Lemma 2 and (10), we write

$$\left.\begin{array}{l} 2p_2 = p_1^2 + x(4-p_1^2) \\ 2q_2 = q_1^2 + y(4-q_1^2) \end{array}\right\} \Rightarrow p_2 - q_2 = \frac{4-p_1^2}{2}(x-y) \tag{15}$$

and

$$\begin{aligned} 4p_3 &= p_1^3 + 2(4-p_1^2)p_1x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2)z \\ 4q_3 &= q_1^3 + 2(4-q_1^2)q_1y - q_1(4-q_1^2)y^2 + 2(4-q_1^2)(1-|y|^2)w \end{aligned}$$

$$p_3 - q_3 = \frac{p_1^3}{2} + \frac{p_1(4-p_1^2)}{2}(x+y) - \frac{p_1(4-p_1^2)}{4}(x^2+y^2) + \frac{4-p_1^2}{2}\left[(1-|x|^2)z - (1-|y|^2)w\right]. \tag{16}$$

Then, using (15) and (16), in (14),

$$\begin{aligned}\left|a_2a_4-a_3^2\right| = \Big| & -\frac{(6\alpha^3+2\alpha^2+3\alpha+1)}{18\alpha^2(\alpha+1)}\frac{16\alpha^4}{(\alpha+1)^4}(1-\beta)^4 p_1^4 + \frac{\alpha^3}{(\alpha+1)^3}(1-\beta)^3 p_1^2\frac{(4-p_1^2)}{2}(x-y) \\ & + \frac{\alpha^2(1-\beta)^2}{3(\alpha+1)^2}p_1^4 + \frac{2\alpha^2(1-\beta)^2}{3(\alpha+1)^2}p_1^2\frac{(4-p_1^2)}{2}(x+y) - \frac{2\alpha^2(1-\beta)^2}{3(\alpha+1)^2}p_1^2\frac{(4-p_1^2)}{4}(x^2+y^2) \\ & + \frac{2\alpha^2(1-\beta)^2}{3(\alpha+1)^2}p_1\frac{(4-p_1^2)}{2}\left[\left(1-|x|^2\right)z - \left(1-|y|^2\right)w\right] - \frac{\alpha^2(1-\beta)^2}{4(\alpha+1)^2}\frac{(4-p_1^2)^2}{4}(x-y)^2\Big|. \\ \leq & \frac{(6\alpha^3+2\alpha^2+3\alpha+1)}{18\alpha^2(\alpha+1)}\frac{16\alpha^4}{(\alpha+1)^4}(1-\beta)^4 p_1^4 + \frac{\alpha^2}{3(\alpha+1)^2}(1-\beta)^2 p_1^4 + \frac{2\alpha^2}{3(\alpha+1)^2}(1-\beta)^2 p_1(4-p_1^2) \\ & + \left[\frac{\alpha^3}{(\alpha+1)^3}(1-\beta)^3 p_1^2\frac{(4-p_1^2)}{2} + \frac{2\alpha^2}{3(\alpha+1)^2}(1-\beta)^2 p_1^2\frac{(4-p_1^2)}{2}\right](|x|+|y|) \\ & + \left[\frac{2\alpha^2(1-\beta)^2}{3(\alpha+1)^2}p_1^2\frac{(4-p_1^2)}{4} - \frac{2\alpha^2(1-\beta)^2}{3(\alpha+1)^2}p_1\frac{(4-p_1^2)}{2}\right](|x|^2+|y|^2) + \frac{\alpha^2(1-\beta)^2}{4(\alpha+1)^2}\frac{(4-p_1^2)^2}{4}(|x|+|y|)^2.\end{aligned} \tag{17}$$

Since $p \in P,\ |p_1| \leq 2$. Letting $|p_1| = p$, we may assume without restriction that $p \in [0,2]$. For $\eta = |x| \leq 1$ and $\mu = |y| \leq 1$, we get

$$\left|a_2a_4 - a_3^2\right| \leq T_1 + (\eta+\mu)\,T_2 + \left(\eta^2+\mu^2\right)T_3 + (\eta+\mu)^2\,T_4 = G(\eta,\mu)$$

where

$$\begin{aligned}
T_1 &= T_1(p) = \frac{\alpha^2}{3\,(\alpha+1)^2}\,(1-\beta)^2\left[\left(\tfrac{8(6\alpha^3+2\alpha^2+3\alpha+1)}{3(\alpha+1)^3}\,(1-\beta)^2+1\right)p^4 - 2p^3 + 8p\right] \geq 0\\
T_2 &= T_2(p) = \frac{\alpha^2}{(\alpha+1)^2}\,(1-\beta)^2\,p^2\,\frac{(4-p^2)}{2}\left[\frac{\alpha}{\alpha+1}\,(1-\beta)+\tfrac{2}{3}\right] \geq 0\\
T_3 &= T_3(p) = \frac{\alpha^2}{3\,(\alpha+1)^2}\,(1-\beta)^2\,\frac{(4-p^2)}{2}p(p-2) \leq 0\\
T_4 &= T_4(p) = \frac{\alpha^2}{4\,(\alpha+1)^2}\,(1-\beta)^2\,\frac{(4-p^2)^2}{4} \geq 0.
\end{aligned}$$

We now need to maximize the function $G(\eta,\mu)$ on the closed square $[0,1]\times[0,1]$. We must investigate the maximum of $G(\eta,\mu)$ according to $p \in (0,2)$, $p=0$ and $p=2$, taking into account the sign of $G_{\eta\eta}.G_{\mu\mu} - (G_{\eta\mu})^2$.

First, let $p \in (0,2)$. Since $T_3 < 0$ and $T_3 + 2T_4 > 0$ for $p \in (0,2)$, we conclude that

$$G_{\eta\eta}.G_{\mu\mu} - (G_{\eta\mu})^2 < 0.$$

Thus, the function G cannot have a local maximum in the interior of the square. Now we investigate the maximum of G on the boundary of the square.

For $\eta = 0$ and $0 \leq \mu \leq 1$ (similarly $\mu = 0$ and $0 \leq \eta \leq 1$), we obtain

$$G(0,\mu) = H(\mu) = (T_3+T_4)\mu^2 + T_2\mu + T_1.$$

i. The case $T_3 + T_4 \geq 0$: In this case for $0 < \mu < 1$ and any fixed p with $0 \leq p < 2$, it is clear that $H'(\mu) = 2(T_3+T_4)\mu + T_2 > 0$; that is, $H(\mu)$ is an increasing function. Hence, for fixed $p \in [0,2)$, the maximum of $H(\mu)$ occurs at $\mu = 1$, and

$$\max H(\mu) = H(1) = T_1 + T_2 + T_3 + T_4.$$

ii. The case $T_3 + T_4 < 0$: Since $T_2 + 2(T_3+T_4) \geq 0$ for $0 < \mu < 1$ and any fixed p with $0 \leq p < 2$, it is clear that $T_2 + 2(T_3+T_4) < 2(T_3+T_4)\mu + T_2 < T_2$ and so $H'(\mu) > 0$. Hence, for fixed $p \in [0,2)$, the maximum of $H(\mu)$ occurs at $\mu = 1$.

Also, for $p = 2$, we obtain

$$G(\eta,\mu) = \frac{16\alpha^2}{3\,(\alpha+1)^2}\,(1-\beta)^2\left[\tfrac{8\left(6\alpha^3+2\alpha^2+3\alpha+1\right)}{3(\alpha+1)^3}\,(1-\beta)^2+1\right]. \tag{18}$$

Taking into account value (18), and cases i and ii, for $0 \leq \mu \leq 1$ and any fixed p with $0 \leq p \leq 2$,

$$\max H(\mu) = H(1) = T_1 + T_2 + T_3 + T_4.$$

For $\eta = 1$ and $0 \le \mu \le 1$ (similarly $\mu = 1$ and $0 \le \eta \le 1$), we obtain

$$G(1,\mu) = F(\mu) = (T_3 + T_4)\mu^2 + (T_2 + 2T_4)\mu + T_1 + T_2 + T_3 + T_4.$$

Similarly to the above cases of $T_3 + T_4$, we get that

$$\max F(\mu) = F(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Since $H(1) \le F(1)$ for $p \in [0,2]$, $\max G(\eta,\mu) = G(1,1)$ on the boundary of the square. Thus, the maximum of G occurs at $\eta = 1$ and $\mu = 1$ in the closed square.

Let $K : [0,2] \to R$.

$$K(p) = \max G(\eta,\mu) = G(1,1) = T_1 + 2T_2 + 2T_3 + 4T_4. \tag{19}$$

Substituting the values of T_1, T_2, T_3, and T_4 in the function K defined by (19) yields

$$\begin{aligned} K(p) &= \frac{\alpha^2}{(\alpha+1)^2}(1-\beta)^2\left\{\left(\tfrac{8(6\alpha^3+2\alpha^2+3\alpha+1)}{9(\alpha+1)^3}(1-\beta)^2 - \tfrac{\alpha}{\alpha+1}(1-\beta) - \frac{5}{12}\right)p^4 \right. \\ &\quad \left. + \left(\tfrac{4\alpha}{\alpha+1}(1-\beta)+2\right)p^2 + 4\right\}. \end{aligned}$$

Assuming that $K(p)$ has a maximum value in an interior of $p \in [0,2]$, by elementary calculation

$$\begin{aligned} K'(p) &= \frac{2\alpha^2}{(\alpha+1)^2}(1-\beta)^2\left\{\left(\tfrac{16(6\alpha^3+2\alpha^2+3\alpha+1)}{9(\alpha+1)^3}(1-\beta)^2 - \tfrac{2\alpha}{\alpha+1}(1-\beta) - \frac{5}{6}\right)p^3 \right. \\ &\quad \left. + \left(\tfrac{4\alpha}{\alpha+1}(1-\beta)+2\right)p\right\}. \end{aligned}$$

As a result of some calculations we can do the following: □

Case 1. Let $\left(\frac{16(6\alpha^3+2\alpha^2+3\alpha+1)}{9(\alpha+1)^3}(1-\beta)^2 - \frac{2\alpha}{\alpha+1}(1-\beta) - \frac{5}{6}\right) \ge 0$. Therefore, $\beta \in \left[0, 1 - \frac{9\alpha(\alpha+1)^2 + (\alpha+1)\sqrt{81\alpha^2(\alpha+1)^2 + 120(\alpha+1)(6\alpha^3+2\alpha^2+3\alpha+1)}}{16(6\alpha^3+2\alpha^2+3\alpha+1)}\right]$ and $K'(p) > 0$ for $p \in (0,2)$. Since K is an increasing function in the interval $(0,2)$, the maximum point of K must be on the boundary of $p \in [0,2]$; that is, $p = 2$. Thus, we have

$$\max K(p) = K(2) = \frac{16\alpha^2}{3(\alpha+1)^2}(1-\beta)^2\left[\tfrac{8(6\alpha^3+2\alpha^2+3\alpha+1)}{3(\alpha+1)^3}(1-\beta)^2 + 1\right].$$

Case 2. Let $\left(\frac{16(6\alpha^3+2\alpha^2+3\alpha+1)}{9(\alpha+1)^3}(1-\beta)^2 - \frac{2\alpha}{\alpha+1}(1-\beta) - \frac{5}{6}\right) < 0$; that is, $\beta \in \left(1 - \frac{9\alpha(\alpha+1)^2 + (\alpha+1)\sqrt{81\alpha^2(\alpha+1)^2 + 120(\alpha+1)(6\alpha^3+2\alpha^2+3\alpha+1)}}{16(6\alpha^3+2\alpha^2+3\alpha+1)}, 1\right)$. Then $K'(p) = 0$ implies the real critical points $p_{0_1} = 0$ or

$$p_{0_2} = \sqrt{\frac{-36\{2\alpha(1-\beta)+(\alpha+1)\}(\alpha+1)^2}{32(6\alpha^3+2\alpha^2+3\alpha+1)(1-\beta)^2 - 36\alpha(\alpha+1)^2(1-\beta) - 15(\alpha+1)^3}}.$$

When

$$\beta \in \begin{bmatrix} 1-\frac{9\alpha(\alpha+1)^2+(\alpha+1)\sqrt{81\alpha^2(\alpha+1)^2+120(\alpha+1)(6\alpha^3+2\alpha^2+3\alpha+1)}}{16(6\alpha^3+2\alpha^2+3\alpha+1)}, \\ 1-\frac{9\alpha(\alpha+1)^2+(\alpha+1)\sqrt{81\alpha^2(\alpha+1)^2+192(\alpha+1)(6\alpha^3+2\alpha^2+3\alpha+1)}}{32(6\alpha^3+2\alpha^2+3\alpha+1)} \end{bmatrix}$$

we observe that $p_{0_2} \geq 2$; that is, p_{0_2} is out of the interval $(0,2)$. Therefore, the maximum value of $K(p)$ occurs at $p_{0_1}=0$ or $p=p_{0_2}$, which contradicts our assumption of having the maximum value at the interior point of $p \in [0,2]$. Since K is an increasing function in the interval $(0,2)$, the maximum point of K must be on the boundary of $p \in [0,2]$; that is, $p=2$. Thus, we have

$$\max K(p) = K(2) = \frac{16\alpha^2}{3\left(\alpha+1\right)^2}\left(1-\beta\right)^2\left[\frac{8\left(6\alpha^3+2\alpha^2+3\alpha+1\right)}{3(\alpha+1)^3}\left(1-\beta\right)^2+1\right].$$

When $\beta \in \left(1-\frac{9\alpha(\alpha+1)^2+(\alpha+1)\sqrt{81\alpha^2(\alpha+1)^2+192(\alpha+1)(6\alpha^3+2\alpha^2+3\alpha+1)}}{32(6\alpha^3+2\alpha^2+3\alpha+1)},1\right)$ we observe that $p_{0_2}<2$; that is, p_{0_2} is the interior of the interval $[0,2]$. Since $K''(p_{02})<0$, the maximum value of $K(p)$ occurs at $p=p_{0_2}$. Thus, we have

$$K(p_{0_2}) = \frac{4\alpha^2}{\left(\alpha+1\right)^2}\left(1-\beta\right)^2\left\{1-\frac{9\{2\alpha(1-\beta)+(\alpha+1)\}^2(\alpha+1)}{32(6\alpha^3+2\alpha^2+3\alpha+1)(1-\beta)^2-36\alpha(\alpha+1)^2(1-\beta)-15(\alpha+1)^3}\right\}.$$

This completes the proof.

Remark 5 *Putting $\alpha=1$ in Theorem 4 we have the second Hankel determinant for the well-known class $S^*_\Sigma(\beta)$ as in [8].*

Corollary 6 *(see [8]) Let f given by (1) be in the class $S^*_\Sigma(\beta)$ and $0 \leq \alpha < 1$. Then*

$$\left|a_2a_4-a_3^2\right| \leq \begin{cases} \frac{4(1-\beta)^2}{3}\left(4\beta^2-8\beta+5\right) & \beta \in \left[0, \frac{29-\sqrt{137}}{32}\right) \\ (1-\beta)^2\left(\frac{13\beta^2-14\beta-7}{16\beta^2-26\beta+5}\right) & \beta \in \left(\frac{29-\sqrt{137}}{32}, 1\right) \end{cases}.$$

Remark 7 *For $\beta=0$ and $\alpha=1$, Theorem 4 readily yields the following coefficient estimates for bi-starlike functions.*

Corollary 8 *(see [8]) Let f given by (1) be in the class S^*_Σ. Then*

$$\left|a_2a_4-a_3^2\right| \leq \frac{20}{3}.$$

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