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# Generalized metric $n$-Leibniz algebras and generalized orthogonal representation of metric Lie algebras

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**Abstract:** We introduce the notion of a generalized metric $n$-Leibniz algebra and show that there is a one-to-one correspondence between generalized metric $n$-Leibniz algebras and faithful generalized orthogonal representations of metric Lie algebras (called Lie triple datas). We further show that there is also a one-to-one correspondence between generalized orthogonal derivations (resp. generalized orthogonal automorphisms) on generalized metric $n$-Leibniz algebras and Lie triple data.

**Key words:** Generalized metric $n$-Leibniz algebra, metric Lie algebra, generalized orthogonal representation, generalized orthogonal derivation, generalized orthogonal automorphism

## 1. Introduction

Ternary Lie algebras (3-Lie algebras) or more generally $n$-ary Lie algebras are the natural generalization of Lie algebras. They were introduced and studied by Filippov in [13] and can be traced back to Nambu [22]. See [15–17, 23] and the review article [9] for more details. This type of algebras appeared also in the algebraic formulation of Nambu mechanics [22] and generalizing Hamiltonian mechanics by considering two Hamiltonians; see [14, 24]. Moreover, 3-Lie algebras appeared in string theory and M-theory. In [3], Basu and Harvey suggested replacing the Lie algebra appearing in the Nahm equation by a 3-Lie algebra for the lifted Nahm equations. Furthermore, in the context of the Bagger–Lambert–Gustavsson model of multiple M2-branes, Bagger and Lambert managed to construct, using a ternary bracket, an $N = 2$ supersymmetric version of the world volume theory of the M-theory membrane; see [1]. These metric 3-Leibniz algebras (generalized 3-Lie algebras) have many applications; see [6, 7, 12, 20] for more details. Metric 3-Lie algebras and metric $n$-Lie algebras were further studied in [2, 21, 25].

The notion of an $n$-Leibniz algebra was introduced in [5] as a generalization of an $n$-Lie algebra and a Leibniz algebra [18, 19]. See also [10] for more results. Through fundamental objects one may represent an $n$-Leibniz algebra by a Leibniz algebra [8]. Motivated by the work in [11], where the authors established a one-to-one correspondence between metric 3-Leibniz algebras and faithful orthogonal representation of metric Lie algebras, it is natural to investigate the $n$-ary case. However, for the usual metric $n$-Leibniz algebras, where $n > 3$, one cannot use the method provided in [11]. We overcome this difficulty by introducing the notion of a generalized metric $n$-Leibniz algebra, where the "metric" is a symmetric nondegenerate $(n-1)$-linear form

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satisfying some compatibility conditions. We also introduce the notion of a generalized orthogonal representation of a Lie algebra and show that there is a one-to-one correspondence between generalized metric $n$-Leibniz algebras and faithful generalized orthogonal representation of metric Lie algebras. We also lift this one-to-one correspondence to the level of generalized orthogonal derivations and generalized orthogonal automorphisms.

The paper is organized as follows. In Section 2, we give a review of $n$-Leibniz algebras and metric Lie algebras. In Section 3, we construct a faithful generalized orthogonal representation of a metric Lie algebra from a generalized metric $n$-Leibniz algebra. In Section 4, we construct a generalized metric $n$-Leibniz algebra from a faithful generalized orthogonal representation of a metric Lie algebra. In Section 5, we show that there is a one-to-one correspondence between generalized orthogonal derivations on generalized metric $n$-Leibniz algebras and Lie triple data. In Section 6, we show that there is a one-to-one correspondence between generalized orthogonal automorphisms on generalized metric $n$-Leibniz algebras and Lie triple data.

In this paper, we work over the real field $\mathbb{R}$ and all the vector spaces are finite-dimensional.

## 2. Preliminaries

**Definition 2.1** ([5]) *An $n$-Leibniz algebra is a vector space $\mathcal{V}$ equipped with an $n$-linear map $[\cdot,\cdots,\cdot]:\mathcal{V}\times\cdots\times\mathcal{V}\to\mathcal{V}$ such that for all $u_1,\cdots,u_{n-1},\ v_1,\cdots,v_n\in\mathcal{V}$, the following* **fundamental** *identity holds:*

$$[u_1,\cdots,u_{n-1},[v_1,\cdots,v_n]]=\sum_{i=1}^{n}[v_1,\cdots,v_{i-1},[u_1,\cdots,u_{n-1},v_i],v_{i+1},\cdots,v_n]. \tag{1}$$

In particular, if $n=2$, we obtain the notion of a Leibniz algebra [18, 19]. If the $n$-linear map $[\cdot,\cdots,\cdot]$ is skew-symmetric, we obtain the notion of an $n$-Lie algebra [13]. In the sequel, when we say an $n$-Leibniz algebra, we always assume that $n\geq 3$.

**Definition 2.2** ([5]) *A* **derivation** *on an $n$-Leibniz algebra $(\mathcal{V},[\cdot,\cdots,\cdot])$ is a linear map $\mathrm{d}_{\mathcal{V}}\in\mathfrak{gl}(\mathcal{V})$, such that for all $u_1,\cdots,u_n\in\mathcal{V}$ the following equality holds:*

$$\mathrm{d}_{\mathcal{V}}[u_1,\cdots,u_n]=\sum_{i=1}^{n}[u_1,\cdots,u_{i-1},\mathrm{d}_{\mathcal{V}}u_i,u_{i+1},\cdots,u_n]. \tag{2}$$

Define $D:\otimes^{n-1}\mathcal{V}\to\mathfrak{gl}(\mathcal{V})$ by

$$D(u_1,\cdots,u_{n-1})u_n=[u_1,\cdots,u_{n-1},u_n],\quad \forall u_1,\cdots,u_{n-1},u_n\in\mathcal{V}. \tag{3}$$

Then the fundamental identity (1) is the condition that $D(u_1,\cdots,u_{n-1})$ is a derivation on the $n$-Leibniz algebra $(\mathcal{V},[\cdot,\cdots,\cdot])$.

On $\otimes^{n-1}\mathcal{V}$, one can define a new bracket operation $[\cdot,\cdot]_{\mathsf{F}}$ by

$$[U,V]_{\mathsf{F}}=\sum_{i=1}^{n-1}v_1\otimes\cdots\otimes v_{i-1}\otimes[u_1,\cdots,u_{n-1},v_i]\otimes v_{i+1}\otimes\cdots\otimes v_{n-1}, \tag{4}$$

for all $U=u_1\otimes\cdots\otimes u_{n-1},\ V=v_1\otimes\cdots\otimes v_{n-1}\in\otimes^{n-1}\mathcal{V}$. It is proved in [8] that $(\otimes^{n-1}\mathcal{V},[\cdot,\cdot]_{\mathsf{F}})$ is a Leibniz algebra. The fundamental identity (1) is equivalent to

$$[D(U),D(V)]=D([U,V]_{\mathsf{F}}). \tag{5}$$

Thus, we obtain that $D$ is a Leibniz algebra homomorphism from $\otimes^{n-1}\mathcal{V}$ to $\mathfrak{gl}(\mathcal{V})$.

**Definition 2.3** ([4, Definition 2]) *Let* $(A, \cdot)$ *be a nonassociative algebra and* $\omega$ *a nondegenerate symmetric bilinear form on* $A$.

(i) *If* $\omega(x \cdot y, z) = \omega(x, y \cdot z)$, *then we say that* $\omega$ *is* **associative-invariant***;*

(ii) *If* $\omega(x \cdot y, z) = -\omega(y, x \cdot z)$, *then we say that* $\omega$ *is* **(left) ad-invariant***;*

(iii) *If* $\omega(x \cdot y, z) = -\omega(x, z \cdot y)$, *then we say that* $\omega$ *is* **(right) ad-invariant***.*

A nondegenerate symmetric bilinear form $\omega$ satisfies at least two of the preceding definitions if and only if $(A, \cdot)$ is an anticommutative algebra. Since a Lie bracket is skew-symmetric, we obtain that left ad-invariant, right ad-invariant, and associative-invariant nondegenerate symmetric bilinear forms on a Lie algebra are the same. See [4] for more details.

Recall that a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ is a said to be **metric** if it is equipped with a symmetric nondegenerate bilinear form $\omega$ that is **(left) ad-invariant**, i.e.:

$$\omega([x, y], z) = -\omega(y, [x, z]), \quad \forall x, y, z \in \mathfrak{g}. \tag{6}$$

Moreover, there is a natural notion of orthogonal derivations and automorphisms on metric Lie algebras.

**Definition 2.4** *Let* $(\mathfrak{g}, [\cdot, \cdot], \omega)$ *be a metric Lie algebra. A derivation* $\mathrm{d}_{\mathfrak{g}}$ *on the Lie algebra* $(\mathfrak{g}, [\cdot, \cdot])$ *is called* **orthogonal** *if the following equality holds:*

$$\omega(\mathrm{d}_{\mathfrak{g}} x, y) + \omega(x, \mathrm{d}_{\mathfrak{g}} y) = 0. \tag{7}$$

**Definition 2.5** *Let* $(\mathfrak{g}, [\cdot, \cdot], \omega)$ *be a metric Lie algebra. An automorphism* $\Phi_{\mathfrak{g}}$ *on the Lie algebra* $(\mathfrak{g}, [\cdot, \cdot])$ *is called* **orthogonal** *if the following equality holds:*

$$\omega(\Phi_{\mathfrak{g}} x, \Phi_{\mathfrak{g}} y) = \omega(x, y). \tag{8}$$

## 3. Construction of Lie triple data from a generalized metric $n$-Leibniz algebra

Let $\mathcal{V}$ be a vector space and $\mathcal{V}^*$ its dual space. Denote by $\mathrm{Sym}^k(\mathcal{V}^*)$ the vector space of symmetric tensors of order $k$ on $\mathcal{V}^*$. Any $\phi \in \mathrm{Sym}^k(\mathcal{V}^*)$ induces a linear map $\phi^\sharp : \mathcal{V} \longrightarrow \mathrm{Sym}^{k-1}(\mathcal{V}^*)$ by

$$\phi^\sharp(u)(v_1, \cdots, v_{k-1}) = \phi(u, v_1, \cdots, v_{k-1}), \quad \forall u, v_1, \cdots, v_{k-1} \in \mathcal{V}.$$

$\phi \in \mathrm{Sym}^k(\mathcal{V}^*)$ is said to be nondegenerate if the induced map $\phi^\sharp : \mathcal{V} \longrightarrow \mathrm{Sym}^{k-1}(\mathcal{V}^*)$ is nondegenerate; that is, $\phi^\sharp(u) = 0$ if and only if $u = 0$.

**Definition 3.1** *A* **generalized metric $n$-Leibniz algebra** *is an $n$-Leibniz algebra* $(\mathcal{V}, [\cdot, \cdots, \cdot])$ *equipped with a symmetric nondegenerate* $(n-1)$*-tensor* $S \in \mathrm{Sym}^{n-1}(\mathcal{V}^*)$ *satisfying the following axioms for all* $u_1, \cdots, u_{n-1}$, $v_1, \cdots, v_{n-1} \in \mathcal{V}$:

(a) *The* **unitarity** *condition*

$$\sum_{i=1}^{n-1} S(v_1, \cdots, v_{i-1}, [u_1, \cdots, u_{n-1}, v_i], v_{i+1}, \cdots, v_{n-1}) = 0; \tag{9}$$

(b) *The* **symmetry** *condition*

$$S([u_1, u_2, \cdots, u_{n-1}, v_1], v_2, \cdots, v_{n-1}) = S([v_1, \cdots, v_{n-1}, u_1], u_2, \cdots, u_{n-1}). \tag{10}$$

We denote a generalized metric $n$-Leibniz algebra by $(\mathcal{V}, [\cdot, \cdots, \cdot], S)$.

**Remark 3.2** *When $n = 3$ in Definition 3.1, we obtain the notion of a generalized metric 3-Leibniz algebra, which is the same as the generalized metric Lie 3-algebra introduced in [11, Definition 1]. See [11] for more applications of generalized metric Lie 3-algebras in the BLG theory.*

**Proposition 3.3** *Let $(\mathcal{V}, [\cdot, \cdots, \cdot], S)$ be a generalized metric $n$-Leibniz algebra. Then we have*

$$\sum_{i=1}^{n-1} [v_i, v_1, \cdots, v_{i-1}, \hat{v_i}, v_{i+1}, \cdots, v_{n-1}, v_n] = 0, \quad \forall v_1, \cdots, v_n \in \mathcal{V}.$$

**Proof** For all $v_1, \cdots, v_n, u_1, \cdots, u_{n-2}$, we have

$$\begin{aligned}
& S([v_1, v_2, \cdots, v_{n-1}, v_n], u_1, \cdots, u_{n-2}) \\
\overset{(10)}{=} \; & S([v_n, u_1, \cdots, u_{n-2}, v_1], v_2, \cdots, v_{n-1}) \\
\overset{(9)}{=} \; & -\sum_{i=2}^{n-1} S([v_n, u_1, \cdots, u_{n-2}, v_i], v_1, v_2, \cdots, v_{i-1}, \hat{v_i}, v_{i+1}, \cdots, v_{n-1}) \\
\overset{(10)}{=} \; & -\sum_{i=2}^{n-1} S([v_i, v_1, v_2, \cdots, v_{i-1}, \hat{v_i}, v_{i+1}, \cdots, v_{n-1}, v_n], u_1, \cdots, u_{n-2}).
\end{aligned}$$

Since $S$ is nondegenerate, we have

$$[v_1, v_2, \cdots, v_{n-1}, v_n] = -\sum_{i=2}^{n-1} [v_i, v_1, \cdots, v_{i-1}, \hat{v_i}, v_{i+1}, \cdots, v_{n-1}, v_n],$$

which finishes the proof. □

**Definition 3.4** *Let $(\mathcal{V}, [\cdot, \cdots, \cdot], S)$ be a generalized metric $n$-Leibniz algebra. A derivation $\mathrm{d}_{\mathcal{V}}$ on the $n$-Leibniz algebra $(\mathcal{V}, [\cdot, \cdots, \cdot])$ is called* **generalized orthogonal** *if the following equality holds:*

$$\sum_{i=1}^{n-1} S(v_1, \cdots, \mathrm{d}_{\mathcal{V}} v_i, \cdots, v_{n-1}) = 0, \tag{11}$$

*for all $v_1, \cdots, v_{n-1} \in \mathcal{V}$.*

**Definition 3.5** *Let $(\mathcal{V}, [\cdot, \cdots, \cdot], S)$ be a generalized metric $n$-Leibniz algebra. An automorphism $\Phi_{\mathcal{V}}$ on the $n$-Leibniz algebra $(\mathcal{V}, [\cdot, \cdots, \cdot])$ is called* **generalized orthogonal** *if the following equality holds:*

$$S(\Phi_{\mathcal{V}} v_1, \cdots, \Phi_{\mathcal{V}} v_{n-1}) = S(v_1, \cdots, v_{n-1}), \tag{12}$$

*for all $v_1, \cdots, v_{n-1} \in \mathcal{V}$.*

**Definition 3.6** *Let $\mathfrak{g}$ be a Lie algebra and $\mathcal{V}$ a vector space equipped with a symmetric nondegenerate $(n-1)$-tensor $S \in \mathrm{Sym}^{n-1}(\mathcal{V}^*)$. A representation $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathcal{V})$ is called* **generalized orthogonal** *if the following equality holds:*

$$\sum_{i=1}^{n-1} S(w_1, \cdots, w_{i-1}, \rho(x)w_i, w_{i+1}, \cdots w_{n-1}) = 0, \tag{13}$$

*for all $x \in \mathfrak{g}$ and $w_1, w_2, \cdots, w_{n-1} \in \mathcal{V}$.*

We denote a generalized orthogonal representation by $(\rho, \mathcal{V}, S)$. When $n = 3$, we recover the usual notion of an orthogonal representation of a Lie algebra.

We introduce the notion of Lie triple data, which is the main object in this paper.

**Definition 3.7** **Lie triple data** *consist of the following structure:*

(i) *a metric Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \omega)$;*

(ii) *a vector space $\mathcal{V}$ equipped with a symmetric nondegenerate $(n-1)$-tensor $S \in \mathrm{Sym}^{n-1}(\mathcal{V}^*)$;*

(iii) *a faithful generalized orthogonal representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathcal{V})$.*

*We will denote a Lie triple data by $(\mathfrak{g}, \mathcal{V}, \rho)$.*

### 3.1. From an $n$-algebra to a Lie algebra

Let $(\mathcal{V}, [\cdot, \cdots, \cdot], S)$ be a generalized metric $n$-Leibniz algebra. Let $\mathfrak{g} = \mathrm{Im}D \subset \mathfrak{gl}(\mathcal{V})$, where $D$ is given by (3).

**Proposition 3.8** *$(\mathfrak{g}, [\cdot,\cdot]_C)$ is a Lie subalgebra of $\mathfrak{gl}(\mathcal{V})$, where $[\cdot,\cdot]_C$ denotes the commutator Lie bracket on $\mathfrak{gl}(\mathcal{V})$.*

**Proof** By the fundamental identity (1), we have

$$\begin{aligned} & D(u_1, \cdots, u_{n-1})(D(v_1, \cdots, v_{n-1})v_n) - D(v_1, \cdots, v_{n-1})(D(u_1, \cdots, u_{n-1})v_n) \\ = & \sum_{i=1}^{n-1} D(v_1, \cdots, v_{i-1}, D(u_1, \cdots, u_{n-1})v_i, v_{i+1}, \cdots, v_{n-1})v_n. \end{aligned}$$

Hence, we have

$$[D(u_1, \cdots, u_{n-1}), D(v_1, \cdots, v_{n-1})]_C = \sum_{i=1}^{n-1} D(v_1, \cdots, v_{i-1}, D(u_1, \cdots, u_{n-1})v_i, v_{i+1}, \cdots, v_{n-1}) \in \mathfrak{g}, \tag{14}$$

which shows that $[\mathfrak{g}, \mathfrak{g}]_C \subset \mathfrak{g}$. The proof is finished. □

Furthermore, we claim that $\mathfrak{g}$ is a metric Lie algebra; that is, there is a symmetric nondegenerate $\mathsf{ad}$-invariant bilinear form $\omega$ on $\mathfrak{g}$. Actually, this bilinear form is defined by*

$$\omega(D(u_1, \cdots, u_{n-1}), D(v_1, \cdots, v_{n-1})) = S(D(u_1, \cdots, u_{n-1})v_1, v_2, \cdots, v_{n-1}). \tag{15}$$

*By $D(u_1, \cdots, u_{n-1}) = 0$, for all $v \in \mathcal{V}$, we have

$$D(u_1, \cdots, u_{n-1})v = [u_1, \cdots, u_{n-1}, v] = 0.$$

Thus, the definition of $\omega$ is well defined.

**Proposition 3.9** *The bilinear form $\omega$ on $\mathfrak{g}$ defined by (15) is symmetric, nondegenerate, and* $\mathsf{ad}$*-invariant. Consequently, $(\mathfrak{g},\omega)$ is a metric Lie algebra.*

**Proof** By the symmetry condition (10) of a generalized metric $n$-Leibniz algebra, we have

$$\begin{aligned}\omega(D(u_1,\cdots,u_{n-1}),D(v_1,\cdots,v_{n-1})) &= S(D(u_1,\cdots,u_{n-1})v_1,v_2,\cdots,v_{n-1})\\ &= S([u_1,\cdots,u_{n-1},v_1],v_2,\cdots,v_{n-1})\\ &= S([v_1,,v_2,\cdots,v_{n-1},u_1],u_2,\cdots,u_{n-1})\\ &= \omega(D(v_1,\cdots,v_{n-1}),D(u_1,\cdots,u_{n-1})).\end{aligned}$$

Thus, the bilinear form $\omega$ is symmetric.

To prove nondegeneracy, let $x\in\mathfrak{g}\subset\mathfrak{gl}(\mathcal{V})$ be such that $\omega(x,D(u_1,\cdots,u_{n-1}))=0$ for all $u_1,\cdots,u_{n-1}\in\mathcal{V}$. Thus, we have

$$S(x(u_1),u_2,\cdots,u_{n-1})=0.$$

By the nondegeneracy of $S$, we have $x(u_1)=0$ for all $u_1\in\mathcal{V}$, which implies that $x=0$.

Finally, we prove the ad-invariance of the bilinear form $\omega$:

$$\begin{aligned}&\omega(D(u_1,\cdots,u_{n-1}),[D(v_1,\cdots,v_{n-1}),D(w_1,\cdots,w_{n-1})]_C)\\ \overset{(14)}{=}\ &\omega(D(u_1,\cdots,u_{n-1}),\sum_{i=1}^{n-1}D(w_1,\cdots,w_{i-1},D(v_1,\cdots,v_{n-1})w_i,w_{i+1},\cdots,w_{n-1}))\\ =\ &\sum_{i=1}^{n-1}\omega(D(u_1,\cdots,u_{n-1}),D(w_1,\cdots,w_{i-1},D(v_1,\cdots,v_{n-1})w_i,w_{i+1},\cdots,w_{n-1}))\\ \overset{(15)}{=}\ &S(D(u_1,\cdots,u_{n-1})(D(v_1,\cdots,v_{n-1})w_1),w_2,\cdots,w_{n-1})\\ &+\sum_{i=2}^{n-1}S(D(u_1,\cdots,u_{n-1})w_1,w_2,\cdots,w_{i-1},D(v_1,\cdots,v_{n-1})w_i,w_{i+1},\cdots,w_{n-1})\\ \overset{(9)}{=}\ &S\Big(\big(D(u_1,\cdots,u_{n-1})\circ D(v_1,\cdots,v_{n-1})-D(v_1,\cdots,v_{n-1})\circ D(u_1,\cdots,u_{n-1})\big)w_1,w_2,\cdots,w_{n-1})\Big)\\ =\ &\omega([D(u_1,\cdots,u_{n-1}),D(v_1,\cdots,v_{n-1})]_C,D(w_1,\cdots,w_{n-1})).\end{aligned}$$

Therefore, the bilinear form $\omega$ on $\mathfrak{g}$ is symmetric, nondegenerate, and ad-invariant. The proof is finished. □

It is obvious that $\mathcal{V}$ is a faithful representation of the Lie algebra $\mathfrak{g}$. Furthermore, we have:

**Proposition 3.10** *$\mathcal{V}$ is a faithful generalized orthogonal representation of the Lie algebra $\mathfrak{g}$.*

**Proof** By the unitarity condition (9) of a generalized metric $n$-Leibniz algebra, we have

$$\begin{aligned}
& S(D(u_1,\cdots,u_{n-1})w_1,w_2,\cdots,w_{n-1}) \\
= \; & S([u_1,\cdots,u_{n-1},w_1],w_2,\cdots,w_{n-1}) \\
= \; & -\sum_{i=2}^{n-1} S(w_1,\cdots,w_{i-1},[u_1,\cdots,u_{n-1},w_i],w_{i+1},\cdots,w_{n-1}) \\
= \; & -\sum_{i=2}^{n-1} S(w_1,\cdots,w_{i-1},D(u_1,\cdots,u_{n-1})w_i,w_{i+1},\cdots,w_{n-1}).
\end{aligned}$$

Thus, $\mathcal{V}$ is a faithful generalized orthogonal representation of $\mathfrak{g}$. □

Summarizing the above discussion, we have:

**Theorem 3.11** *Let $(\mathcal{V},[\cdot,\cdots,\cdot],S)$ be a generalized metric $n$-Leibniz algebra. Then $(\mathfrak{g},\mathcal{V},\mathrm{Id})$ is Lie triple data, i.e. $(\mathfrak{g},\omega)$ is a metric Lie algebra and $(\mathrm{Id},\mathcal{V},S)$ is its faithful generalized orthogonal representation.*

**Example 3.12** ([11, Example 4]) Consider the 4-dimensional 3-Lie algebra $A_4$ on $\mathbb{R}^4$ with the standard Euclidean inner product $\langle\cdot,\cdot\rangle$. With respect to an orthogonal basis $\{e_1,e_2,e_3,e_4\}$, the 3-Lie bracket is given by

$$[e_1,e_2,e_3]=e_4,\quad [e_2,e_3,e_4]=-e_1,\quad [e_1,e_3,e_4]=e_2,\quad [e_1,e_2,e_4]=-e_3.$$

It is a generalized metric 3-Leibniz algebra. It is obvious that $\wedge^2\mathbb{R}^4$ is 6-dimensional and generated by

$$e_1\wedge e_2,\quad e_1\wedge e_3,\quad e_1\wedge e_4,\quad e_2\wedge e_3,\quad e_2\wedge e_4,\quad e_3\wedge e_4.$$

Denote $D(e_i\wedge e_j)$ by $D_{ij}$. We have

$$D_{12}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\; D_{13}=\begin{pmatrix}0&0&0&0\\0&0&0&-1\\0&0&0&0\\0&1&0&0\end{pmatrix},\; D_{14}=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix},$$

$$D_{23}=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\-1&0&0&0\end{pmatrix},\; D_{24}=\begin{pmatrix}0&0&-1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},\; D_{34}=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

It is obvious that $\{D_{ij},\ i<j\}$ are the basis of $\mathfrak{so}(4)$. Therefore, $\mathrm{Im}(D)=\mathfrak{so}(4)=\{A\in\mathbb{R}^{4\times 4}|A^T=-A\}$.

Next we consider the induced nondegenerate bilinear form $\omega$ on $\mathfrak{so}(4)$. The nonzero ones are given by

$$\omega(D_{12},D_{34})=1,\quad \omega(D_{13},D_{24})=-1,\quad \omega(D_{14},D_{23})=1,$$

which implies that $\omega$ is not positive definite, but has signature $(3,3)$. Thus, we obtain that $(\mathfrak{so}(4),\mathbb{R}^4,\mathrm{Id})$ is Lie triple data.

### 3.2. From the Leibniz algebra to Lie algebra

Let $(\mathcal{V},[\cdot,\cdots,\cdot],S)$ be a generalized metric $n$-Leibniz algebra. In the middle of the $n$-Leibniz algebra $(\mathcal{V},[\cdot,\cdots,\cdot])$ and the Lie algebra $\mathfrak{g}$, we have the Leibniz algebra $(\otimes^{n-1}\mathcal{V},[\cdot,\cdot]_{\mathsf{F}})$. Moreover, $D$ is a Leibniz algebra epimorphism from $\otimes^{n-1}\mathcal{V}$ to $\mathfrak{g}$. In this section, we analyze the metric structure on the Leibniz algebra $(\otimes^{n-1}\mathcal{V},[\cdot,\cdot]_{\mathsf{F}})$. We define a bilinear form $B$ on $\otimes^{n-1}\mathcal{V}$ by

$$B(u_1\otimes\cdots\otimes u_{n-1},v_1\otimes\cdots\otimes v_{n-1})=S([u_1,\cdots,u_{n-1},v_1],v_2,\cdots,v_{n-1}). \tag{16}$$

**Proposition 3.13** *The bilinear form $B$ on $\otimes^{n-1}\mathcal{V}$ defined by (16) is symmetric and associative-invariant.*

**Proof** By the symmetry condition (10) of a generalized metric $n$-Leibniz algebra, we have

$$\begin{aligned}B(u_1\otimes\cdots\otimes u_{n-1},v_1\otimes\cdots\otimes v_{n-1}) &= S([u_1,\cdots,u_{n-1},v_1],v_2,\cdots,v_{n-1})\\ &= S([v_1,v_2,\cdots,v_{n-1},u_1],u_2,\cdots,u_{n-1})\\ &= B(v_1\otimes\cdots\otimes v_{n-1},u_1\otimes\cdots\otimes u_{n-1}).\end{aligned}$$

Moreover, we prove the associative-invariance of the bilinear form $B$:

$$\begin{aligned}&B(u_1\otimes\cdots\otimes u_{n-1},[v_1\otimes\cdots\otimes v_{n-1},w_1\otimes\cdots\otimes w_{n-1}]_{\mathsf{F}})\\ \overset{(4)}{=}\;& B(u_1\otimes\cdots\otimes u_{n-1},\sum_{i=1}^{n-1}w_1\otimes\cdots w_{i-1}\otimes[v_1,\cdots,v_{n-1},w_i]\otimes w_{i+1}\cdots\otimes w_{n-1}))\\ \overset{(16)}{=}\;& S([u_1,\cdots,u_{n-1},[v_1,\cdots,v_{n-1},w_1]],w_2,\cdots,w_{n-1})\\ &+\sum_{i=2}^{n-1}S([u_1,\cdots,u_{n-1},w_1],w_2,\cdots,w_{i-1},[v_1,\cdots,v_{n-1},w_i],w_{i+1},\cdots,w_{n-1})\\ \overset{(9)}{=}\;& S([u_1,\cdots,u_{n-1},[v_1,\cdots,v_{n-1},w_1]],w_2,\cdots,w_{n-1})\\ &-S([v_1,\cdots,v_{n-1},[u_1,\cdots,u_{n-1},w_1]],w_2,\cdots,w_{n-1})\\ \overset{(1)}{=}\;& B([u_1\otimes\cdots\otimes u_{n-1},v_1\otimes\cdots\otimes v_{n-1}]_{\mathsf{F}},w_1\otimes\cdots\otimes w_{n-1}).\end{aligned}$$

Therefore, the bilinear form $B$ on $\otimes^{n-1}\mathcal{V}$ is symmetric and associative-invariant. The proof is finished. □

**Remark 3.14** *For a skew-symmetric multiplication, being associative-invariant and* $\mathsf{ad}$*-invariant are the same. See Definition 2.3 and its explanation. The bilinear form $B$ defined by (16) is not* $\mathsf{ad}$*-invariant in general since the bracket operation $[\cdot,\cdot]_{\mathsf{F}}$ in the Leibniz algebra $(\otimes^{n-1}\mathcal{V},[\cdot,\cdot]_{\mathsf{F}})$ is not skew-symmetric.*

**Proposition 3.15** *The bilinear form $B$ on $\otimes^{n-1}\mathcal{V}$ is nondegenerate if and only if* $\ker D=0$.

**Proof** Let $V=\sum_i v_{i,1}\otimes\cdots v_{i,n-1}\in\otimes^{n-1}\mathcal{V}$ be such that $B(V,w_1\otimes\cdots\otimes w_{n-1})=0$ for all $w_1,w_2,\cdots,w_{n-1}\in\mathcal{V}$. Therefore, we have

$$S([V,w_1],w_2,\cdots,w_{n-1})=0.$$

Since $S$ is nondegenerate, we have $[V,w_1]=0$ for all $w_1\in\mathcal{V}$ and hence $V\in\ker D$. The proof is finished. □

**Proposition 3.16** *The Leibniz algebra morphism* $D : \otimes^{n-1}\mathcal{V} \longrightarrow \mathfrak{g}$ *preserves the metric.*

**Proof** For all $u_1, \cdots, u_{n-1}, v_1, \cdots, v_{n-1} \in \mathcal{V}$, we have

$$\begin{aligned}\omega(D(u_1, \cdots, u_{n-1}), D(v_1, \cdots, v_{n-1})) &= S([u_1, \cdots, u_{n-1}, v_1], v_2 \cdots, v_{n-1}) \\ &= B(u_1 \otimes \cdots \otimes u_{n-1}, v_1 \otimes \cdots \otimes v_{n-1}).\end{aligned}$$

Thus, $D$ preserves the metric. □

## 4. Construction of a generalized metric $n$-Leibniz algebra from Lie triple data

Let $(\mathfrak{g}, [\cdot, \cdot], \omega)$ be a metric Lie algebra and $(\rho, \mathcal{V}, S)$ a faithful generalized orthogonal representation of $\mathfrak{g}$ as defined in Definition 3.6. We start by defining an $(n-1)$-linear map $D : \mathcal{V} \times \cdots \times \mathcal{V} \to \mathfrak{g}$, by transposing the $\mathfrak{g}$-action. That is, for given $v_1, \cdots, v_{n-1} \in \mathcal{V}$, define $D(v_1, \cdots, v_{n-1}) \in \mathfrak{g}$ by

$$\omega(x, D(v_1, \cdots, v_{n-1})) = S(\rho(x)v_1, v_2, \cdots, v_{n-1}), \quad \forall x \in \mathfrak{g}. \tag{17}$$

**Proposition 4.1** *With the above notations, for all* $v_1, v_2, \cdots, v_{n-1} \in \mathcal{V}$, *we have*

$$\sum_{i=1}^{n-1} D(v_i, v_1, \cdots, v_{i-1}, \hat{v}_i, v_{i+1}, \cdots, v_{n-1}) = 0.$$

**Proof** Since $(\rho, \mathcal{V}, S)$ is a generalized orthogonal representation of $\mathfrak{g}$, we have

$$\begin{aligned}\omega(x, D(v_1, \cdots, v_{n-1})) &= S(\rho(x)v_1, v_2, \cdots, v_{n-1}) \\ &\overset{(13)}{=} -\sum_{i=2}^{n-1} S(v_1, \cdots, v_{i-1}, \rho(x)v_i, v_{i+1}, \cdots, v_{n-1}) \\ &= -\sum_{i=2}^{n-1} S(\rho(x)v_i, v_1, \cdots, v_{i-1}, \hat{v}_i, v_{i+1}, \cdots, v_{n-1}) \\ &= -\sum_{i=2}^{n-1} \omega(x, D(v_i, v_1, \cdots, v_{i-1}, \hat{v}_i, v_{i+1}, \cdots, v_{n-1})).\end{aligned}$$

By the nondegeneracy of $\omega$, we have

$$D(v_1, v_2, \cdots, v_{n-1}) = -\sum_{i=2}^{n-1} D(v_i, v_1, \cdots, v_{i-1}, \hat{v}_i, v_{i+1}, \cdots, v_{n-1}).$$

Thus, the proof is finished. □

**Proposition 4.2** *The* $(n-1)$*-linear map* $D : \mathcal{V} \times \cdots \times \mathcal{V} \to \mathfrak{g}$ *is surjective.*

**Proof** We denote by $(\mathrm{Im}D)^{\perp}$ the orthogonal compliment space of $\mathrm{Im}D$, i.e.

$$(\mathrm{Im}D)^{\perp} := \{x \in \mathfrak{g} | \omega(x, y) = 0, \ \forall y \in \mathrm{Im}D\}.$$

Let $x \in (\mathrm{Im}D)^{\perp}$. Then for all $v_1, \cdots, v_{n-1} \in \mathcal{V}$, we have

$$\omega(x, D(v_1, \cdots, v_{n-1})) = 0.$$

Therefore, by (17) we obtain $S(\rho(x)v_1, v_2, \cdots, v_{n-1}) = 0$. The nondegeneracy of $S$ implies that $\rho(x)v_1 = 0$ for all $v_1 \in \mathcal{V}$, which in turn implies that $x = 0$ since the representation of $\mathfrak{g}$ on $\mathcal{V}$ is faithful. Therefore, $(\mathrm{Im}D)^{\perp} = 0$ and $D$ is surjective. □

We define an $n$-linear map $[\cdot, \cdots, \cdot] : \mathcal{V} \times \cdots \times \mathcal{V} \to \mathcal{V}$ by

$$[v_1, \cdots, v_{n-1}, v_n] = \rho(D(v_1, \cdots, v_{n-1}))v_n. \tag{18}$$

By Proposition 4.1, it is straightforward to obtain the following.

**Lemma 4.3** *For all $v_1, \cdots, v_n \in \mathcal{V}$, there holds*

$$\sum_{i=1}^{n-1} [v_i, v_1, \cdots, v_{i-1}, \hat{v_i}, v_{i+1}, \cdots, v_{n-1}, v_n] = 0.$$

**Remark 4.4** *For $n = 3$, we obtain that the 3-bracket is skew-symmetric in the first two entries.*

The following theorem says that the converse of Theorem 3.11 also holds. Thus, there is a one-to-one correspondence between generalized metric $n$-Leibniz algebras and faithful generalized orthogonal representations of metric Lie algebras.

**Theorem 4.5** *Let $(\rho, \mathcal{V}, S)$ be a faithful generalized orthogonal representation of a metric Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \omega)$. Then $(\mathcal{V}, [\cdot, \cdots, \cdot], S)$ is a generalized metric $n$-Leibniz algebra, where the $n$-bracket $[\cdot, \cdots, \cdot]$ is defined by (18).*

**Proof** For all $x, y \in \mathfrak{g}$ and $u_1, \cdots, u_{n-1} \in \mathcal{V}$, we have

$$\begin{aligned}
\omega([D(u_1, \cdots, u_{n-1}), x], y) &\overset{(6)}{=} \omega(D(u_1, \cdots, u_{n-1}), [x, y]) \\
&= \omega([x, y], D(u_1, \cdots, u_{n-1})) \\
&\overset{(17)}{=} S(\rho([x, y])u_1, u_2, \cdots, u_{n-1}) \\
&= S(\rho(x)\rho(y)u_1, u_2, \cdots, u_{n-1}) - S(\rho(y)\rho(x)u_1, u_2, \cdots, u_{n-1}) \\
&\overset{(13)}{=} -\sum_{i=2}^{n-1} S(\rho(y)u_1, u_2, \cdots, u_{i-1}, \rho(x)u_i, u_{i+1}, \cdots, u_{n-1}) \\
&\quad -S(\rho(y)\rho(x)u_1, u_2, \cdots, u_{n-1}) \\
&\overset{(17)}{=} -\sum_{i=2}^{n-1} \omega(y, D(u_1, u_2, \cdots, u_{i-1}, \rho(x)u_i, u_{i+1}, \cdots, u_{n-1})) \\
&\quad -\omega(y, D(\rho(x)u_1, u_2, \cdots, u_{n-1})).
\end{aligned}$$

By the nondegeneracy of the bilinear form $\omega$ on $\mathfrak{g}$, we have

$$[x, D(u_1, \cdots, u_{n-1})] = \sum_{i=1}^{n-1} D(u_1, \cdots, u_{i-1}, \rho(x)u_i, u_{i+1}, \cdots, u_{n-1}).$$

By substituting $x = D(v_1, \cdots, v_{n-1})$ and applying both sides of the above equation to $u_n$, we have

$$\begin{aligned} & [v_1, \cdots, v_{n-1}, [u_1, \cdots, u_{n-1}, u_n]] - [u_1, \cdots, u_{n-1}, [v_1, \cdots, v_{n-1}, u_n]] \\ = & \sum_{i=1}^{n-1} [u_1, \cdots, u_{i-1}, [v_1, \cdots, v_{n-1}, u_i], u_{i+1}, \cdots, u_{n-1}, u_n]. \end{aligned}$$

Thus, $(\mathcal{V}, [\cdot, \cdots, \cdot])$ is an $n$-Leibniz algebra.

By (13), we have

$$\begin{aligned} & \sum_{i=1}^{n-1} S(v_1, \cdots, v_{i-1}, [u_1, \cdots, u_{n-1}, v_i], v_{i+1}, \cdots v_{n-1}) \\ = & \sum_{i=1}^{n-1} S(v_1, \cdots, v_{i-1}, \rho(D(u_1, \cdots, u_{n-1}))v_i, v_{i+1}, \cdots v_{n-1}) \\ = & \ 0. \end{aligned}$$

Thus, the unitarity condition in Definition 3.1 holds.

Since the bilinear form $\omega$ on $\mathfrak{g}$ is symmetric, we have

$$\begin{aligned} S([u_1, \cdots, u_{n-1}, v_1], v_2, \cdots, v_{n-1}) &= S(\rho(D(u_1, \cdots, u_{n-1}))v_1, v_2, \cdots, v_{n-1}) \\ &\overset{(17)}{=} \omega(D(u_1, \cdots, u_{n-1}), D(v_1, v_2, \cdots, v_{n-1})) \\ &= \omega(D(v_1, v_2, \cdots, v_{n-1}), D(u_1, \cdots, u_{n-1})) \\ &\overset{(17)}{=} S(\rho(D(v_1, v_2, \cdots, v_{n-1}))u_1, u_2, \cdots, u_{n-1}) \\ &= S([v_1, v_2, \cdots, v_{n-1}, u_1], u_2, \cdots, u_{n-1}), \end{aligned}$$

which implies that the symmetry condition in Definition 3.1 holds.

Thus, $(\mathcal{V}, [\cdot, \cdots, \cdot], S)$ is a generalized metric $n$-Leibniz algebra. The proof is finished. □

## 5. Generalized orthogonal derivations

In this section, we introduce the notion of a generalized orthogonal derivation on Lie triple data and show that there is a one-to-one correspondence between generalized orthogonal derivations on generalized metric $n$-Leibniz algebras and Lie triple data.

**Definition 5.1** *A generalized orthogonal derivation on Lie triple data $(\mathfrak{g}, \mathcal{V}, \rho)$ is a pair $(\mathrm{d}_{\mathfrak{g}}, \mathrm{d}_{\mathcal{V}})$, where $\mathrm{d}_{\mathfrak{g}}$ is an orthogonal derivation on the metric Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \omega)$ and $\mathrm{d}_{\mathcal{V}} \in \mathfrak{gl}(\mathcal{V})$ is a linear map satisfying the following conditions:*

$$\mathrm{d}_{\mathcal{V}} \circ \rho(x) = \rho(\mathrm{d}_{\mathfrak{g}}(x)) + \rho(x) \circ \mathrm{d}_{\mathcal{V}}, \tag{19}$$

$$\sum_{i=1}^{n-1} S(w_1, \cdots, \mathrm{d}_{\mathcal{V}} w_i, \cdots, w_{n-1}) = 0, \tag{20}$$

*for all $x \in \mathfrak{g}$ and $w_1, w_2, \cdots, w_{n-1} \in \mathcal{V}$.*

**Example 5.2** Consider the Lie triple data $(\mathfrak{so}(4),\mathbb{R}^4,\mathrm{Id})$ given in Example 3.12. For any $A\in\mathfrak{so}(4)$, define $\mathsf{ad}_A\in\mathfrak{gl}(\mathfrak{so}(4))$ by $\mathsf{ad}_A B=:[A,B]_C$ for all $B\in\mathfrak{so}(4)$. Then $(\mathrm{d}_{\mathfrak{so}(4)}=\mathsf{ad}_A,\mathrm{d}_{\mathbb{R}^4}=A)$ is a generalized orthogonal derivation on $(\mathfrak{so}(4),\mathbb{R}^4,\mathrm{Id})$.

Let $(\mathcal{V},[\cdot,\cdots,\cdot],S)$ be a generalized metric $n$-Leibniz algebra with a generalized orthogonal derivation $\mathrm{d}_{\mathcal{V}}$. Let $(\mathfrak{g},[\cdot,\cdot]_C,\omega)$ be the corresponding metric Lie algebra given in Proposition 3.9. Define $\mathrm{d}_{\mathfrak{g}}:\mathfrak{g}\longrightarrow\mathfrak{g}$ by

$$\mathrm{d}_{\mathfrak{g}}\big(D(w_1,\cdots,w_{n-1})\big)=\sum_{i=1}^{n-1}D(w_1,\cdots,\mathrm{d}_{\mathcal{V}}w_i,\cdots,w_{n-1}).\tag{21}$$

Equivalently,

$$\mathrm{d}_{\mathfrak{g}}\big(D(w_1,\cdots,w_{n-1})\big)=[\mathrm{d}_{\mathcal{V}},D(w_1,\cdots,w_{n-1})]_C.$$

**Proposition 5.3** *Let $\mathrm{d}_{\mathcal{V}}$ be a generalized orthogonal derivation on a generalized metric $n$-Leibniz algebra $(\mathcal{V},[\cdot,\cdots,\cdot],S)$. Then $(\mathrm{d}_{\mathfrak{g}},\mathrm{d}_{\mathcal{V}})$ is a generalized orthogonal derivation on the Lie triple data $(\mathfrak{g},\mathcal{V},\mathrm{Id})$ given by Theorem 3.11.*

**Proof** For all $u_1,\cdots,u_{n-1},v_1,\cdots,v_{n-1}\in\mathcal{V}$, we have

$$\begin{aligned}
&\mathrm{d}_{\mathfrak{g}}[D(u_1,\cdots,u_{n-1}),D(v_1,\cdots,v_{n-1})]_C\\
=\;&[\mathrm{d}_{\mathcal{V}},[D(u_1,\cdots,u_{n-1}),D(v_1,\cdots,v_{n-1})]_C]_C\\
=\;&[[\mathrm{d}_{\mathcal{V}},D(u_1,\cdots,u_{n-1})]_C,D(v_1,\cdots,v_{n-1})]_C+[D(u_1,\cdots,u_{n-1}),[\mathrm{d}_{\mathcal{V}},D(v_1,\cdots,v_{n-1})]_C]_C\\
=\;&[\mathrm{d}_{\mathfrak{g}}\big(D(u_1,\cdots,u_{n-1})\big),D(v_1,\cdots,v_{n-1})]_C+[D(u_1,\cdots,u_{n-1}),\mathrm{d}_{\mathfrak{g}}\big(D(v_1,\cdots,v_{n-1})\big)]_C,
\end{aligned}$$

which implies that $\mathrm{d}_{\mathfrak{g}}$ is a derivation of the Lie algebra $(\mathfrak{g},[\cdot,\cdot]_C)$.

Since $\mathrm{d}_{\mathcal{V}}$ is generalized orthogonal, for all $D(u_1,\cdots,u_{n-1}),D(v_1,\cdots,v_{n-1})\in\mathfrak{g}$, we have

$$\begin{aligned}
&\omega(\mathrm{d}_{\mathfrak{g}}D(u_1,\cdots,u_{n-1}),D(v_1,\cdots,v_{n-1}))+\omega(D(u_1,\cdots,u_{n-1}),\mathrm{d}_{\mathfrak{g}}D(v_1,\cdots,v_{n-1}))\\
=\;&\sum_{i=1}^{n-1}S([u_1,\cdots,\mathrm{d}_{\mathcal{V}}u_i,\cdots,u_{n-1},v_1],v_2,\cdots,v_{n-1})+S([u_1,\cdots,u_{n-1},\mathrm{d}_{\mathcal{V}}v_1],v_2,\cdots,v_{n-1})\\
&+\sum_{i=2}^{n-1}S([u_1,\cdots,u_{n-1},v_1],v_2,\cdots,\mathrm{d}_{\mathcal{V}}v_i,\cdots,v_{n-1})\\
=\;&S(\mathrm{d}_{\mathcal{V}}[u_1,\cdots,u_{n-1},v_1],v_2,\cdots,v_{n-1})+\sum_{i=2}^{n-1}S([u_1,\cdots,u_{n-1},v_1],v_2,\cdots,\mathrm{d}_{\mathcal{V}}v_i,\cdots,v_{n-1})\\
=\;&0.
\end{aligned}$$

Thus, $\mathrm{d}_{\mathfrak{g}}$ is an orthogonal derivation on the metric Lie algebra $(\mathfrak{g},[\cdot,\cdot]_C,\omega)$.

Moreover, for all $D(u_1,\cdots,u_{n-1})\in\mathfrak{g}$, we have

$$\begin{aligned}
\mathrm{d}_{\mathfrak{g}}(D(u_1,\cdots,u_{n-1}))+D(u_1,\cdots,u_{n-1})\circ\mathrm{d}_{\mathcal{V}}&=[\mathrm{d}_{\mathcal{V}},D(u_1,\cdots,u_{n-1})]_C+D(u_1,\cdots,u_{n-1})\circ\mathrm{d}_{\mathcal{V}}\\
&=\mathrm{d}_{\mathcal{V}}\circ D(u_1,\cdots,u_{n-1}).
\end{aligned}$$

Thus, equality (19) holds. Furthermore, (20) holds automatically. The proof is finished. □

The converse of the above result also holds.

**Proposition 5.4** *Let* $(\mathrm{d}_{\mathfrak{g}}, \mathrm{d}_{\mathcal{V}})$ *be a generalized orthogonal derivation on the Lie triple data* $(\mathfrak{g}, \mathcal{V}, \rho)$*. Then* $\mathrm{d}_{\mathcal{V}}$ *is a generalized orthogonal derivation on the corresponding generalized metric* $n$*-Leibniz algebra* $(\mathcal{V}, [\cdot, \cdots, \cdot], S)$ *given in Theorem 4.5.*

**Proof** We only need to prove that $\mathrm{d}_{\mathcal{V}}$ is a derivation on the $n$-Leibniz algebra $(\mathcal{V}, [\cdot, \cdots, \cdot])$. For all $v_1, \cdots, v_{n-1} \in \mathcal{V}$ and $x \in \mathfrak{g}$, we have

$$\begin{aligned}
&\omega(\mathrm{d}_{\mathfrak{g}} D(v_1, \cdots, v_{n-1}) - \sum_{i=1}^{n-1} D(v_1, \cdots, \mathrm{d}_{\mathcal{V}} v_i, \cdots, v_{n-1}), x) \\
\overset{(7)}{=}\;& -\omega(D(v_1, \cdots, v_{n-1}), \mathrm{d}_{\mathfrak{g}} x) - \sum_{i=2}^{n-1} S(\rho(x) v_1, v_2, \cdots, \mathrm{d}_{\mathcal{V}} v_i, \cdots, v_{n-1}) - S(\rho(x)(\mathrm{d}_{\mathcal{V}} v_1), v_2, \cdots, v_{n-1}) \\
\overset{(20)}{=}\;& -S(\rho(\mathrm{d}_{\mathfrak{g}} x) v_1, v_2, \cdots, v_{n-1}) + S(\mathrm{d}_{\mathcal{V}}(\rho(x) v_1), v_2, \cdots, v_i, \cdots, v_{n-1}) \\
& -S(\rho(x)(\mathrm{d}_{\mathcal{V}} v_1), v_2, \cdots, v_{n-1}) \\
\overset{(19)}{=}\;& 0.
\end{aligned}$$

Thus, we have

$$\mathrm{d}_{\mathfrak{g}} D(v_1, \cdots, v_{n-1}) = \sum_{i=1}^{n-1} D(v_1, \cdots, \mathrm{d}_{\mathcal{V}} v_i, \cdots, v_{n-1}). \tag{22}$$

For all $v_1, \cdots, v_n, u_1, \cdots, u_{n-2} \in \mathcal{V}$, we have

$$\begin{aligned}
& S(\mathrm{d}_{\mathcal{V}}[v_1, \cdots, v_n] - \sum_{i=1}^{n} [v_1, \cdots, \mathrm{d}_{\mathcal{V}} v_i, \cdots, v_n], u_1, \cdots, u_{n-2}) \\
=\;& S(\mathrm{d}_{\mathcal{V}}(\rho(D(v_1, \cdots, v_{n-1})) v_n), u_1, \cdots, u_{n-2}) \\
& -\sum_{i=1}^{n-1} S(\rho(D(v_1, \cdots, \mathrm{d}_{\mathcal{V}} v_i, \cdots, v_{n-1})) v_n, u_1, \cdots, u_{n-2}) \\
& -S(\rho(D(v_1, \cdots, v_{n-1}))(\mathrm{d}_{\mathcal{V}} v_n), u_1, \cdots, u_{n-2}) \\
\overset{(19)}{=}\;& S(\rho(\mathrm{d}_{\mathfrak{g}} D(v_1, \cdots, v_{n-1})) v_n, u_1, \cdots, u_{n-2}) \\
& -\sum_{i=1}^{n-1} S(\rho(D(v_1, \cdots, \mathrm{d}_{\mathcal{V}} v_i, \cdots, v_{n-1})) v_n, u_1, \cdots, u_{n-2}) \\
\overset{(22)}{=}\;& 0.
\end{aligned}$$

Therefore, $\mathrm{d}_{\mathcal{V}}$ is a derivation of the $n$-Leibniz algebra $(\mathcal{V}, [\cdot, \cdots, \cdot])$. □

## 6. Generalized orthogonal automorphisms

In this section, we introduce the notion of a generalized orthogonal automorphism on Lie triple data and show that there is a one-to-one correspondence between generalized orthogonal automorphisms on generalized metric $n$-Leibniz algebras and Lie triple datas.

**Definition 6.1** *A generalized orthogonal automorphism on the Lie triple data $(\mathfrak{g}, \mathcal{V}, \rho)$ is a pair $(\Phi_{\mathfrak{g}}, \Phi_{\mathcal{V}})$, where $\Phi_{\mathfrak{g}}$ is an orthogonal automorphism on the metric Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \omega)$ and $\Phi_{\mathcal{V}} \in \mathfrak{gl}(\mathcal{V})$ is an invertible linear map satisfying the following conditions:*

$$\Phi_{\mathcal{V}}(\rho(x)w) = \rho(\Phi_{\mathfrak{g}}(x))(\Phi_{\mathcal{V}}w), \tag{23}$$

$$S(\Phi_{\mathcal{V}}w_1, \cdots, \Phi_{\mathcal{V}}w_{n-1}) = S(w_1, \cdots, w_{n-1}), \tag{24}$$

*for all $x \in \mathfrak{g}$ and $w, w_1, w_2, \cdots, w_{n-1} \in \mathcal{V}$.*

**Example 6.2** Consider the Lie triple data $(\mathfrak{so}(4), \mathbb{R}^4, \mathrm{Id})$ given in Example 3.12. For any $A \in \mathfrak{so}(4)$, define $\mathsf{Ad}_{e^A} \in \mathfrak{gl}(\mathfrak{so}(4))$ by $\mathsf{Ad}_{e^A}B =: e^A B e^{-A}$ for all $B \in \mathfrak{so}(4)$. Then $(\Phi_{\mathfrak{so}(4)} = \mathsf{Ad}_{e^A}, \Phi_{\mathbb{R}^4} = e^A)$ is a generalized orthogonal automorphism on $(\mathfrak{so}(4), \mathbb{R}^4, \mathrm{Id})$.

Let $(\mathcal{V}, [\cdot, \cdots, \cdot], S)$ be a generalized metric $n$-Leibniz algebra with a generalized orthogonal automorphism $\Phi_{\mathcal{V}}$. Let $(\mathfrak{g}, [\cdot,\cdot]_C, \omega)$ be the corresponding metric Lie algebra given in Proposition 3.9. Define $\Phi_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\Phi_{\mathfrak{g}}\big(D(w_1, \cdots, w_{n-1})\big) = D(\Phi_{\mathcal{V}}w_1, \cdots, \Phi_{\mathcal{V}}w_{n-1}). \tag{25}$$

Equivalently,

$$\Phi_{\mathfrak{g}}\big(D(w_1, \cdots, w_{n-1})\big) = \Phi_{\mathcal{V}} \circ D(w_1, \cdots, w_{n-1}) \circ \Phi_{\mathcal{V}}^{-1}.$$

**Proposition 6.3** *Let $\Phi_{\mathcal{V}}$ be a generalized orthogonal automorphism on a generalized metric $n$-Leibniz algebra $(\mathcal{V}, [\cdot, \cdots, \cdot], S)$. Then $(\Phi_{\mathfrak{g}}, \Phi_{\mathcal{V}})$ is a generalized orthogonal automorphism on the Lie triple data $(\mathfrak{g}, \mathcal{V}, \mathrm{Id})$ given by Theorem 3.11.*

**Proof** For all $u_1, \cdots, u_{n-1}, v_1, \cdots, v_{n-1} \in \mathcal{V}$, we have

$$\begin{aligned}
& \Phi_{\mathfrak{g}}[D(u_1, \cdots, u_{n-1}), D(v_1, \cdots, v_{n-1})]_C \\
= \; & \Phi_{\mathcal{V}} \circ D(u_1, \cdots, u_{n-1}) \circ D(v_1, \cdots, v_{n-1}) \circ \Phi_{\mathcal{V}}^{-1} \\
& -\Phi_{\mathcal{V}} \circ D(v_1, \cdots, v_{n-1}) \circ D(u_1, \cdots, u_{n-1}) \circ \Phi_{\mathcal{V}}^{-1} \\
= \; & [\Phi_{\mathfrak{g}} D(u_1, \cdots, u_{n-1}), \Phi_{\mathfrak{g}} D(v_1, \cdots, v_{n-1})]_C.
\end{aligned}$$

Thus, $\Phi_{\mathfrak{g}}$ is an automorphism of the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_C)$. Since $\Phi_{\mathcal{V}}$ is generalized orthogonal, for all

$D(u_1,\cdots,u_{n-1}), D(v_1,\cdots,v_{n-1})\in\mathfrak{g}$, we have

$$\begin{aligned}
&\omega(\Phi_{\mathfrak{g}}D(u_1,\cdots,u_{n-1}),\Phi_{\mathfrak{g}}D(v_1,\cdots,v_{n-1}))\\
=\;& S([\Phi_{\mathcal{V}}u_1,\cdots,\Phi_{\mathcal{V}}u_{n-1},\Phi_{\mathcal{V}}v_1],\Phi_{\mathcal{V}}v_2,\cdots,\Phi_{\mathcal{V}}v_{n-1})\\
=\;& S(\Phi_{\mathcal{V}}[u_1,\cdots,u_{n-1},v_1],\Phi_{\mathcal{V}}v_2,\cdots,\Phi_{\mathcal{V}}v_{n-1})\\
=\;& S([u_1,\cdots,u_{n-1},v_1],v_2,\cdots,v_{n-1})\\
=\;& \omega(D(u_1,\cdots,u_{n-1}),D(v_1,\cdots,v_{n-1})).
\end{aligned}$$

Thus, $\Phi_{\mathfrak{g}}$ is an orthogonal automorphism on the metric Lie algebra $(\mathfrak{g},[\cdot,\cdot]_C,\omega)$.

Moreover, for all $D(u_1,\cdots,u_{n-1})\in\mathfrak{g}$ and $w\in\mathcal{V}$, we have

$$\begin{aligned}
\Phi_{\mathcal{V}}(D(u_1,\cdots,u_{n-1})w) &= [\Phi_{\mathcal{V}}u_1,\cdots,\Phi_{\mathcal{V}}u_{n-1},\Phi_{\mathcal{V}}w]\\
&= (\Phi_{\mathfrak{g}}D(u_1,\cdots,u_{n-1}))(\Phi_{\mathcal{V}}w).
\end{aligned}$$

Thus, equality (23) holds. Furthermore, (24) holds automatically. The proof is finished. □

The converse of the above result also holds.

**Proposition 6.4** *Let $(\Phi_{\mathfrak{g}},\Phi_{\mathcal{V}})$ be a generalized orthogonal automorphism on a Lie triple data $(\mathfrak{g},\mathcal{V},\rho)$. Then $\Phi_{\mathcal{V}}$ is a generalized orthogonal automorphism on the corresponding generalized metric $n$-Leibniz algebra $(\mathcal{V},[\cdot,\cdots,\cdot],S)$ given in Theorem 4.5.*

**Proof** We only need to prove that $\Phi_{\mathcal{V}}$ is an automorphism on the $n$-Leibniz algebra $(\mathcal{V},[\cdot,\cdots,\cdot])$. For all $v_1,\cdots,v_{n-1}\in\mathcal{V}$ and $x\in\mathfrak{g}$, we have

$$\begin{aligned}
&\omega(\Phi_{\mathfrak{g}}D(v_1,\cdots,v_{n-1})-D(\Phi_{\mathcal{V}}v_1,\cdots,\Phi_{\mathcal{V}}v_{n-1}),x)\\
\overset{(8)}{=}\;& \omega(D(v_1,\cdots,v_{n-1}),\Phi_{\mathfrak{g}}^{-1}x)-\omega(D(\Phi_{\mathcal{V}}v_1,\cdots,\Phi_{\mathcal{V}}v_{n-1}),x)\\
=\;& S(\rho(\Phi_{\mathfrak{g}}^{-1}x)v_1,v_2,\cdots,v_{n-1})-S(\rho(x)(\Phi_{\mathcal{V}}v_1),\Phi_{\mathcal{V}}v_2,\cdots,\Phi_{\mathcal{V}}v_{n-1})\\
\overset{(24)}{=}\;& S(\Phi_{\mathcal{V}}(\rho(\Phi_{\mathfrak{g}}^{-1}x)v_1),\Phi_{\mathcal{V}}v_2,\cdots,\Phi_{\mathcal{V}}v_{n-1})-S(\rho(x)(\Phi_{\mathcal{V}}v_1),\Phi_{\mathcal{V}}v_2,\cdots,\Phi_{\mathcal{V}}v_{n-1})\\
\overset{(23)}{=}\;& S(\rho(x)(\Phi_{\mathcal{V}}v_1),\Phi_{\mathcal{V}}v_2,\cdots,\Phi_{\mathcal{V}}v_{n-1})-S(\rho(x)(\Phi_{\mathcal{V}}v_1),\Phi_{\mathcal{V}}v_2,\cdots,\Phi_{\mathcal{V}}v_{n-1})\\
=\;& 0.
\end{aligned}$$

Thus, we have

$$\Phi_{\mathfrak{g}}D(v_1,\cdots,v_{n-1})=D(\Phi_{\mathcal{V}}v_1,\cdots,\Phi_{\mathcal{V}}v_{n-1}). \tag{26}$$

For all $v_1, \cdots, v_n, u_1, \cdots, u_{n-2} \in \mathcal{V}$, we have

$$\begin{aligned} & S(\Phi_{\mathcal{V}}[v_1, \cdots, v_n] - [\Phi_{\mathcal{V}} v_1, \cdots, \Phi_{\mathcal{V}} v_i, \cdots, \Phi_{\mathcal{V}} v_n], u_1, \cdots, u_{n-2}) \\ = \; & S(\Phi_{\mathcal{V}}(\rho(D(v_1, \cdots, v_{n-1}))v_n), u_1, \cdots, u_{n-2}) \\ & -S(\rho(D(\Phi_{\mathcal{V}} v_1, \cdots, \Phi_{\mathcal{V}} v_i, \cdots, \Phi_{\mathcal{V}} v_{n-1}))\Phi_{\mathcal{V}} v_n, u_1, \cdots, u_{n-2}) \\ \overset{(23)}{=} \; & S(\rho(\Phi_{\mathfrak{g}} D(v_1, \cdots, v_{n-1}))(\Phi_{\mathcal{V}} v_n), u_1, \cdots, u_{n-2}) \\ & -S(\rho(D(\Phi_{\mathcal{V}} v_1, \cdots, \Phi_{\mathcal{V}} v_i, \cdots, \Phi_{\mathcal{V}} v_{n-1}))(\Phi_{\mathcal{V}} v_n), u_1, \cdots, u_{n-2}) \\ \overset{(26)}{=} \; & 0. \end{aligned}$$

Therefore, $\Phi_{\mathcal{V}}$ is an automorphism of the $n$-Leibniz algebra $(\mathcal{V}, [\cdot, \cdots, \cdot])$. □

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