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Jackson and Stechkin type inequalities of trigonometric approximation in $A^{p,q(\cdot)}_{w,\vartheta}$

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Abstract: In this paper, we study Jackson and Stechkin type theorems of trigonometric polynomial approximation in the space $A^{p,q(\cdot)}_{w,\vartheta}$ by considering a modulus of smoothness defined by virtue of the Steklov operator.

Key words: Modulus of smoothness, trigonometric polynomials, Muckenhoupt weight, direct and inverse theorems

1. Introduction and main results

Let $\mathbb{T} := [-\pi, \pi]$. The space $A^{p,q(\cdot)}_{w,\vartheta}(\mathbb{T})$ is the intersection of the well-known weighted Lebesgue space $L^p_w(\mathbb{T})$ and the variable exponent weighted Lebesgue space $L^{q(\cdot)}_\vartheta(\mathbb{T})$, i.e. $A^{p,q(\cdot)}_{w,\vartheta}(\mathbb{T}) = L^p_w(\mathbb{T}) \cap L^{q(\cdot)}_\vartheta(\mathbb{T})$. In the literature, the variable exponent Lebesgue spaces were first investigated by Orlicz [23]. In the space $L^p_w(\mathbb{T})$ the direct and inverse theorems of trigonometric approximation were proved in the papers [1, 2, 5, 6, 19, 20, 29]. In the space $L^{q(\cdot)}_\vartheta(\mathbb{T})$, similar theorems were obtained in the papers [3, 7, 10, 18, 23, 25]. Variable exponent weighted Lebesgue spaces have important applications in the theory of elasticity, fluid dynamics, and differential equations [15, 24]. Also, in weighted Lorentz spaces $L^{p,q}_w(\mathbb{T})$, which is a generalization of $L^p_w(\mathbb{T})$, direct and inverse theorems of approximation theory were proved in [8, 9, 21, 28]. In this paper, we prove the direct and inverse theorems of trigonometric approximation in $A^{p,q(\cdot)}_{w,\vartheta}(\mathbb{T})$. First, we give some definitions and notations.

Let $\wp(\mathbb{T})$ be the class of all measurable functions $p(\cdot) : \mathbb{T} \to [1, \infty]$ called the variable exponent on $\mathbb{T}$. In this paper, the notation $p(\cdot)$ is used always for a variable exponent. Let $p(\cdot) \in \wp(\mathbb{T})$ and we set

$$p^- = \underset{x\in\mathbb{T}}{ess}\inf p(x),\ p^+ = \underset{x\in\mathbb{T}}{ess}\sup p(x).$$

A measurable function $w : \mathbb{T} \to [0, \infty]$ is called a weight function if the preimage set $w^{-1}(\{0, \infty\})$ has the Lebesgue measure zero.

Let $1 \le p < \infty$ and f be a measurable function on $\mathbb{T}$. The weighted Lebesgue space L^p_w is defined as the set of all measurable functions f such that

$$\|f\|_{p,w} := \left(\int_{\mathbb{T}} |f(x)|^p w(x)dx\right)^{\frac{1}{p}} < \infty.$$

It is known that L^p_w is a Banach space with respect to the norm $\|f\|_{p,w}$.

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Weighted variable exponent Lebesgue space $L_{\vartheta}^{p(\cdot)}$ is defined as the set of all measurable functions f on $\mathbb{T}$ such that

$$\|f\|_{p(\cdot),\vartheta} = \left\| f\vartheta^{\frac{1}{p(\cdot)}} \right\|_{p(\cdot)} < \infty$$

where

$$\|f\|_{p(\cdot)} := \inf\left\{ \alpha > 0 : \int_{\mathbb{T}} \left| \frac{f(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}.$$

It is a Banach space with respect to the norm $\|f\|_{p(\cdot),\vartheta}$.

Let $1 < p < \infty,\ q(\cdot) \in \wp(\mathbb{T})$. We set

$$A_{w,\vartheta}^{p,q(\cdot)} := \left\{ f : f \in L_w^p \cap L_{\vartheta}^{q(\cdot)} \right\}$$

and equip this Banach space with the norm

$$\|f\|_{w,\vartheta}^{p,q(\cdot)} = \|f\|_{p,w} + \|f\|_{q(\cdot),\vartheta}$$

for any $f \in A_{w,\vartheta}^{p,q(\cdot)}$.

We denote by $C_c(\mathbb{T})$ the space of all continuous, complex valued functions on $\mathbb{T}$. $C^{\infty}(\mathbb{T})$ consists of functions that are continuously differentiable arbitrarily many times. The set $C_0^{\infty}(\mathbb{T})$ is the subset of $C^{\infty}(\mathbb{T})$ of functions that have compact support. $C(\mathbb{T})$ is the set of all continuous functions on $\mathbb{T}$.

Some properties of $A_{w,\vartheta}^{p,q(\cdot)}$ were given in [26]:

I) $\left(A_{w,\vartheta}^{p,q(\cdot)}, \|\cdot\|_{w,\vartheta}^{p,q(\cdot)} \right)$ is a Banach space with $q^+ < \infty$,

II) The space $A_{w,\vartheta}^{p,q(\cdot)}$ is a Banach function space on $\mathbb{T}$.

The set $C(\mathbb{T})$ is dense subset of $A_{w,\vartheta}^{p,q(\cdot)}(\mathbb{T})$.

Indeed: The set $C_0^{\infty}(\mathbb{T})$ is dense subset of $A_{w,\vartheta}^{p,q(\cdot)}(\mathbb{T})$ [26, Corollary 2], so for every $f \in A_{w,\vartheta}^{p,q(\cdot)}(\mathbb{T})$ there exists $g \in C_0^{\infty}(\mathbb{T})$ such that $\|f-g\|_{A_{w,\vartheta}^{p,q(\cdot)}} < \varepsilon$. On the other hand, as the embedding $C_0^{\infty}(\mathbb{T}) \hookrightarrow C(\mathbb{T})$ is valid [13, p. 15], we have $g \in C(\mathbb{T})$. Thus, the set $C(\mathbb{T})$ is a dense subset of $A_{w,\vartheta}^{p,q(\cdot)}(\mathbb{T})$.

The weight functions used in this paper satisfy Muckenhoupt condition A_p [22] defined as

$$\sup \left(\frac{1}{|I|} \int_I w(x)dx \right) \left(\frac{1}{|I|} \int_I w^{1-p'}(x)dx \right)^{p-1} = C_{A_p} < \infty, \quad p' = \frac{p}{p-1}, \quad 1 < p < \infty,$$

where the supremum is taken with respect to all intervals I, which are any subintervals of $\mathbb{T}$. The constant C_{A_p} is called the Muckenhoupt constant of w.

For weights w, the class $A_{q(\cdot)}$ is defined as

$$\|w\|_{q(\cdot)} := \sup_{I \in \varsigma} |I|^{-q_I} \|w\|_{L^1(I)} \left\| \frac{1}{w} \right\|_{L^{\frac{q'(\cdot)}{q(\cdot)}}(I)} < \infty,$$

where ς denotes the set of all intervals of $\mathbb{T}$, $q_I = \left(\frac{1}{|I|} \int_I \frac{1}{q(x)} dx \right)$ and $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$ [11, 14].

We define the modulus of smoothness of a function $f \in A^{p,q(\cdot)}_{w,\vartheta}$ as

$$\Omega_r(f,\delta)_{A^{p,q(\cdot)}_{w,\vartheta}} := \sup_{0\le h\le\delta} \|(I-\sigma_h)^r f\|_{A^{p,q(\cdot)}_{w,\vartheta}}, \ r\in\mathbb{N} \tag{1.1}$$

where

$$\sigma_h f(x) := \frac{1}{2h}\int_{x-h}^{x+h} f(u)du, \quad x\in\mathbb{T}.$$

If the condition

$$|q(x)-q(y)| \le \frac{C}{-\ln|x-y|}, \quad x,y\in\Omega, \quad |x-y|\le\frac{1}{2}$$

holds, then we say that the exponent $q(x)$ satisfies the Log–Hölder continuous function where $\Omega\subset\mathbb{T}$. We denote the family of all Log–Hölder continuous functions by the symbol $P^{\log}(\mathbb{T})$.

Denote by Mf the Hardy–Littlewood maximal operator, defined for $f\in L^1(\mathbb{T})$ by

$$Mf(x) := \sup_I \frac{1}{|I|}\int_I |f(y)|\,dy, \ x\in\mathbb{T}$$

where the supremum is taken with respect to all the intervals I, which are any subintervals of $\mathbb{T}$.

Let $1<p<\infty$, $q\in P^{\log}$ and $1<q^-\le q^+<\infty$. The maximal operator $M: A^{p,q(\cdot)}_{w,\vartheta}\to A^{p,q(\cdot)}_{w,\vartheta}$ is bounded iff $w\in A_p$ and $\vartheta\in A_{q(\cdot)}$ [26, Theorem 10]. Therefore, the shift operator $\sigma_h(f)$ belongs to $A^{p,q(\cdot)}_{w,\vartheta}$ and thus $\Omega_r(f,\delta)_{A^{p,q(\cdot)}_{w,\vartheta}}$ makes sense under these conditions.

The best approximation $E_n(f)_{A^{p,q(\cdot)}_{w,\vartheta}}$ of $f\in A^{p,q(\cdot)}_{w,\vartheta}$ is given by

$$E_n(f)A^{p,q(\cdot)}_{w,\vartheta} = \inf_{T_n\in\mathbf{T}_n} \|f-T_n\|_{A^{p,q(\cdot)}_{w,\vartheta}},$$

where $\mathbf{T}_n$ is the set of trigonometric polynomials T_n of degree not more than n.

For an $f\in A^{p,q(\cdot)}_{w,\vartheta}$ the K-functional is defined as

$$K\left(f,t;A^{p,q(\cdot)}_{w,\vartheta},W^r_{p,q(\cdot),w,\vartheta}\right) := \inf_{g\in W^r_{p,q(\cdot),w,\vartheta}} \left\{\|f-g\|_{A^{p,q(\cdot)}_{w,\vartheta}} + t^r\left\|\frac{d^r}{dx^r}g\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}\right\}, \quad t>0$$

where

$$W^r_{p,q(\cdot),w,\vartheta} := \left\{g\in A^{p,q(\cdot)}_{w,\vartheta} : \frac{d^r}{dx^r}g\in A^{p,q(\cdot)}_{w,\vartheta}\right\}.$$

The notation $\lesssim$ indicates that "$A\lesssim B$ iff there exists a positive constant C, independent of essential parameters, such that $A\le CB$".

If $A\lesssim B$ and $B\lesssim A$, simultaneously, we will write $A\approx B$.

The aim of this paper is to prove the Jackson and Stechkin type theorems of polynomial approximation in $A^{p,q(\cdot)}_{w,\vartheta}$ by considering modulus of smoothness (1.1).

Theorem 1.1 *Let* $f \in A^{p,q(\cdot)}_{w,\vartheta}$, $w \in A_p$, $\vartheta \in A_{q(\cdot)}$, $1 < p < \infty$, $q \in P^{\log}$, $1 < q^- \leq q^+ < \infty$, *and* $n, r \in \mathbb{N}$. *Then*

$$E_n(f)_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim \Omega_r\left(f, \frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}} \tag{1.2}$$

holds for some constant depending only on $r, p, q(\cdot)$ *and* C_{A_p}.

Theorem 1.2 *Let* $f \in A^{p,q(\cdot)}_{w,\vartheta}$, $1 < p < \infty$, $q \in P^{\log}$, $1 < q^- \leq q^+ < \infty$, $w \in A_p$, $\vartheta \in A_{q(\cdot)}$, *and* $n, r \in \mathbb{N}$. *Then*

$$\Omega_r\left(f, \frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim \frac{1}{n^{2r}} \sum_{k=0}^{n} (k+1)^{2r-1} E_k(f)_{A^{p,q(\cdot)}_{w,\vartheta}}$$

holds with some constant depending only on $r, p, q(\cdot)$ *and* C_{A_p} *and* $C_{A_{q(\cdot)}}$ *are constants corresponding to* L_p *and* $L_{q(\cdot)}$, *respectively.*

From Theorem 1.1 and Theorem 1.2, we obtain the following Marchaud type inequality.

Corollary 1.3 *Let* $f \in A^{p,q(\cdot)}_{w,\vartheta}$, $1 < p < \infty$, $q \in P^{\log}$, $1 < q^- \leq q^+ < \infty$, $w \in A_p$, $\vartheta \in A_{q(\cdot)}$. *Then we have*

$$\Omega_r(f, \delta)_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim \delta^{2r} \int_{\delta}^{1} u^{-2r-1} \Omega_k(f, u)_{A^{p,q(\cdot)}_{w,\vartheta}}\, du, \quad 0 < \delta < 1,$$

for $r, k \in \mathbb{N}$ *with* $r < k$.

Corollary 1.4 *Let* $f \in A^{p,q(\cdot)}_{w,\vartheta}$, $w \in A_p$, $\vartheta \in A_{q(\cdot)}$, $1 < p < \infty$, $q \in P^{\log}$, $1 < q^- \leq q^+ < \infty$. *If*

$$E_n(f)_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim n^{-\alpha}, \quad n \in \mathbb{N}$$

for some $\alpha > 0$, *then, for a given* $r \in \mathbb{N}$, *we have the estimations*

$$\Omega_r(f, \delta)_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim \begin{cases} \delta^{\alpha} & , \quad r > \alpha/2; \\ \delta^{2r} \log \frac{1}{\delta} & , \quad r = \alpha/2; \\ \delta^{2r} & , \quad r < \alpha/2. \end{cases}$$

If we define the generalized Lipschitz class $Lip\left(\alpha, A^{p,q(\cdot)}_{w,\vartheta}\right)$ *for* $\alpha > 0$ *as*

$$Lip\left(\alpha, A^{p,q(\cdot)}_{w,\vartheta}\right) := \left\{ f \in A^{p,q(\cdot)}_{w,\vartheta} : \Omega_k(f, \delta)_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim \delta^{\alpha},\ \delta > 0 \right\},$$

then by virtue of Theorem 1.1 and Corollary 1.3 we obtain the following result, which gives a constructive characterization of the Lipschitz classes $Lip\left(\alpha, A^{p,q(\cdot)}_{w,\vartheta}\right)$ in the case of $k := [\alpha/2]+1$, $[x] := \max\{n \in \mathbb{Z} : n \leq x\}$, $\alpha > 0$.

Corollary 1.5 *Let* $f \in A^{p,q(\cdot)}_{w,\vartheta}$, $w \in A_p$, $\vartheta \in A_{q(\cdot)}$, $1 \le \alpha < \infty$, $1 < p < \infty$, $q \in P^{\log}$, *and* $1 < q^- \le q^+ < \infty$. *The following conditions are equivalent:*

$$(i)\, f \in Lip\left(\alpha, A^{p,q(\cdot)}_{w,\vartheta}\right); \quad (ii)\, E_n(f)_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim n^{-\alpha}, \quad n \in \mathbb{N}.$$

2. Auxiliary results

To prove the above theorems we need the following auxiliary results.

Lemma 2.1 *Let* $f \in A^{p,q(\cdot)}_{w,\vartheta}$, $1 < p < \infty$, $q \in P^{\log}$, $1 < q^- \le q^+ < \infty$, $w \in A_p$, $\vartheta \in A_{q(\cdot)}$, *and* $t, k > 0$. *Then*

$$\Omega_1(f,t)_{A^{p,q(\cdot)}_{w,\vartheta}} \approx K\left(f,t; A^{p,q(\cdot)}_{w,\vartheta}, W^2_{p,q(\cdot),w,\vartheta}\right)$$

and

$$\Omega_1(f,kt)_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim (1+[k])^2\, \Omega_1(f,t)_{A^{p,q(\cdot)}_{w,\vartheta}}$$

hold for some constants depending only on $p, q(\cdot)$, *and* C_{A_p} *and* $C_{A_{q(\cdot)}}$ *are constants corresponding to* L_p *and* $L_{q(\cdot)}$, *respectively.*

Proof Let $t > 0$. Then there exists an $n \in \mathbb{N}$ such that $(1/n) < t \le (2/n)$.

For this $n \in \mathbb{N}$, we shall consider an operator L_n on $A^{p,q(\cdot)}_{w,\vartheta}$ as follows:

$$(L_n f)(x) := 3n^3 \int_0^{1/n}\int_0^t\int_{-u}^u f(x+s)\,dsdudt, \quad x \in \mathbb{T},\ f \in A^{p,q(\cdot)}_{w,\vartheta}.$$

Then from [16] and [3, p. 14], with some constant $c \in \mathbb{R}$,

$$\frac{d^2}{dx^2}(L_n f)(x) = cn^2\left(\left(I - \sigma_{1/n}\right)\right) f(x) \tag{2.1}$$

is valid for a.e. $x \in \mathbb{T}$.

The operator L_n is bounded in $A^{p,q(\cdot)}_{w,\vartheta}$. In fact, using the uniform boundedness of the operator $f \to \sigma_{1/n} f$ in $A^{p,q(\cdot)}_{w,\vartheta}$ we obtain that

$$\frac{d^2}{dx^2} L_n f(x) \in A^{p,q(\cdot)}_{w,\vartheta}$$

and so $L_n f \in W^2_{p,q(\cdot),w,\vartheta}$. From the generalized Minkowski inequality and the boundedness of $\sigma_{1/n}$ we obtain

$$\|L_n f\|_{A^{p,q(\cdot)}_{w,\vartheta}} = \left\| 3n^3 \int_0^{1/n}\int_0^t\int_{-u}^u f(x+s)\,dsdudt \right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim 3n^3 \int_0^{1/n}\int_0^t 2u \|\sigma_u f\|_{A^{p,q(\cdot)}_{w,\vartheta}}\, dudt$$

$$\lesssim 3n^3 \|f\|_{A^{p,q(\cdot)}_{w,\vartheta}} \int_0^{1/n}\int_0^t 2ududt = \|f\|_{A^{p,q(\cdot)}_{w,\vartheta}}. \tag{2.2}$$

By (2.2) we have $f - L_n f \in A^{p,q(\cdot)}_{w,\vartheta}$ for $f \in A^{p,q(\cdot)}_{w,\vartheta}$ and

$$\begin{aligned} K\left(f,t;A^{p,q(\cdot)}_{w,\vartheta},W^2_{p,q(\cdot),w,\vartheta}\right) &\leq 4K\left(f,1/n;A^{p,q(\cdot)}_{w,\vartheta},W^2_{p,q(\cdot),w,\vartheta}\right) \\ &\lesssim \|f-L_nf\|_{A^{p,q(\cdot)}_{w,\vartheta}} + n^{-2}\left\|\frac{d^2}{dx^2}L_nf(x)\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \\ &= :I_1+I_2. \end{aligned} \tag{2.1}$$

Using the generalized Minkowski inequality, we have

$$\begin{aligned} I_1 &= \|f-L_nf\|_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim n^3\int_0^{1/n}\int_0^t 2u\left\|(I-\sigma_u)f\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}dudt \\ &\lesssim \sup_{0\leq u\leq 1/n}\left\|(I-\sigma_u)f\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} 3n^3\int_0^{1/n}\int_0^t 2ududt \\ &\lesssim \sup_{0\leq u\leq 1/n}\left\|(I-\sigma_u)f\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} = \Omega_1\left(f,\frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}. \end{aligned} \tag{2.2}$$

Using (2.1), we have

$$I_2 = n^{-2}\left\|\frac{d^2}{dx^2}L_nf(x)\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} = \left\|n^{-2}\frac{d^2}{dx^2}L_nf(x)\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}$$

$$= \left\|C\left(I-\sigma_{1/n}\right)f\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim \sup_{0\leq u\leq 1/n}\left\|(I-\sigma_u)f\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} = \Omega_1\left(f,\frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}. \tag{2.5}$$

From (2.1)–(2.5),

$$K\left(f,t;A^{p,q(\cdot)}_{w,\vartheta},W^2_{p,q(\cdot),w,\vartheta}\right) \lesssim \Omega_1\left(f,\frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim \Omega_1(f,t)_{A^{p,q(\cdot)}_{w,\vartheta}}.$$

Now we obtain the converse estimate of the last inequality. For $g \in W^2_{p,q(\cdot),w,\vartheta}$,

$$(I-\sigma_h)g(x) = \frac{1}{2h}\int_{-h}^{h}(g(x)-g(x+t))\,dt = -\frac{1}{8h}\int_0^h\int_0^t\int_{-u}^{u}\left(\frac{d^2}{dx^2}g\right)(x+s)\,dsdudt.$$

Then

$$\begin{aligned}
&\left\|(I-\sigma_h)\,g\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}\\
\leq\ & \frac{1}{8h}\int_0^h\int_0^t 2u\left\|\frac{1}{2u}\int_{-u}^{u}\left(\frac{d^2}{dx^2}g\right)(x+s)\,ds\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}dudt\\
\lesssim\ & \frac{1}{8h}\int_0^h\int_0^t 2u\left\|\frac{d^2}{dx^2}g\,(x)\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}dudt = h^2\left\|\frac{d^2}{dx^2}g\,(x)\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \qquad (2.3)
\end{aligned}$$

and we get

$$\Omega_1\,(g,t)_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim t^2\left\|\frac{d^2}{dx^2}g\,(x)\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}$$

for $g\in W^2_{p,q(\cdot),w,\vartheta}$.

Then for $g\in W^2_{p,q(\cdot),w,\vartheta}$,

$$\Omega_1\,(f,t)_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim \|f-g\|_{A^{p,q(\cdot)}_{w,\vartheta}} + t^2\left\|\frac{d^2}{dx^2}g\,(x)\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}.$$

If we take the infimum on $g\in W^2_{p,q(\cdot),w,\vartheta}$ in the last inequality,

$$\Omega_1\,(f,t)_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim K\left(f,t;A^{p,q(\cdot)}_{w,\vartheta},W^2_{p,q(\cdot),w,\vartheta}\right).$$

Consequently,

$$\Omega_1\,(f,t)_{A^{p,q(\cdot)}_{w,\vartheta}} \approx K\left(f,t;A^{p,q(\cdot)}_{w,\vartheta},W^2_{p,q(\cdot),w,\vartheta}\right).$$

Using the last equivalence we have

$$\begin{aligned}
\Omega_1\,(f,kt)_{A^{p,q(\cdot)}_{w,\vartheta}} &\lesssim \inf_{g\in W^2_{p,q(\cdot),w,\vartheta}}\left\{\|f-g\|_{A^{p,q(\cdot)}_{w,\vartheta}} + (kt)^2\left\|\frac{d^2}{dx^2}g\,(x)\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}\right\}\\
&\lesssim (1+[k])^2\inf_{g\in W^2_{p,q(\cdot),w,\vartheta}}\left\{\|f-g\|_{A^{p,q(\cdot)}_{w,\vartheta}} + t^2\left\|\frac{d^2}{dx^2}g\,(x)\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}\right\}\\
&\lesssim (1+[k])^2\,\Omega_1\,(f,t)_{A^{p,q(\cdot)}_{w,\vartheta}}
\end{aligned}$$

and the lemma is proved. □

Lemma 2.2 (a) *Let* $F\in C(\mathbb{T})$, $w\in A_p$, $\vartheta\in A_{q(\cdot)}$, $1<p<\infty$, $q\in P^{\log}$, $1<q^-\leq q^+<\infty$, *and* n, m, $r\in\mathbb{N}$. *Then there exists a number* $\delta\in(0,1)$, *depending only on* $p,q(\cdot)$ *and* C_{A_p}, *such that*

$$\Omega_r\,(F,t)_{A^{p,q(\cdot)}_{w,\vartheta}} \leq C_1\delta^{mr}\,\|F\|_{C(\mathbb{T})} + C_2\Omega_{r+1}\,(F,t)_{A^{p,q(\cdot)}_{w,\vartheta}}$$

holds for any $t \in (0, 1/n)$, *where the constant* $C_1 > 0$ *depends only on* r, $p, q(\cdot)$ *and* C_{A_p}, *and the constant* $C_2 > 0$ *depends only on* $r, m, p, q(\cdot)$ *and* C_{A_p}.

(b) *Let* $f \in A^{p,q(\cdot)}_{w,\vartheta}$. *Then there exists* $F \in C(\mathbb{T})$ *so that*

$$\Omega_r(f,t)_{A^{p,q(\cdot)}_{w,\vartheta}} \le C_1 \delta^{mr} \|F\|_{C(\mathbb{T})} + C_2 \Omega_{r+1}(f,t)_{A^{p,q(\cdot)}_{w,\vartheta}}$$

where the constant $C_1 > 0$ *depends only on* r, $p, q(\cdot)$ *and* C_{A_p}, *and the constant* $C_2 > 0$ *depends only on* $r, m, p, q(\cdot)$ *and* C_{A_p}.

Proof *(a)* We will follow as in Lemma 3.2 of [4]. There exists a constant $\mathbf{C} > 1$, such that boundedness of $\sigma_h F$ in $A^{p,q(\cdot)}_{w,\vartheta}$ implies

$$\|\sigma_h F\|_{A^{p,q(\cdot)}_{w,\vartheta}} \le \mathbf{C} \|F\|_{A^{p,q(\cdot)}_{w,\vartheta}}$$

for any $h > 0$. We define $\delta := \mathbf{C} / (\mathbf{C} + \mathbf{1})$.

We can write the equalities

$$I - \sigma_h = \frac{1}{2}(I - \sigma_h)(I + \sigma_h) + \frac{1}{2}(I - \sigma_h)^2$$

and

$$\sigma_h (I - \sigma_h) = \frac{1}{2}(I - \sigma_h)(I + \sigma_h) - \frac{1}{2}(I - \sigma_h)^2, \quad h > 0.$$

Hence,

$$\begin{aligned} & \|(I - \sigma_h) g\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \|\sigma_h (I - \sigma_h) g\|_{A^{p,q(\cdot)}_{w,\vartheta}} \\ \le\ & \|(I - \sigma_h)(I + \sigma_h) g\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \left\|(I - \sigma_h)^2 g\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}, \end{aligned} \tag{2.4}$$

for $g \in C(\mathbb{T})$ and $h > 0$. On the other hand,

$$\begin{aligned} & \|(I - \sigma_h)^r F\|_{A^{p,q(\cdot)}_{w,\vartheta}} \\ =\ & \delta \left((1/\mathbf{C}) \|(I - \sigma_h)^r F\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \|(I - \sigma_h)^r F\|_{A^{p,q(\cdot)}_{w,\vartheta}} \right) \\ \le\ & \delta \left(\|(I - \sigma_h)^r F\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \|(I - \sigma_h)^r F\|_{A^{p,q(\cdot)}_{w,\vartheta}} \right) \\ =\ & \delta \left(\left\|(I - \sigma_h)(I - \sigma_h)^{r-1} F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \|(I - \sigma_h)^r F\|_{A^{p,q(\cdot)}_{w,\vartheta}} \right) \\ =\ & \delta \left(\left\| \left(\sigma_h (I - \sigma_h) + (I - \sigma_h)^2 \right) (I - \sigma_h)^{r-1} F \right\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \|(I - \sigma_h)^r F\|_{A^{p,q(\cdot)}_{w,\vartheta}} \right) \\ \le\ & \delta \left(\left\| \sigma_h (I - \sigma_h)(I - \sigma_h)^{r-1} F \right\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \left\| (I - \sigma_h)^2 (I - \sigma_h)^{r-1} F \right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \right) \\ & + \delta \|(I - \sigma_h)^r F\|_{A^{p,q(\cdot)}_{w,\vartheta}} \\ \le\ & \delta \left(\|\sigma_h (I - \sigma_h)^r F\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \left\|(I - \sigma_h)^{r+1} F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \|(I - \sigma_h)^r F\|_{A^{p,q(\cdot)}_{w,\vartheta}} \right). \end{aligned} \tag{2.5}$$

If we take $g := (I-\sigma_h)^{r-1} F$ in (2.5), then

$$\begin{aligned} & \left\|\sigma_h (I-\sigma_h)^r F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \left\|(I-\sigma_h)^r F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \\ \leq \; & \left\|(I-\sigma_h)^r (I+\sigma_h) F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \left\|(I-\sigma_h)^{r+1} F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}. \end{aligned}$$

Applying this in (2.4),

$$\begin{aligned} & \left\|(I-\sigma_h)^r F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \\ \leq \; & \delta \left(\left\|\sigma_h (I-\sigma_h)^r F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \left\|(I-\sigma_h)^{r+1} F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \left\|(I-\sigma_h)^r F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \right) \\ \leq \; & \delta \left(\left\|(I-\sigma_h)^r (I+\sigma_h) F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \left\|(I-\sigma_h)^{r+1} F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \right) \\ & + \delta \left\|(I-\sigma_h)^{r+1} F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \\ \leq \; & \delta \left\|(I-\sigma_h)^r (I+\sigma_h) F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} + 2\delta \left\|(I-\sigma_h)^{r+1} F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}. \end{aligned} \tag{2.6}$$

If we repeat r times the last inequality, then

$$\begin{aligned} \left\|(I-\sigma_h)^r F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \leq \; & \delta \left\|(I-\sigma_h)^r (I+\sigma_h) F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} + 2\delta \left\|(I-\sigma_h)^{r+1} F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \\ \leq \; & \delta^2 \left\|(I-\sigma_h)^r (I+\sigma_h)^2 F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \\ & + 2\delta^2 \left\|(I-\sigma_h)^{r+1} (I+\sigma_h) F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} + 2\delta \left\|(I-\sigma_h)^{r+1} F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \\ \leq \; & ... \leq \delta^r \left\|(I-\sigma_h)^r (I+\sigma_h)^r F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \\ & + 2\sum_{k=1}^{r} \delta^k \left\|(I-\sigma_h)^{r+1} (I+\sigma_h)^{k-1} F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \\ = \; & \delta^r \left\|\left(I-\sigma_h^2\right)^r F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} + 2\sum_{k=1}^{r} \delta^k \left\|(I-\sigma_h)^{r+1} (I+\sigma_h)^{k-1} F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}. \end{aligned}$$

Hence, the last inequality gives

$$\left\|(I-\sigma_h)^r F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} \leq \delta^r \left\|\left(I-\sigma_h^2\right)^r F\right\|_{A^{p,q(\cdot)}_{w,\vartheta}} + C\left(r, A^{p,q(\cdot)}_{w,\vartheta}\right) \Omega_{r+1}(F,h)_{A^{p,q(\cdot)}_{w,\vartheta}} \tag{2.10}$$

for $0 < h \leq 1/n$. From (2.10) and applying

$$\|F\|_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim \|F\|_{C(\mathbb{T})},$$

we get

$$\left\|\left(I-\sigma_h\right)^r F\right\|_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim$$

$$\lesssim \delta^r \left\|\left(I-\sigma_h^2\right)^r F\right\|_{A_{w,\vartheta}^{p,q(\cdot)}} + C\left(r, A_{w,\vartheta}^{p,q(\cdot)}\right) \Omega_{r+1}(F,h)_{A_{w,\vartheta}^{p,q(\cdot)}}$$

$$\lesssim \delta^{2r} \left\|\left(I-\sigma_h^4\right)^r F\right\|_{A_{w,\vartheta}^{p,q(\cdot)}} + \left(\delta^r+1\right) C\left(r, A_{w,\vartheta}^{p,q(\cdot)}\right) \Omega_{r+1}(F,h)_{A_{w,\vartheta}^{p,q(\cdot)}}$$

$$\lesssim \ldots \lesssim \delta^{mr} \left\|\left(I-\sigma_h^{2^m}\right)^r F\right\|_{A_{w,\vartheta}^{p,q(\cdot)}} + C\left(r, A_{w,\vartheta}^{p,q(\cdot)}, m\right) \Omega_{r+1}(F,h)_{A_{w,\vartheta}^{p,q(\cdot)}}$$

$$\lesssim \delta^{mr} \left\|\left(I-\sigma_h^{2^m}\right)^r F\right\|_{C(\mathbb{T})} + C\left(r, A_{w,\vartheta}^{p,q(\cdot)}, m\right) \Omega_{r+1}(F,h)_{A_{w,\vartheta}^{p,q(\cdot)}}$$

$$\lesssim \delta^{mr} \|F\|_{C(\mathbb{T})} + C\left(r, A_{w,\vartheta}^{p,q(\cdot)}, m\right) \Omega_{r+1}(F,h)_{A_{w,\vartheta}^{p,q(\cdot)}}. \tag{2.11}$$

Taking the supremum, from the last inequality we obtain

$$\Omega_r(F,t)_{A_{w,\vartheta}^{p,q(\cdot)}} \lesssim \delta^{mr} \|F\|_{C(\mathbb{T})} + \Omega_{r+1}(F,t)_{A_{w,\vartheta}^{p,q(\cdot)}},$$

since $\left\|\left(I-\sigma_h^{2^m}\right)^r F\right\|_{C(\mathbb{T})} \leq 2^r \|F\|_{C(\mathbb{T})}$.

(b) Let $f \in A_{w,\vartheta}^{p,q(\cdot)}$. For any $\varepsilon > 0$, there is $F \in C(\mathbb{T})$ such that $\|f-F\|_{C(\mathbb{T})} < \varepsilon$. From (a) and density of $C(\mathbb{T})$ in $A_{w,\vartheta}^{p,q(\cdot)}$ we get

$$\Omega_r(f,t)_{A_{w,\vartheta}^{p,q(\cdot)}} \leq C_1 \delta^{mr} \|F\|_{C(\mathbb{T})} + C_2 \Omega_{r+1}(f,t)_{A_{w,\vartheta}^{p,q(\cdot)}}.$$

□

Lemma 2.3 *([27, p. 344]) Let σ^* be the Fejer operator, $M(x)$ be a maximal function, and $f \in L^1$. Then there exists an absolute constant c such that*

$$\sigma^*(x,f) \leq cM(x).$$

Lemma 2.4 *Let $1 < p < \infty$, $q \in P^{\log}$, $1 < q^- \leq q^+ < \infty$. If $w \in A_p$, $\vartheta \in A_{q(\cdot)}$, then for any trigonometric polynomial T_n of degree n the following inequality holds:*

$$\|T_n'\|_{A_{w,\vartheta}^{p,q(\cdot)}} \leq cn \|T_n\|_{A_{w,\vartheta}^{p,q(\cdot)}},$$

with a constant c independent of n.

Proof It is known that $|T_n'(x)| \leq 2n\sigma^*\left(|T_n'(x)|\right)$.

From the last inequality, Lemma 2.3, and [26, Theorem 10] we can finish the proof of lemma. □

3. Proofs of main results

Proof of Theorem 1.1. The case $r=1$. Let $n\in\mathbb{N}$ and $f\in A^{p,q(\cdot)}_{w,\vartheta}$ be fixed. We will use the operator L_nf. We know that the relations $L^p_w\subset L^1$ and $L^{q(\cdot)}_\vartheta\subset L^1$ hold, so $A^{p,q(\cdot)}_{w,\vartheta}\subset L^1$. Let $S_n(f)$ be the nth partial sum of the Fourier series of f. Using (2.2), (2.5), and the uniformly boundedness (in n) of the operator $f\to S_n$ on $A^{p,q(\cdot)}_{w,\vartheta}$ ([3, 17]), we get

$$\begin{aligned} E_n(f)_{A^{p,q(\cdot)}_{w,\vartheta}} &= E_n(f-L_nf+L_nf)_{A^{p,q(\cdot)}_{w,\vartheta}}\leq E_n(f-L_nf)_{A^{p,q(\cdot)}_{w,\vartheta}}+E_n(L_nf)_{A^{p,q(\cdot)}_{w,\vartheta}} \\ &\lesssim \|f-L_nf\|_{A^{p,q(\cdot)}_{w,\vartheta}}+n^{-2}\left\|\frac{d^2}{dx^2}L_nf\left(x\right)\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}\lesssim\Omega_1\left(f,\frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}. \end{aligned}\tag{3.1}$$

The case $r\geq 2$. We will use induction on r as in [12]. We obtained above that the estimate (1.2) holds for $r=1$. We suppose that the inequality (1.2) holds for some $r=2,3,4,\ldots$. We will show that the inequality (1.2) holds for $r+1$.

We set $g(\cdot):=f(\cdot)-S_nf(\cdot)$. Then

$$S_n(g)=S_n(f-S_n(f))=S_n(f)-S_n\left(S_n(f)\right)=S_n(f)-S_n(f)=0$$

and

$$\|S_nf\|_{A^{p,q(\cdot)}_{w,\vartheta}}\lesssim\|S_nf\|_{p,w}+\|S_nf\|_{q(\cdot),\vartheta}\lesssim\|f\|_{p,w}+\|f\|_{q(\cdot),\vartheta}\lesssim\|f\|_{A^{p,q(\cdot)}_{w,\vartheta}}.$$

Using the induction hypothesis, [17], and [3],

$$\|g\|_{A^{p,q(\cdot)}_{w,\vartheta}}=\|g-S_n(g)\|_{A^{p,q(\cdot)}_{w,\vartheta}}\leq CE_n(g)_{A^{p,q(\cdot)}_{w,\vartheta}}\leq\mathbf{C}\Omega_r\left(g,\frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}.$$

From Lemma 2.2 we have that

$$\Omega_r\left(f,\frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}\leq C_1\delta^{mr}\left\|f\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}\left\|F\right\|_{C(\mathbb{T})}+C_2\Omega_{r+1}\left(f,\frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}\tag{2.13}$$

for any $f\in A^{p,q(\cdot)}_{w,\vartheta}$.

Indeed: If $\|f\|_{A^{p,q(\cdot)}_{w,\vartheta}}=1$, then

$$\begin{aligned} \Omega_r\left(f,\frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}} &\leq C_1\delta^{mr}\left\|F\right\|_{C(\mathbb{T})}+C_2\Omega_{r+1}\left(f,\frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}} \\ &= C_1\delta^{mr}\left\|F\right\|_{C(\mathbb{T})}\left\|f\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}+C_2\Omega_{r+1}\left(f,\frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}. \end{aligned}\tag{3.2}$$

If $\|f\|_{A^{p,q(\cdot)}_{w,\vartheta}}=0$, then

$$\Omega_r\left(f,\frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}=C_1\delta^{mr}\left\|f\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}\left\|F\right\|_{C(\mathbb{T})}+C_2\Omega_{r+1}\left(f,\frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}.$$

In the case of $\|f\|_{A^{p,q(\cdot)}_{w,\vartheta}} > 0$ and $\|f\|_{A^{p,q(\cdot)}_{w,\vartheta}} \neq 1$, let $f^* = \frac{f}{\|f\|_{A^{p,q(\cdot)}_{w,\vartheta}}}$. Then $\|f^*\|_{A^{p,q(\cdot)}_{w,\vartheta}} = 1$. Thus, from (3.2) we get

$$\Omega_r\left(f^*, \frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}} \leq C_1\delta^{mr}\|F\|_{C(\mathbb{T})}\|f^*\|_{A^{p,q(\cdot)}_{w,\vartheta}} + C_2\Omega_{r+1}\left(f^*, \frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}.$$

Hence,

$$\frac{1}{\|f\|_{A^{p,q(\cdot)}_{w,\vartheta}}}\Omega_r\left(f, \frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}} \leq C_1\delta^{mr}\|F\|_{C(\mathbb{T})}\frac{1}{\|f\|_{A^{p,q(\cdot)}_{w,\vartheta}}}\|f\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \frac{C_2}{\|f\|_{A^{p,q(\cdot)}_{w,\vartheta}}}\Omega_{r+1}\left(f, \frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}$$

and (2.13) follows.

If we choose m so big that $\mathbf{C}C_1\delta^{mr}\|F\|_{C(\mathbb{T})} < 1/2$, we obtain

$$\|g\|_{A^{p,q(\cdot)}_{w,\vartheta}} \leq \mathbf{C}\Omega_r\left(g, \frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}} \leq \mathbf{C}C_1\delta^{mr}\|F\|_{C(\mathbb{T})}\|g\|_{A^{p,q(\cdot)}_{w,\vartheta}} + \mathbf{C}C_2\Omega_{r+1}\left(g, \frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}.$$

Then

$$\|g\|_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim \Omega_{r+1}\left(g, \frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}.$$

From [17] and [3] and using

$$\Omega_r\left(S_nf; \frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}} \leq C\Omega_r\left(f; \frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}$$

we have

$$\Omega_{r+1}\left(f - S_nf, \frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim \Omega_{r+1}\left(f, \frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}.$$

Consequently,

$$E_n(f)_{A^{p,q(\cdot)}_{w,\vartheta}} \leq \|f - S_n(f)\|_{A^{p,q(\cdot)}_{w,\vartheta}} = \|g\|_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim \Omega_{r+1}\left(f, \frac{1}{n}\right)_{A^{p,q(\cdot)}_{w,\vartheta}}.$$

This completes the proof. □

Proof of Theorem 1.2. Using (2.3) and the equalities [3]

$$\begin{aligned}\frac{d^2}{dx^2}\left(\left(I - \sigma_h\right)g\right)(x) &= \left(I - \sigma_h\right)\frac{d^2}{dx^2}g(x)\\ \left(I - \sigma_h\right)^{m+n}f(x) &= \left(I - \sigma_h\right)^m\left(I - \sigma_h\right)^n f(x), \quad m, n \in \mathbb{N}\end{aligned}$$

we obtain

$$\Omega_r(g, t)_{A^{p,q(\cdot)}_{w,\vartheta}} \lesssim \delta^{2r}\left\|g^{(2r)}\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}, \quad r \in \mathbb{N}, \tag{2.15}$$

for $g^{(2r)} \in A^{p,q(\cdot)}_{w,\vartheta}$ and $\delta > 0$. On the other hand, for any $m \in \mathbb{N}$

$$\Omega_r(f, \delta)_{A^{p,q(\cdot)}_{w,\vartheta}} \leq \Omega_r\left(f - S_{2^{m+1}}(f), \delta\right)_{A^{p,q(\cdot)}_{w,\vartheta}} + \Omega_r\left(S_{2^{m+1}}(f), \delta\right)_{A^{p,q(\cdot)}_{w,\vartheta}} \tag{2.16}$$

and

$$\Omega_r\left(f-S_{2^{m+1}}(f),\delta\right)_{A^{p,q(\cdot)}_{w,\vartheta}}\lesssim\left\|f-S_{2^{m+1}}(f)\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}\lesssim E_{2^{m+1}}(f)_{A^{p,q(\cdot)}_{w,\vartheta}}. \tag{2.17}$$

Then by (2.15) and the weighted version of Bernstein's inequality in $A^{p,q(\cdot)}_{w,\vartheta}$ [28, Lemma 2.2],

$$\begin{aligned}\Omega_r\left(S_{2^{m+1}}(f),\delta\right)_{A^{p,q(\cdot)}_{w,\vartheta}} &\lesssim \delta^{2r}\left\|S^{(2r)}_{2^{m+1}}(f)\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}\\ &\lesssim \delta^{2r}\left\{\left\|S^{(2r)}_{1}(f)-S^{(2r)}_{0}(f)\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}+\sum_{i=1}^{m}\left\|S^{(2r)}_{2^{i+1}}(f)-S^{(2r)}_{2^{i}}(f)\right\|_{A^{p,q(\cdot)}_{w,\vartheta}}\right\}\\ &\lesssim \delta^{2r}\left\{E_0(f)_{A^{p,q(\cdot)}_{w,\vartheta}}+\sum_{i=1}^{m}2^{(i+1)2r}E_{2^i}(f)_{A^{p,q(\cdot)}_{w,\vartheta}}\right\}\\ &\lesssim \delta^{2r}\left\{E_0(f)_{A^{p,q(\cdot)}_{w,\vartheta}}+2^{2r}E_1(f)_{A^{p,q(\cdot)}_{w,\vartheta}}+\sum_{i=1}^{m}2^{(i+1)2r}E_{2^i}(f)_{A^{p,q(\cdot)}_{w,\vartheta}}\right\}.\end{aligned}$$

Applying here the inequality and choosing m as $2^m\le n<2^{m+1}$, from (2.16) $-$ (2.19), we obtain the following result:

$$2^{(i+1)2r}E_{2^i}(f)_{A^{p,q(\cdot)}_{w,\vartheta}}\lesssim\sum_{k=2^{i-1}+1}^{2^m}k^{2r-1}E_k(f)_{A^{p,q(\cdot)}_{w,\vartheta}},\quad i\ge 1, \tag{2.18}$$

and we obtain

$$\Omega_r\left(S_{2^{m+1}}(f),\delta\right)_{A^{p,q(\cdot)}_{w,\vartheta}}\lesssim\delta^{2r}\left\{E_0(f)_{A^{p,q(\cdot)}_{w,\vartheta}}+2^{2r}E_1(f)_{A^{p,q(\cdot)}_{w,\vartheta}}+\sum_{k=2}^{2^m}k^{2r-1}E_k(f)_{A^{p,q(\cdot)}_{w,\vartheta}}\right\}$$

$$\lesssim\delta^{2r}\left\{E_0(f)_{A^{p,q(\cdot)}_{w,\vartheta}}+\sum_{k=1}^{2^m}k^{2r-1}E_k(f)_{A^{p,q(\cdot)}_{w,\vartheta}}\right\}, \tag{2.19}$$

using the estimate

$$E_{2^{m+1}}(f)_{A^{p,q(\cdot)}_{w,\vartheta}}\lesssim\frac{1}{n^{2r}}\sum_{k=2^{m-1}+1}^{2^m}k^{2r-1}E_k(f)_{A^{p,q(\cdot)}_{w,\vartheta}}.$$

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