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Existence of solution for some two-point boundary value fractional differential equations

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Abstract: Using a fixed point theorem, we establish the existence of a solution for a class of boundary value fractional differential equation. Secondly, we will adopt the method of successive approximations to obtain an approximate solution to our problem. Furthermore, using the Laplace transform technique, an explicit solution to a particular case of our problem is obtained. Finally, some examples are given to illustrate our results.

Key words: Fractional differential equations, Riemann–Liouville fractional derivative, existence of solutions, fixed point theory, Laplace transform

1. Introduction

In the last three decades, fractional differential equations have attracted the attention of many researchers. They appear in the models of many phenomena in various fields, such as physics, mechanics, biology, dynamical systems, and nonlinear oscillations of earthquakes. For more on the fundamentals and a holistic review of the theory and applications of fractional calculus, see [2,7,9,10,12] and the references therein. According to [10], fractional order models are more adequate than integer order models. For instance, results obtained from [1] show that the fractional order of damping has a significant effect on the dynamic behaviors of motion when compared to that of the integer order case. In the theory of (classical and fractional) differential equations, fixed point theorems are frequently used in obtaining some qualitative properties of solutions of nonlinear differential equations (see [3–5,13–15]).

In [1], the existence and uniqueness of solution for the fractional differential equation

$$D^{\alpha}u(t) = f(t, u(t)),\ t \in [0, T],\ 1 < \alpha \leq 2,$$

with the boundary conditions

$$D^{\alpha-2}u(0^+) = b_0 D^{\alpha-2}u(T^-)$$

and

$$D^{\alpha-1}u(0^+) = b_1 D^{\alpha-1}u(T^-),$$

were considered. Using a standard fixed point theorem, the authors were able to prove the existence and uniqueness of the solution to the above differential equation in the space of $C_{2-\alpha}[0, T]$. Here, D^{α} denotes the

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Riemann–Liouville fractional derivative of order α and $b_0 \neq 1, b_1 \neq 1$, and f is a continuous function on $[0,T] \times \mathbb{R}$.

In [6], the Laplace transform method was applied in solving a linear fractional order equation. The result was obtained by transforming the fractional differential equation into an algebraic equation. Accordingly, [8] applied Laplace transform to a fractional order system. By using Gronwall and Hölder inequalities, the authors established that solutions of fractional differential equations are of exponential order. Hence, the Laplace transform technique was proved to be applicable in fractional equations.

Motivated by these works, and some results in [3–5,13–15], we are going to study the existence of the solution to the two-point boundary value fractional differential equation (FDE),

$$D^{\alpha}_{a_+} x(t) + k\, D^{\beta}_{a_+} x(t) + g(t, x(t)) = h(t),\ t \in [a,b], \tag{1}$$

$$D^{\alpha-1}_{a_+} x(a_+) = D^{\alpha-1}_{a_+} x(b_-), \tag{2}$$

$$I^{2-\alpha}_{a_+} x(a_+) = I^{2-\alpha}_{a_+} x(b_-), \tag{3}$$

$$I^{1-\beta}_{a_+} x(a_+) = I^{1-\beta}_{a_+} x(b_-), \tag{4}$$

in the space $W^{\alpha,\beta}(a,b)$, where $1 < \alpha < 2,\ 0 < \beta < 1,\ k$ is a positive constant, $D^{\alpha}_{a_+} x(t)$ is the left Riemann–Liouville fractional derivative of a function x of order α, $h \in L^{\frac{1}{\beta}}(a,b)$, and g is an $L^{\infty}(a,b)$-Carathéodory function.

$$W^{\alpha,\beta}(a,b) = \left\{ x \in C_{2-\alpha}[a,b] : D^{\alpha}_{a_+} x(t), D^{\beta}_{a_+} x(t) \in L^{\frac{1}{\beta}}(a,b), \right\}$$

and $C_{2-\alpha}[a,b] = \left\{ x : x(t)(t-a)^{2-\alpha} \in C^0[a,b] \right\}$, endowed with the norm

$$\|x\|_{2-\alpha} = \sup \left\{ x(t)(t-a)^{2-\alpha} : t \in [a,b] \right\}.$$

In addition, an approximate solution to our problem shall be obtained. Lastly, we will seek an explicit solution to a special case of (1)–(4).

This work is organized as follows. In Section 2, some basic terms of fractional calculus and some useful lemmas related to our work will be discussed. The existence theorem for the given fractional boundary value problem, using a standard fixed point theorem, is the subject matter of Section three. Section 4 is devoted to the use of the associated Volterra integral equation in finding an approximate solution to our problem. Finally, in the last section, the Laplace transform technique will be used to obtain an explicit solution to a particular case of our problem.

2. Preliminaries

Definition 2.1 *The left Riemann–Liouville fractional derivative of a function x, of order α, with lower limit a is defined as*

$$D^{\alpha}_{a_+} x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} x(s) ds, \tag{5}$$

with $n-1<\alpha<n,\ n=[\alpha]+1$*, while the Riemann–Liouville fractional integral of a function* x*, of order* $\alpha>0$*, is defined as*

$$I_{a_+}^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)ds. \tag{6}$$

Observe from equations (5) and (6) that

$$D_{a_+}^{\alpha}x(t)=\frac{d^n}{dt^n}I_{a_+}^{n-\alpha}x(t). \tag{7}$$

Definition 2.2 *The two-parameter Mittag–Leffler function of* $z\in\mathbb{C}$*, denoted by* $E_{\alpha,\beta}(z)$*, is defined as*

$$E_{\alpha,\beta}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\beta)},$$

where $\alpha>0,\ \beta>0$.

Lemma 2.3 (7) *The space* $AC^n[a,b]$ *consists of those and only those functions* f *that can be represented in the form*

$$f(x)=I_{a_+}^{n}\varphi(x)+\sum_{k=0}^{n-1}c_k(x-a)^k, \tag{8}$$

where $\varphi\in L_1(a,b),\ c_k(k=0,1,2,\cdots,n-1)$ *are arbitrary constants.*

Lemma 2.4 (7) *If* $f\in L_1(a,b)$ *and* $I_{a_+}^{n-\alpha}f(t)\in AC^n[a,b]$*, then the equality*

$$I_{a_+}^{\alpha}(D_{a_+}^{\alpha})=f(x)-\sum_{j=1}^{n}\frac{D_{a_+}^{\alpha-j}f(a_+)(x-a)^{\alpha-j}}{\Gamma(\alpha-j+1)} \tag{9}$$

holds almost everywhere on $[a,b]$.

Corollary 2.5 *If* $0<\beta<1<\alpha<2$*, then*

$$I_{a_+}^{\alpha}D_{a_+}^{\beta}x(t)=I_{a_+}^{\alpha-\beta}x(t)-\frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1}I_{a_+}^{1-\beta}x(a_+). \tag{10}$$

Proof

$$\begin{aligned}I_{a_+}^{\alpha}D_{a_+}^{\beta}x(t)&=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}D_{a_+}^{\beta}x(s)ds\\&=\frac{1}{\Gamma(\alpha+1)}\frac{d}{dt}\int_a^t(t-s)^{\alpha}D_{a_+}^{\beta}x(s)ds.\end{aligned}$$

However,

$$\frac{1}{\Gamma(\alpha+1)}\int_a^t(t-s)^{\alpha}D_{a_+}^{\beta}x(s)ds=\frac{1}{\Gamma(\alpha+1)}\int_a^t(t-s)^{\alpha}\frac{d}{ds}I_{a_+}^{1-\beta}x(s)ds.$$

Using integration by parts on the right-hand side of the above equation, one obtains

$$\frac{1}{\Gamma(\alpha+1)}\int_a^t (t-s)^\alpha \frac{d}{ds} I_{a_+}^{1-\beta} x(s)ds = \frac{1}{\Gamma(\alpha+1)}(t-s)^\alpha I_{a_+}^{1-\beta} x(s) \mid_a^t + \frac{\alpha}{\Gamma(\alpha+1)}\int_a^t (t-s)^\alpha I_{a_+}^{1-\beta} x(s)ds$$

$$= -\frac{1}{\Gamma(\alpha+1)}(t-a)^\alpha I_{a_+}^{1-\beta} x(a_+) + I_{a_+}^{\alpha} I_{a_+}^{1-\beta} x(t).$$

Thus,

$$I_{a_+}^{\alpha} D_{a_+}^{\beta} x(t) = \frac{d}{dt}\{-\frac{1}{\Gamma(\alpha+1)}(t-a)^\alpha I_{a_+}^{1-\beta} x(a_+) + I_{a_+}^{\alpha} I_{a_+}^{1-\beta} x(t)\}$$

$$= -\frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1} I_{a_+}^{1-\beta} x(a_+) + I_{a_+}^{\alpha-1} I_{a_+}^{1-\beta} x(t)$$

$$= I_{a_+}^{\alpha-\beta} x(t) - \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1} I_{a_+}^{1-\beta} x(a_+).$$

Therefore, our corollary is established. □

Lemma 2.6 *If $x \in W^{\alpha,\beta}(a,b)$, then x satisfies the relations (1)–(4) if and only if x satisfies the integral equation*

$$x(t) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} D_{a_+}^{\alpha-1} x(b_-) - \frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)} I_{a_+}^{2-\alpha} x(a_+) + k I_{a_+}^{\alpha-\beta} x(t) - k\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} I_{a_+}^{1-\beta} x(b_-)$$

$$= I_{a_+}^{\alpha}[h(t) - g(t, x(t))].$$

Remark 1 *We note that the above equation is the Volterra integral equation associated with equations (1)–(4).*

Proof We first prove the necessity. Let $x \in W^{\alpha,\beta}(a,b)$ satisfy equations (1)–(4). Then by definition of $W^{\alpha,\beta}(a,b)$, we have that $D_{a_+}^{\alpha} x(t), D_{a_+}^{\beta} x(t) \in L^{\frac{1}{\beta}}(a,b)$. Hence, $D_{a_+}^{\alpha} x(t), D_{a_+}^{\beta} x(t), g(t,x(t)) \in L^{\frac{1}{\beta}}(a,b) \subset L_1(a,b)$. From (7), we have that

$$D_{a_+}^{\alpha} x(t) = \frac{d^2}{dt^2} I_{a_+}^{2-\alpha} x(t). \tag{11}$$

By Lemma 2.3, $I_{a_+}^{2-\alpha} x(t) \in AC^2[a,b]$. Thus, we can apply Lemma 2.4 on x. Furthermore, we operate $I_{a_+}^{\alpha}$ on both sides of (1), by using equations (9) and (10) and boundary conditions (2)–(4), and then our necessary result is established.

To prove the sufficiency, let $x \in W^{\alpha,\beta}(a,b)$ satisfy the Volterra integral equation above, and then applying the operator $D_{a_+}^{\alpha}$ to it, we obtain (1). Moreover, applying $D_{a_+}^{\alpha-1}$, $I_{a_+}^{2-\alpha}$, $I_{a_+}^{1-\beta}$ to the Volterra equation, we obtain the respective boundary conditions. Thus, the sufficiency result is obtained. Therefore, our theorem is proved. □

Theorem 2.7 (11) *Suppose that $f, f', \cdots, f^{(n-1)}$ are continuous on $(0,\infty)$ and are of exponential order, while $f^{(n)}$ is piecewise continuous on $(0,\infty)$ for each $n \geq 1$. Then,*

$$\mathcal{L}(f^{(n)}(t))(s) = s^n \mathcal{L}(f(t)) - s^{n-1} f(0_+) - \cdots - f^{(n-1)}(0_+).$$

Lemma 2.8 (7) *Let $\Re(\alpha) > 0$ and $f \in L_1(0,b)$, for any $b > 0$. Also, let*

$$|f(t)| \leq Ae^{p_0 t},\ (0 < t < b)$$

hold for some constant $A > 0$ and $p_0 > 0$, and then the relation $\mathcal{L}(I^{\alpha}_{0_+} f(t)) = s^{-\alpha}\mathcal{L}(f(t))$ is valid for $\Re(s) > p_0$.

Theorem 2.9 (8) *If $\alpha > 0$, $n = [\alpha] + 1$ and $x(t), I^{n-\alpha}_{0_+}x(t), \frac{d}{dt}I^{n-\alpha}_{0_+}x(t), \cdots, \frac{d^{n-1}}{dt^{n-1}}I^{n-\alpha}_{0_+}x(t)$ are continuous on $(0,\infty)$ and of exponential order, while $D^{\alpha}_{a_+}x(t)$ is piecewise continuous on $(0,\infty)$. Then,*

$$\mathcal{L}(D^{\alpha}_{a_+}x(t)) = s^{\alpha}\mathcal{L}(x) - \sum_{k=0}^{n-1} s^{n-k-1}\frac{d^{(k-1)}}{dt^{(k-1)}}I^{n-\alpha}_{0_+}x(0_+).$$

Lemma 2.10 (7) *If $\Re(s) > 0$, $\lambda \in \mathbb{C}$, $|\lambda s^{-\alpha}| < 1$, then*

$$\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})(s)\right\} = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}. \tag{12}$$

Theorem 2.11 (1) *Let E be a Banach space. Assume that $T : E \longrightarrow E$ is a completely continuous operator and the set $V = \{x \in E : x = \mu T x, 0 \leq \mu \leq 1\}$ is bounded. Then T has a fixed point in E.*

3. Existence result

In this section, we shall use the fixed point theory of Theorem 2.11 to obtain an existence result for equations (1)–(4).

Theorem 3.1 *Assuming that g is an L^{∞}-Carathéodory function and $h \in L^{\frac{1}{\beta}}(a,b)$, then equations (1)- (4) have a solution in $W^{\alpha,\beta}(a,b)$.*

Proof First, we recall that the space $C^0[a,b]$ of all continuous functions on $[a,b]$ is complete. By [1], $C^{2-\alpha}[a,b]$ is a complete space when endowed with the norm

$$\|x\|_{2-\alpha} = \sup\left\{x(t)(t-a)^{2-\alpha} : t \in [a,b]\right\}.$$

Consequently, $W^{\alpha,\beta}(a,b)$ is a complete space. Therefore, $W^{\alpha,\beta}(a,b)$ is a Banach space. Now we define an operator $T : W^{\alpha,\beta}(a,b) \longrightarrow W^{\alpha,\beta}(a,b)$ by

$$Tx(t) = \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}D^{\alpha-1}_{a_+}x(b_-) + \frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)}I^{2-\alpha}_{a_+}x(b_-) - kI^{\alpha-\beta}_{a_+}x(t) + k\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}I^{1-\beta}_{a_+}x(b_-)$$

$$+I^{\alpha}_{a_+}[h(t) - g(t,x(t))].$$

Then our T is well defined. It suffices to show that T is uniformly bounded on every bounded subset of $W^{\alpha,\beta}(a,b)$, continuous and equicontinuous on $W^{\alpha,\beta}(a,b)$. We make the following observations:

1. Taking a bounded set V in $W^{\alpha,\beta}(a,b)$, then V is bounded in $C^{2-\alpha}[a,b]$. Thus,

$$(t-a)^{2-\alpha}|Tx(t)| = |\frac{(t-a)}{\Gamma(\alpha)}D^{\alpha-1}_{a_+}x(b_-) + \frac{1}{\Gamma(\alpha-1)}I^{2-\alpha}_{a_+}x(b_-) - k(t-a)^{2-\alpha}I^{\alpha-\beta}_{a_+}x(t)$$

$$+k\frac{(t-a)}{\Gamma(\alpha)}I^{1-\beta}_{a_+}x(b_-) + (t-a)^{2-\alpha}I^{\alpha}_{a_+}[h(t)-g(t,x(t))|$$

$$\leq \frac{\gamma}{\Gamma(\alpha)}|D^{\alpha-1}_{a_+}x(b_-)| + \frac{1}{\Gamma(\alpha-1)}|I^{2-\alpha}_{a_+}x(b_-)| + k\gamma^{2-\alpha}|I^{\alpha-\beta}_{a_+}x(t)|$$

$$+k\frac{\gamma}{\Gamma(\alpha)}|I^{1-\beta}_{a_+}x(b_-)| + \gamma^{2-\alpha}|I^{\alpha}_{a_+}[h(t)-g(t,x(t))|$$

where $\gamma = \max_{a\leq t\leq b}|t-a|$. However,

$$|I^{\alpha-\beta}_{a_+}x(t)| \ = |\frac{1}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{\alpha-\beta-1}x(s)ds|$$

$$\leq \frac{1}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{-\beta}(t-s)^{\alpha-1}|x(s)|ds$$

$$\leq \frac{M}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{-\beta}ds$$

$$\leq \frac{-M}{\Gamma(\alpha-\beta)(1-\beta)}(t-s)^{1-\beta}|_a^t$$

$$= \frac{M\gamma^{1-\beta}}{(1-\beta)\Gamma(\alpha-\beta)}$$

for some $M > 0$.
This implies that

$$|I^{\alpha-\beta}_{a_+}x(t)| \ \leq \frac{M\gamma^{1-\beta}}{(1-\beta)\Gamma(\alpha-\beta)}. \tag{13}$$

Equally,

$$|I^{\alpha}_{a_+}[h(t)-g(t,x(t))| \leq |\left(\int_a^t (t-s)^{\frac{\alpha-1}{1-\beta}}\right)^{1-\beta}\left(\int_a^t (h(s)-g(s,\eta(s))^{\frac{1}{\beta}}\right)^{\beta}$$

$$\leq \left(\frac{1-\beta}{\alpha-2\beta}\right)^{1-\beta}\gamma^{\alpha-2\beta}\|(h-g)\|_{L^{\frac{1}{\beta}}}.$$

Thus,

$$|I^{\alpha}_{a_+}[h(t)-g(t,x(t))| \leq \left(\frac{1-\beta}{\alpha-2\beta}\right)^{1-\beta}\gamma^{\alpha-2\beta}\|(h-g)\|_{L^{\frac{1}{\beta}}}. \tag{14}$$

In view of (13) and (14), we have that

$$|(t-a)^{2-\alpha}|Tx(t)| \leq \frac{\gamma}{\Gamma(\alpha)}|D^{\alpha-1}_{a_+}x(b_-)| + \frac{1}{\Gamma(\alpha-1)}|I^{2-\alpha}_{a_+}x(b_-)| + \frac{Mk\gamma^{3-\alpha-\beta}}{(1-\beta)\Gamma(\alpha-\beta)}$$

$$+k\frac{\gamma}{\Gamma(\alpha)}|I_{a_+}^{1-\beta}x(b_-)|+\gamma^{2(1-\beta)}\left(\frac{1-\beta}{\alpha-2\beta}\right)^{1-\beta}\|(h-g)\|_{L^{\frac{1}{\beta}}}, \tag{15}$$

which implies that $T(V)$ is uniformly bounded.

2. Secondly, we show that T is continuous on $W^{\alpha,\beta}(a,b)$. We pick a sequence of functions $x_n \in V$, such that

$$\lim_{n\longrightarrow\infty} x_n(t) = x$$

for all $t \in [a,b]$, and then

$$(t-a)^{2-\alpha}|Tx_n - Tx| \le \frac{k(t-a)^{2-\alpha}}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{\alpha-\beta-1}|x_n(s)-x(s)|ds$$

$$+\frac{(t-a)^{2-\alpha}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|g(s,x_n(s))-g(s,x(s))|ds$$

$$\longrightarrow 0$$

as $n \longrightarrow \infty$, which implies that

$$\|Tx_n - Tx\| \longrightarrow 0$$

as $n \longrightarrow \infty$.

3. For any $t_1, t_2 \in [a,b]$, with $t_1 < t_2$ and $x \in V$, we have

$$|(t_2-a)^{2-\alpha}Tx(t_2)-(t_1-a)^{2-\alpha}Tx(t_1)| \le$$

$$(t_2-t_1)\frac{|D_{a_+}^{\alpha-1}x(b_-)|}{\Gamma(\alpha)} + k|\frac{(t_2-a)^{2-\alpha}}{\Gamma(\alpha-\beta)}\int_a^{t_2}(t_2-s)^{\alpha-\beta-1}x(s)ds$$

$$-\frac{(t_1-a)^{2-\alpha}}{\Gamma(\alpha-\beta)}\int_a^{t_1}(t_1-s)^{\alpha-\beta-1}x(s)ds| + k(t_2-t_1)I_{a_+}^{1-\beta}x(b_-)$$

$$+|\frac{(t_2-a)^{2-\alpha}}{\Gamma(\alpha)}\int_a^{t_2}(t_2-s)^{\alpha-1}(h(s)-g(s,x(s))ds - \frac{(t_1-a)^{2-\alpha}}{\Gamma(\alpha)}\int_a^{t_1}(h(s)-g(s,x(s))ds|$$

$$\le (t_2-t_1)D_{a_+}^{\alpha-1}x(b_-) + \frac{k}{\Gamma(\alpha-\beta)}\int_a^{t_1}(t_2-a)^{2-\alpha}(t_2-s)^{\alpha-\beta-1} - (t_1-a)^{2-\alpha}(t_1-s)^{\alpha-\beta-1}|x(s)|ds$$

$$+\frac{k}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-a)^{2-\alpha}(t_2-s)^{\alpha-\beta-1}|x(s)|ds + k(t_2-t_1)I_{a_+}^{1-\beta}x(b_-)+$$

$$\frac{1}{\Gamma(\alpha)}\int_a^{t_1}(t_2-a)^{2-\alpha}(t_2-s)^{\alpha-\beta-1} - (t_1-a)^{2-\alpha}(t_1-s)^{\alpha-\beta-1}|h(s)-g(s,x(s))|ds+$$

$$\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-a)^{2-\alpha}(t_2-s)^{\alpha-\beta-1}|h(s)-g(s,x(s))|ds \longrightarrow 0$$

as $t_2 \longrightarrow t_1$.

This implies that $\|Tx(t_2)-Tx(t_1)\| \longrightarrow 0$. Hence, T is equicontinuous on V. □

From our observations, we conclude that T is compact. Hence, it is a completely continuous operator, and by Theorem 2.11, T has a fixed point, which is equal to the solution of our problem.

4. Approximate result

Having obtained the Volterra integral equation in Lemma 2.5, which is equivalent to the boundary value problem (1)–(4), we wil apply the former to find an approximate solution to equations (1)–(4). We shall apply the method of successive approximations known in this context as the Picard iteration technique in arriving at our result. That is, from the Volterra integral equation associated with our differential equation, we have that

$$x_m(t) = x_0(t) - kI_{a_+}^{\alpha-\beta}x_{m-1}(t) + I_{a_+}^{\alpha}(h-g), \tag{16}$$

where

$$x_0(t) = \frac{A(t-a)^{\alpha-1}}{\Gamma(\alpha)} + \frac{B(t-a)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{kC(t-a)^{\alpha-1}}{\Gamma(\alpha)},$$

$$A = D_{b_-}^{\alpha-1}x(a_+),\ B = I_{a_+}^{2-\alpha}x(b_-),\ C = I_{a_+}^{1-\beta}x(b_-).$$

Now, from (16), we have that

$$x_1(t) = x_0(t) - kI_{a_+}^{\alpha-\beta}x_0(t) + I_{a_+}^{\alpha}(h-g). \tag{17}$$

However,

$$I_{a_+}^{\alpha-\beta}x_0(t) = I_{a_+}^{\alpha-\beta}\{\frac{A(t-a)^{\alpha-1}}{\Gamma(\alpha)} + \frac{B(t-a)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{kC(t-a)^{\alpha-1}}{\Gamma(\alpha)}\}$$

$$= \frac{A(t-a)^{2\alpha-\beta-1}}{\Gamma(2\alpha-\beta)} + \frac{B(t-a)^{2\alpha-\beta-2}}{\Gamma(2\alpha-\beta-1)} + \frac{kC(t-a)^{2\alpha-\beta-1}}{\Gamma(2\alpha-\beta)}.$$

Thus, from equation (17), we obtain that

$$x_1(t) = \frac{A(t-a)^{\alpha-1}}{\Gamma(\alpha)} + \frac{B(t-a)^{\alpha-2}}{\Gamma(\alpha-1)}$$

$$+\frac{kC(t-a)^{\alpha-1}}{\Gamma(\alpha)} - k\left[\frac{A(t-a)^{2\alpha-\beta-1}}{\Gamma(2\alpha-\beta)} + \frac{B(t-a)^{2\alpha-\beta-2}}{\Gamma(2\alpha-\beta-1)} + \frac{kC(t-a)^{2\alpha-\beta-1}}{\Gamma(2\alpha-\beta)}\right] + I_{a_+}^{\alpha}(g-h)$$

$$= A\sum_{j=0}^{1}\frac{(-1)^j k^j (t-a)^{\alpha(j+1)-\beta j-1}}{\Gamma(\alpha(j+1)-\beta j)} + B\sum_{j=0}^{1}\frac{(-1)^j k^j (t-a)^{\alpha(j+1)-\beta j-2}}{\Gamma(\alpha(j+1)-\beta j-1)}$$

$$+C\sum_{j=0}^{1}\frac{(-1)^j k^{j+1} (t-a)^{\alpha(j+1)-\beta j-1}}{\Gamma(\alpha(j+1)-\beta j)} + I_{a_+}^{\alpha}(g-h).$$

Thus,

$$x_1(t) = A\sum_{j=0}^{1}\frac{(-1)^j k^j (t-a)^{\alpha(j+1)-\beta j-1}}{\Gamma(\alpha(j+1)-\beta j)} + B\sum_{j=0}^{1}\frac{(-1)^j k^j (t-a)^{\alpha(j+1)-\beta j-2}}{\Gamma(\alpha(j+1)-\beta j-1)}$$

$$+C\sum_{j=0}^{1}\frac{(-1)^j k^{j+1} (t-a)^{\alpha(j+1)-\beta j-1}}{\Gamma(\alpha(j+1)-\beta j)} + I_{a_+}^{\alpha}(g-h). \tag{18}$$

Next,

$$x_2(t) = x_0(t) - kI_{a_+}^{\alpha-\beta}x_1(t) + I_{a_+}^{\alpha}(g-h), \tag{19}$$

but

$$I_{a_+}^{\alpha-\beta}x_1(t) = A\sum_{j=0}^{1}\frac{(-1)^j k^j (t-a)^{\alpha(j+2)-\beta(j+1)-1}}{\Gamma(\alpha(j+2)-\beta(j+1))} + B\sum_{j=0}^{1}\frac{(-1)^j k^j (t-a)^{\alpha(j+2)-\beta(j+1)-2}}{\Gamma(\alpha(j+2)-\beta(j+1)-1)}$$

$$+C\sum_{j=0}^{1}\frac{(-1)^j k^{j+2} (t-a)^{\alpha(j+2)-\beta(j+1)-1}}{\Gamma(\alpha(j+2)-\beta(j+1))} + I^{\alpha-\beta}I_{a_+}^{\alpha}(g-h).$$

Therefore,

$$x_2(t) = A\sum_{j=0}^{2}\frac{(-1)^j k^j (t-a)^{\alpha(j+1)-\beta j-1}}{\Gamma(\alpha(j+1)-\beta j)} + B\sum_{j=0}^{2}\frac{(-1)^j k^j (t-a)^{\alpha(j+1)-\beta j-2}}{\Gamma(\alpha(j+1)-\beta j-1)}$$

$$+C\sum_{j=0}^{2}\frac{(-1)^j k^{j+1} (t-a)^{\alpha(j+1)-\beta j-1}}{\Gamma(\alpha(j+1)-\beta j)} + I^{\alpha-\beta}I_{a_+}^{\alpha}(h-g)$$

$$= A\sum_{j=0}^{2}\frac{(-1)^j k^j (t-a)^{\alpha(j+1)-\beta j-1}}{\Gamma(\alpha(j+1)-\beta j)} + B\sum_{j=0}^{2}\frac{(-1)^j k^j (t-a)^{\alpha(j+1)-\beta j-2}}{\Gamma(\alpha(j+1)-\beta j-1)}$$

$$+C\sum_{j=0}^{2}\frac{(-1)^j k^{j+1} (t-a)^{\alpha(j+1)-\beta j-1}}{\Gamma(\alpha(j+1)-\beta j)} + \int_a^t\sum_{j=0}^{1}\frac{(-1)^j k^j (t-s)^{\alpha(j+1)-\beta j}}{\Gamma(\alpha(j+1)-\beta j)}(h-g)ds.$$

Continuing the process, we derive the following relation for $x_m(t)$:

$$x_m(t) = A\sum_{j=0}^{m}\frac{(-1)^j k^j (t-a)^{\alpha(j+1)-\beta j-1}}{\Gamma(\alpha(j+1)-\beta j)} + B\sum_{j=0}^{m}\frac{(-1)^j k^j (t-a)^{\alpha(j+1)-\beta j-2}}{\Gamma(\alpha(j+1)-\beta j-1)}$$

$$+C\sum_{j=0}^{m}\frac{(-1)^j k^{j+1} (t-a)^{\alpha(j+1)-\beta j-1}}{\Gamma(\alpha(j+1)-\beta j)} + \int_a^t\sum_{j=0}^{m-1}\frac{(-1)^j k^j (t-s)^{\alpha(j+1)-\beta j}}{\Gamma(\alpha(j+1)-\beta j)}(h-g)ds.$$

Letting $m \longrightarrow \infty$, we obtain that

$$x(t) = A\sum_{j=0}^{\infty}\frac{(-1)^j k^j (t-a)^{\alpha(j+1)-\beta j-1}}{\Gamma(\alpha(j+1)-\beta j)} + B\sum_{j=0}^{\infty}\frac{(-1)^j k^j (t-a)^{\alpha(j+1)-\beta j-2}}{\Gamma(\alpha(j+1)-\beta j-1)}$$

$$+C\sum_{j=0}^{\infty}\frac{(-1)^j k^{j+1} (t-a)^{\alpha(j+1)-\beta j-1}}{\Gamma(\alpha(j+1)-\beta j)} + \int_a^t\sum_{j=0}^{\infty}\frac{(-1)^j k^j (t-s)^{\alpha(j+1)-\beta j}}{\Gamma(\alpha(j+1)-\beta j)}(h-g)ds$$

$$= A(t-a)^{\alpha-1}E_{\alpha-\beta,\alpha}[-k(t-a)^{\alpha-\beta}] + B(t-a)^{\alpha-2}E_{\alpha-\beta,\alpha-1}[-k(t-a)^{\alpha-\beta}]$$

$$+kC(t-a)^{\alpha-1}E_{\alpha-\beta,\alpha}[-k(t-a)^{\alpha-\beta}] + \int_a^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}[-k(t-s)^{\alpha-\beta}](h-g)ds.$$

Therefore, the explicit solution to the Volterra integral equation as well as the boundary value problem (1)–(4) is given as

$$x(t) = A(t-a)^{\alpha-1}E_{\alpha-\beta,\alpha}[-k(t-a)^{\alpha-\beta}] + B(t-a)^{\alpha-2}E_{\alpha-\beta,\alpha-1}[-k(t-a)^{\alpha-\beta}]$$

$$+kC(t-a)^{\alpha-1}E_{\alpha-\beta,\alpha}[-k(t-a)^{\alpha-\beta}] + \int_a^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}[-k(t-s)^{\alpha-\beta}(h-g)]ds.$$

5. Solution by Laplace transform

According to Liang [8], the Laplace transform method is one of the useful tools for solving fractional order differential equations. Under some appropriate conditions, we will apply the Laplace transform method to solve equations (1)–(4).

Now, supposing that the conditions of Theorem 2.9 are satisfied and $h(t)-g(t,x(t))$ satisfies the conditions of Lemma 2.8, then by applying the Laplace transform operator to both sides of (1), with the lower limit equal to 0, we have

$$\mathcal{L}\left\{D^{\alpha}_{0_+}x(t)+k\;D^{\beta}_{0_+}x(t)+g(t,x(t))=h(t)\right\} \tag{20}$$

$$\Longrightarrow s^{\alpha}\mathcal{L}x(t)-D^{\alpha-1}x(0_+)-sI^{2-\alpha}x(a_+)+ks^{\beta}\mathcal{L}x(t)-kI^{1-\beta}x(0_+)=\mathcal{L}h(t)-\mathcal{L}g(t,x(t))$$

$$\Longrightarrow (s^{\alpha}+ks^{\beta})\mathcal{L}\left\{x(t)\right\}=D^{\alpha-1}x(0_+)+kI^{1-\beta}x(0_+)+sI^{2-\alpha}x(a_+)+\mathcal{L}h(t)-\mathcal{L}g(t,x(t))$$

$$\Longrightarrow \mathcal{L}\left\{x(t)\right\}=\frac{D^{\alpha-1}x(0_+)+kI^{1-\beta}x(0_+)}{s^{\alpha}+ks^{\beta}}+\frac{sI^{2-\alpha}x(0_+)}{s^{\alpha}+ks^{\beta}}+\frac{(\mathcal{L}h(t)-\mathcal{L}g(t,x(t))}{s^{\alpha}+ks^{\beta}}$$

$$\Longrightarrow \mathcal{L}\left\{x(t)\right\}=\frac{(D^{\alpha-1}x(0_+)+kI^{1-\beta}x(0_+))s^{-\beta}}{s^{\alpha-\beta}+k}+\frac{s^{1-\beta}I^{2-\alpha}x(0_+)}{s^{\alpha-\beta}+k}+\frac{(\mathcal{L}h(t)-\mathcal{L}g(t,x(t)))s^{-\beta}}{s^{\alpha-\beta}+k}.$$

Taking the inverse Laplace transform of the preceding equation and using Lemma 2.10, we obtain that

$$x(t)=(D^{\alpha-1}x(0_+)+kI^{1-\beta}x(0_+)t^{\alpha-1}E_{\alpha-\beta,\alpha}(-kt^{\alpha-\beta})+I^{2-\alpha}x(0_+)t^{2-\alpha}E_{\alpha-\beta,\alpha-1}(-kt^{\alpha-\beta})$$

$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(-k(t-s)^{\alpha-\beta})[h(s)-g(t,x(s))]ds.$$

On applying the boundary conditions, we have

$$x(t)=(D^{\alpha-1}x(b_-)+kI^{1-\beta}x(b_-)t^{\alpha-1}E_{\alpha-\beta,\alpha}(-kt^{\alpha-\beta})+I^{2-\alpha}x(b_-)t^{2-\alpha}E_{\alpha-\beta,\alpha-1}(-kt^{\alpha-\beta})$$

$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(-k(t-s)^{\alpha-\beta})[h(s)-g(t,x(s))]ds.$$

This satisfies our given problem.

6. Some examples

Example 1 *Considering the problem*

$$D^{\frac{8}{5}}_{a_+}x(t)+kD^{\frac{4}{5}}_{a_+}x(t)+\frac{\sin(t)}{1+x^{\frac{4}{5}}(t)}=t^{\frac{-4}{3}}, \tag{21}$$

$$D^{\frac{3}{5}}_{a_+}x(t)\mid_{t=a}=D^{\frac{3}{5}}_{a_+}x(t)\mid_{t=b}=\pi, \tag{22}$$

$$I^{\frac{2}{5}}_{a_+}x(t)\mid_{t=a}=I^{\frac{2}{5}}_{a_+}x(t)\mid_{t=b}=\delta, \tag{23}$$

$$I^{\frac{1}{5}}_{a_+}x(t)\mid_{t=a}=I^{\frac{1}{5}}_{a_+}x(t)\mid_{t=b}=\epsilon, \tag{24}$$

then by Theorem 3.1, the problem (21)–(24) has a solution in $W^{\frac{8}{5},\frac{4}{5}}(a,b)$.

Its approximate solution according to Section 4 is given as

$$x(t) = (k\epsilon + \pi)\,(t-a)^{\frac{3}{5}} E_{\frac{4}{5},\frac{8}{5}}[-k(t-a)^{\frac{4}{5}}] + \delta(t-a)^{\frac{-2}{5}} E_{\frac{4}{5},\frac{3}{5}}[-k(t-a)^{\frac{4}{5}}] +$$

$$\int_a^t (t-s)^{\frac{3}{5}} E_{\frac{4}{5},\frac{8}{5}}[-k(t-s)^{\frac{4}{5}}] \left(s^{\frac{\beta}{1-\alpha}} - \frac{\sin(-s)}{1+x(s)^{\frac{4}{5}}} \right) ds.$$

Example 2 *Consider the fractional boundary value problem*

$$D_{0_+}^{\frac{\pi}{2}} x(t) + kD_{0_+}^{\frac{\pi}{4}} x(t) + e^{-t} = t^{\frac{4}{\pi}}, t \in [0,T], T > 0, \tag{25}$$

$$D_{0_+}^{\frac{\pi}{2}-1} x(t) \mid_{t=0} = D_{0_+}^{\frac{\pi}{2}-1} x(t) \mid_{t=T} = A, \tag{26}$$

$$I_{0_+}^{2-\frac{\pi}{2}} x(t) \mid_{t=0} = I_{0_+}^{2-\frac{\pi}{2}} x(t) \mid_{t=T} = B, \tag{27}$$

$$I_{0_+}^{1-\frac{\pi}{4}} x(t) \mid_{t=0} = I_{0_+}^{1-\frac{\pi}{4}} x(t) \mid_{t=T} = C, \tag{28}$$

with A, B, C *real numbers and* k *a positive real number. Then, according to Theorem 3.1, our problem has a solution in* $W^{\frac{\pi}{2},\frac{\pi}{4}}(0,T)$. *Its exact solution according to Section 5 is given as*

$$x(t) = (A + Ck)t^{\frac{\pi}{2}-1} E_{\frac{\pi}{4},\frac{\pi}{2}}(-kt^{\frac{\pi}{4}}) + Bt^{2-\frac{\pi}{2}} E_{\frac{\pi}{4},\frac{\pi}{2}}(-kt^{\frac{\pi}{4}}) +$$

$$\int_0^t (t-s)^{\frac{\pi}{2}-1} E_{\frac{\pi}{4},\frac{\pi}{2}}[-k(t-s)^{\frac{\pi}{4}}](s^{\frac{\pi}{4}} - e^{-s}) ds.$$

7. Conclusion

Using a fixed point theorem, we have succeeded in proving the existence of the solution to problem (1)–(4) in the space of $W^{\alpha,\beta}(a,b)$. In addition, by applying the method of successive approximations, we obtained an approximate solution to our given problem. Moreover, an explicit solution to a special case of our problem was obtained by using the Laplace transform method. Finally, some examples were used to illustrate our results.

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