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Free modules and crossed modules of R-algebroids

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Abstract: In this paper, first, we construct the free modules and precrossed modules of R-algebroids. Then we introduce the Peiffer ideal of a precrossed module and use it to construct the free crossed module.

Key words: R-category, R-algebroid, crossed modules, free modules

1. Introduction

Crossed modules, algebraic models of two types, were first invented by Whitehead [23, 24] in his study on homotopy groups and have been studied by many mathematicians. Various studies on crossed modules over groups and groupoids can be found in papers and books such as [7, 8, 21], and those over algebras in [4, 5, 19, 20, 22] and in [11, 13, 14] in different names. Kassell and Loday [12] studied crossed modules of Lie algebras and higher dimensional analogues were proposed by Ellis [10] for use in homotopical and homological algebras. Mosa [18] studied crossed modules of R-algebroids and double algebroids. Pullback and pushout crossed modules of algebroids can be found in [1] and [2], respectively. Provided that P is a group and K is a set, the construction of the free P-group on K and the constructions of the free precrossed and crossed modules on a function $\omega : \mathrm{K} \longrightarrow \mathrm{P}$ were handled in [7]. Shammu constructed the free crossed module on a function $f : \mathrm{K} \longrightarrow \mathrm{A}$ where, with our notations, K is a set and A is an R-algebra for a commutative ring R in [22].

The basic goal of this paper is to construct the free R-algebroid crossed module. For this goal, after giving some basic data in the second section, we define the category $\mathrm{Sets}_0/\mathrm{Alg}(R)$ whose objects are all functions $\omega : \mathrm{K} \longrightarrow \mathrm{A}_0 \times \mathrm{A}_0$, where K is a set and A is an R-algebroid, and its subcategory $\mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A})$ formed by a fixed R-algebroid A in the third section. Then we construct the free R-agebroid A-module determined by an object $\omega : \mathrm{K} \longrightarrow \mathrm{A}_0 \times \mathrm{A}_0$ of $\mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A})$ in the same section. In the fourth section we define the category $\mathrm{Sets}/\mathrm{Alg}(R)$, whose objects are formed by all functions of the form $\omega : \mathrm{K} \longrightarrow \mathrm{A}$ where K is a set and A is an R-algebroid and its subcategory $\mathrm{Sets}/(\mathrm{Alg}(R)/\mathrm{A})$, for a fixed R-algebroid A. Then, in the same section, we construct the free R-algebroid precrossed A-module determined by an object of $\mathrm{Sets}/(\mathrm{Alg}(R)/\mathrm{A})$.

In Section 5, we introduce the Peiffer ideal for an R-algebroid precrossed module to construct a crossed module and this procedure gives us the functor $(-)^{\mathrm{cr}}$ from the category of precrossed to the category of crossed modules of R-algebroids.

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In the last section, we construct the free R-algebroid crossed A-module determined by an object $\omega : \mathrm{K} \longrightarrow \mathrm{A}$ of $\mathrm{Sets}/\left(\mathrm{Alg}(R)/\mathrm{A}\right)$, from the corresponding precrossed module, using the functor $(-)^{\mathrm{cr}}$.

2. Preliminaries

R-algebroids were especially studied by Mitchell [15–17] and by Amgott [3]. Mitchell gave a categorical definition of R-algebroids and obtained some interesting results. Mosa defined crossed modules of R-algebroids and proved the equivalence of crossed modules of algebroids and special double algebroids with connections in [18]. Alp constructed the pullback and pushout crossed modules of algebroids in [1] and [2], respectively. In this section, we give some basic definitions concerning crossed modules of R-algebroids.

Definition 1 *[15–17]. Let R be a commutative ring. A category of which each homset has an R-module structure and of which composition is R-bilinear is called an '*R-category*'. A small R-category is called an '*R-algebroid*'. Moreover, if we omit the axiom of the existence of identities from an R-algebroid structure then the remaining structure is called a '*pre-R-algebroid*'.*

A pre-R-algebroid A comes with an object set $Ob\left(\mathrm{A}\right) = \mathrm{A}_0$, a morphism set $Mor\left(\mathrm{A}\right)$, and two functions $s, t : Mor\left(\mathrm{A}\right) \longrightarrow Ob\left(\mathrm{A}\right)$, the source and target functions respectively, such that if $sa = x$ and $ta = y$ then we say that 'a is from x to y' and write $a \in \mathrm{A}\left(x, y\right)$ where $\mathrm{A}\left(x, y\right)$ is a homset, the set of all morphisms of A from x to y. From the definition, $\mathrm{A}\left(x, y\right)$ is an R-module for all $x, y \in \mathrm{A}_0$. Moreover, we say that A is over A_0.

Definition 2 *[15–17]. An R-linear functor between two R-categories is called an '*R-functor*' and an R-functor between two R-algebroids is called an '*R-algebroid morphism*'. Moreover, an assignment between two pre-R-algebroids satisfying all axioms of an R-functor except for the identity preservation axiom is called a '*pre-R-algebroid morphism*'.*

All R-algebroids and their morphisms form the category $\mathrm{Alg}\left(R\right)$.

Remark 3 *Throughout this paper, for a (pre-)R-algebroid A, $a \in \mathrm{A}$ will mean that a is a morphism of A. Moreover, if $a, a' \in \mathrm{A}$ with $ta = sa'$ then their composition will be denoted by aa'.*

Definition 4 *[18]. Let A be a pre-R-algebroid and*

$$\mathrm{I} = \left\{\mathrm{I}\left(x, y\right) \subseteq \mathrm{A}\left(x, y\right) : x, y \in \mathrm{A}_0\right\}$$

*be a family of R-submodules of A. For all $w, x, y, z \in \mathrm{A}_0$, $a' \in \mathrm{A}\left(w, x\right)$, $a'' \in \mathrm{A}\left(y, z\right)$ and $a \in \mathrm{I}\left(x, y\right)$ if $a'a \in \mathrm{I}\left(w, y\right)$, and $aa'' \in \mathrm{I}\left(x, z\right)$ then I is said to be a '*two-sided ideal*' of A.*

Definition 5 *[18]. Let A be an R-algebroid and M be a pre-R-algebroid with the same object set A_0. A family of maps defined for all $x, y, z \in \mathrm{A}_0$ as*

$$\begin{array}{ccc} \mathrm{M}\left(x, y\right) \times \mathrm{A}\left(y, z\right) & \longrightarrow & \mathrm{M}\left(x, z\right) \\ (m, a) & \longmapsto & m^a \end{array}$$

*is called a '*right action*' of A on M, if the conditions*

$$\begin{array}{llll} 1. & \left(m^a\right)^{a'} = m^{aa'} & 4. & \left(m_1 + m_2\right)^a = m_1^a + m_2^a \\ 2. & m^{a_1 + a_2} = m^{a_1} + m^{a_2} & 5. & \left(r \cdot m\right)^a = r \cdot m^a = m^{r \cdot a} \\ 3. & \left(m'm\right)^a = m'm^a & 6. & m^{1_{tm}} = m \end{array}$$

are satisfied for all $r \in R$, $a, a', a_1, a_2 \in \mathrm{A}$, $m, m', m_1, m_2 \in \mathrm{M}$ with compatible sources and targets.

A 'left action' *of* A *on* M *can be defined in a similar way.*

If A *has a right and a left action on* M *and if the condition*

$$\left({}^{a}m\right)^{a'} = {}^{a}\left(m^{a'}\right)$$

is satisfied for all $m \in \mathrm{M}$ *and* $a, a' \in \mathrm{A}$ *with* $ta = sm,\ tm = sa'$ *then* A *is said to have an* 'associative action' *on* M.

Definition 6 *Let* A *be an* R*-algebroid and* M *be a pre-*R*-algebroid with the same object set* A_0. *If* A *has an associative action on* M *then* M *is called an* 'A-module'. *If* M *is an* A*-module we usually write* (M, A) *and call it an '*R*-algebroid module' or an '*R*-algebroid* A*-module'. Moreover, for any two* R*-algebroid modules* (M, A) *and* (N, B) *a pair* $(f, g) : (\mathrm{M}, \mathrm{A}) \longrightarrow (\mathrm{N}, \mathrm{B})$ *is called an* R*-algebroid module morphism if* $f : \mathrm{M} \longrightarrow \mathrm{N}$ *is a pre-*R*-algebroid morphism,* $g : \mathrm{A} \longrightarrow \mathrm{B}$ *is an* R*-algebroid morphism and the conditions*

$$\begin{array}{ll} 1. & fm \in \mathrm{N}\left(g\left(sm\right), g\left(tm\right)\right), \\ 2. & f\left({}^{a}m\right) = {}^{ga}\left(fm\right) \ \text{and}\ f\left(m^{a'}\right) = \left(fm\right)^{ga'} \end{array}$$

are satisfied for all $m \in \mathrm{M}$, $a, a' \in \mathrm{A}$ *with* $ta = sm$, $tm = sa'$.

Thus, we get a category, denoted by $\mathrm{ModAlg}(R)$, whose objects are all R-algebroid modules and morphisms are all R-algebroid module morphisms. Furthermore, all R-algebroid A-modules with the identity morphism I_{A} on A form a subcategory $\mathrm{ModAlg}(R)/\mathrm{A}$ of $\mathrm{ModAlg}(R)$.

Definition 7 *[18]. Let* A *be an* R*-algebroid and* M *be a pre-*R*-algebroid with the same set of objects* A_0 *and let* A *have an associative action on* M. *A pre-*R*-algebroid morphism* $\mu : \mathrm{M} \longrightarrow \mathrm{A}$ *is called an "*R*-algebroid precrossed module" or an "*R*-algebroid precrossed* A*-module" if the condition*

$$\mathrm{CM1)} \quad \mu\left({}^{a}m\right) = a\left(\mu m\right) \quad \text{and} \quad \mu\left(m^{a'}\right) = \left(\mu m\right) a'$$

is satisfied, and $\mu : \mathrm{M} \longrightarrow \mathrm{A}$ *is called an "*R*-algebroid crossed module" or an "*R*-algebroid crossed* A*-module" if a second condition,*

$$\mathrm{CM2)} \quad m^{\mu m'} = mm' = {}^{\mu m}m',$$

is satisfied, for all $a, a' \in \mathrm{A}$ *and* $m, m' \in \mathrm{M}$ *with* $ta = sm, tm = sa' = sm'$. *Thus, a crossed module is a precrossed module satisfying* CM2.

Let $\mathcal{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A})$ *and* $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{B})$ *be two (pre)crossed modules of* R*-algebroids and let* $f : \mathrm{M} \longrightarrow \mathrm{N}$ *be a pre-R-algebroid morphism and* $g : \mathrm{A} \longrightarrow \mathrm{B}$ *be an* R*-algebroid morphism. The pair* $(f, g) : \mathcal{M} \longrightarrow \mathcal{N}$ *is called a (pre)crossed module morphism if the conditions*

$$\begin{array}{ll} 1. & f\left({}^{a}m\right) = {}^{ga}\left(fm\right) \quad \text{and} \quad f\left(m^{a'}\right) = \left(fm\right)^{ga'} \\ 2. & \left(\eta f\right)\left(m\right) = \left(g\mu\right)\left(m\right) \end{array}$$

are satisfied, for all $a, a' \in \mathrm{A}$ *and* $m \in \mathrm{M}$ *with* $ta = sm, tm = sa'$. *The meaning of the second condition is that the diagram in Figure 1 is commutative.*

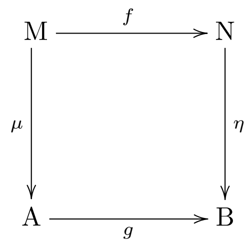

Figure 1

Note, also, that if $\mu : \mathrm{M} \longrightarrow \mathrm{A}$ *is a (pre)crossed module then* M *is an* A*-module and a (pre)crossed module morphism is a module morphism satisfying the second condition.*

Thus, all R-algebroid precrossed modules and their morphisms form a category denoted by $\mathrm{PXAlg}(R)$. Moreover, all R-algebroid precrossed A-modules with the identity morphism on A form a subcategory $\mathrm{PXAlg}(R)/\mathrm{A}$ of $\mathrm{PXAlg}(R)$. Similarly, all R-algebroid crossed modules form the category $\mathrm{XAlg}(R)$ and all R-algebroid crossed A-modules form the category $\mathrm{XAlg}(R)/\mathrm{A}$, which is a subcategory of $\mathrm{XAlg}(R)$. Obviously, $\mathrm{XAlg}(R)$ is a full subcategory of $\mathrm{PXAlg}(R)$ and $\mathrm{XAlg}(R)/\mathrm{A}$ is a full subcategory of $\mathrm{PXAlg}(R)/\mathrm{A}$.

Example 8 *[18]. If* A *is an* R*-algebroid and* I *is a two-sided ideal of* A*, then the inclusion morphism*

$$i : \mathrm{I} \longrightarrow \mathrm{A}$$

is a crossed module with the action of A *on* I *defined by*

$${}^{a}b = ab \quad \textit{and} \quad b^{a'} = ba'$$

for all $a, a' \in \mathrm{A}$*,* $b \in \mathrm{I}$ *with* $ta = sb$*,* $tb = sa'$*.*

3. Free R-algebroid modules

Clearly an R-algebroid module (M,A) comes with a function $\xi_{\mathrm{M}} : Mor(\mathrm{M}) \longrightarrow \mathrm{A}_0 \times \mathrm{A}_0$ defined as $\xi_{\mathrm{M}} m = (sm, tm)$ for all $m \in \mathrm{M}$. This motivates us to form a category, $\mathrm{Sets}_0/\mathrm{Alg}(R)$, whose objects are all functions $\omega : \mathrm{K} \longrightarrow \mathrm{A}_0 \times \mathrm{A}_0$ defined as $\omega k = (\omega_1 k, \omega_2 k)$, where K is a set and A is an R-algebroid, and whose morphisms are all pairs $(f, g_0 \times g_0) : \omega \longrightarrow \omega'$ where if $\omega' : \mathrm{K}' \longrightarrow \mathrm{B}_0 \times \mathrm{B}_0$ then $f : \mathrm{K} \longrightarrow \mathrm{K}'$ is a function, $g : \mathrm{A} \longrightarrow \mathrm{B}$ is an R-algebroid morphism, g_0 is the restriction of g on A_0, and $g_0 \times g_0 : \mathrm{A}_0 \times \mathrm{A}_0 \longrightarrow \mathrm{B}_0 \times \mathrm{B}_0$ is defined as $(g_0 \times g_0)(x, y) = (g_0 x, g_0 y)$ for all $x, y \in \mathrm{A}_0$, making the diagram in Figure 2 commutative.

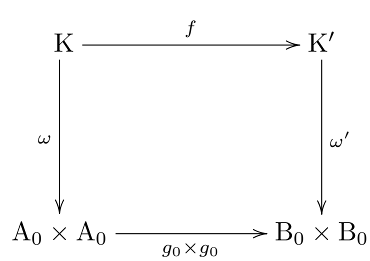

Figure 2

By fixing the R-algebroid A and taking $g_0 \times g_0$ as $I_{\mathrm{A}_0 \times \mathrm{A}_0}$, the identity function on $\mathrm{A}_0 \times \mathrm{A}_0$, we obtain a subcategory $\mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A})$ of $\mathrm{Sets}_0/\mathrm{Alg}(R)$.

Note that, for each R-algebroid module (M,A), the function ξ_{M} is an object of $\mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A})$.

Proposition 9 *For any object* $\omega : \mathrm{K} \longrightarrow \mathrm{A}_0 \times \mathrm{A}_0$ *of* $\mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A})$ *there exists an* R*-algebroid* A*-module* $(\mathrm{F}(\omega), \mathrm{A})$ *and a morphism* $(i_m, I_{\mathrm{A}_0\times\mathrm{A}_0}) : \omega \longrightarrow \xi_{\mathrm{F}(\omega)}$ *such that for all* R*-algebroid* A*-modules* (N, A) *and for all morphisms* $(f, I_{\mathrm{A}_0\times\mathrm{A}_0}) : \omega \longrightarrow \xi_{\mathrm{N}}$ *there exists a unique* A*-module morphism* $(\alpha, I_{\mathrm{A}}) : (\mathrm{F}(\omega), \mathrm{A}) \longrightarrow (\mathrm{N}, \mathrm{A})$ *satisfying* $f = \alpha i_m$*, which means that the diagram in Figure 3 is commutative.*

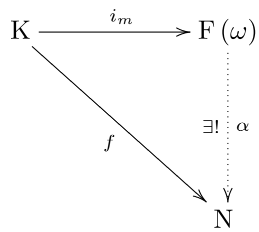

Figure 3

$(\mathrm{F}(\omega), \mathrm{A})$*, with the morphism* $(i_m, I_{\mathrm{A}_0\times\mathrm{A}_0})$*, is called the free* R*-algebroid* A*-module determined by* ω*. The free module is unique up to isomorphism.*

Proof Provided that $n \in \mathbb{N}^+$, $k, k_1, ..., k_n \in \mathrm{K}$ and $a, a_1, ..., a_n, a', a'_1, ..., a'_n \in \mathrm{A}$, consider all elements of the form aka' under the conditions $ta = \omega_1 k$ and $sa' = \omega_2 k$, and tying such elements construct all words of the form $a_1k_1a'_1a_2k_2a'_2...a_nk_na'_n$ under the conditions $ta'_1 = sa_2, ..., ta'_{n-1} = sa_n$. For any word $p_i = a_{i_1}k_{i_1}a'_{i_1}...a_{i_n}k_{i_n}a'_{i_n}$ define its source as $sp_i = sa_{i_1}$ and its target as $tp_i = ta'_{i_n}$, and for all $x, y \in \mathrm{A}_0$ denote the free additive abelian group generated by all words with source x and target y by $\mathrm{G}(\omega)(x, y)$. Obviously, each element of $\mathrm{G}(\omega)(x, y)$ is of the form $\sum_i p_i$ where p_is are words with source x and target y. Now we consider the normal subgroup $\mathrm{N}(x, y)$ of $\mathrm{G}(\omega)(x, y)$ generated by all elements of forms

$$\begin{gathered}
a_1k_1a'_1...(a_i + a''_i)\,k_ia'_i...a_nk_na'_n - a_1k_1a'_1...a_ik_ia'_i...a_nk_na'_n - a_1k_1a'_1...a''_ik_ia'_i...a_nk_na'_n \\
a_1k_1a'_1...a_ik_i\,(a'_i + a'''_i)...a_nk_na'_n - a_1k_1a'_1...a_ik_ia'_i...a_nk_na'_n - a_1k_1a'_1...a_ik_ia'''_i...a_nk_na'_n \\
(r \cdot a_1)\,k_1a'_1...a_ik_ia'_i...a_nk_na'_n - a_1k_1a'_1...(r \cdot a_i)\,k_ia'_i...a_nk_na'_n \\
(r \cdot a_1)\,k_1a'_1...a_ik_ia'_i...a_nk_na'_n - a_1k_1a'_1...a_ik_i\,(r \cdot a'_i)...a_nk_na'_n
\end{gathered}$$

for all $r \in R$. If we divide $\mathrm{G}(\omega)(x, y)$ by $\mathrm{N}(x, y)$ then we get an abelian quotient group $[\mathrm{G}(\omega)(x, y)]$ of which elements are cosets of $\mathrm{N}(x, y)$. We denote $[\mathrm{G}(\omega)(x, y)]$ with $\mathrm{F}(\omega)(x, y)$, and the cosets $p_i + \mathrm{N}(x, y)$ and $\sum_i p_i + \mathrm{N}(x, y)$ with $[p_i]$ and $\left[\sum_i p_i\right]$, respectively, for all $p_i, \sum_i p_i \in \mathrm{G}(\omega)(x, y)$. It is obvious that $\left[\sum_i p_i\right] = \sum_i [p_i]$.

Now we can define an R-action on $\mathrm{F}(\omega)(x, y)$ as $r \cdot [p_i] = \left[(r \cdot a_{i_1})\,k_{i_1}a'_{i_1}...a_{i_n}k_{i_n}a'_{i_n}\right]$ and $r \cdot \left(\sum_i [p_i]\right) = \sum_i [r \cdot p_i]$ for all $r \in R$, and with this action the quotient group $\mathrm{F}(\omega)(x, y)$ is clearly an R-module.

Hence, the family $\mathrm{F}(\omega) = \{\mathrm{F}(\omega)(x, y) : x, y \in \mathrm{A}_0\}$ becomes a pre-R-algebroid by the composition defined for all $x, y, z \in \mathrm{A}_0$ as

$$\begin{array}{ccc}
\mathrm{F}(\omega)(x, y) \times \mathrm{F}(\omega)(y, z) & \longrightarrow & \mathrm{F}(\omega)(x, z) \\
\left(\sum_i [p_i], \sum_j [p_j]\right) & \longmapsto & \left(\sum_i [p_i]\right)\left(\sum_j [p_j]\right) = \sum_{i,j} [p_ip_j] = \sum_i \sum_j [p_ip_j]
\end{array}$$

where if $p_i = a_{i_1}k_{i_1}a'_{i_1}...a_{i_n}k_{i_n}a'_{i_n}$ and $p_j = a_{j_1}k_{j_1}a'_{j_1}...a_{j_{n'}}k_{j_{n'}}a'_{j_{n'}}$ then $p_ip_j = a_{i_1}k_{i_1}a'_{i_1}...a_{i_n}k_{i_n}a'_{i_n}a_{j_1}k_{j_1}a'_{j_1}...$ $a_{j_{n'}}k_{j_{n'}}a'_{j_{n'}}$.

Moreover, an associative A-action on $\mathrm{F}(\omega)$ can be defined as ${}^{a}\left(\sum_i [p_i]\right) = \sum_i\ [{}^{a}p_i]$ and $\left(\sum_i [p_i]\right)^{a'} = \sum_i \left[p_i^{a'}\right]$ where ${}^{a}p_i = (aa_{i_1})\,k_{i_1}a'_{i_1}...a_{i_n}k_{i_n}a'_{i_n}$ and $p_i^{a'} = a_{i_1}k_{i_1}a'_{i_1}...a_{i_n}k_{i_n}\left(a'_{i_n}a'\right)$ under the condition $ta = sp_i$, $tp_i = sa'$, and this action makes $\mathrm{F}(\omega)$ an A-module.

Define $i_m : \mathrm{K} \longrightarrow \mathrm{F}(\omega)$ as $i_m(k) = [1k1]$ $(= [1_{\omega_1 k}k1_{\omega_2 k}])$ and $\alpha : \mathrm{F}(\omega) \longrightarrow \mathrm{N}$ as $\alpha[aka'] = {}^{a}(fk)^{a'}$, $\alpha[p_i] = \alpha\left[a_{i_1}k_{i_1}a'_{i_1}\right]...\alpha\left[a_{i_n}k_{i_n}a'_{i_n}\right]$ and $\alpha\left(\sum_i [p_i]\right) = \sum_i \alpha[p_i]$ for all $(f, I_{\mathrm{A}_0\times\mathrm{A}_0}) : \omega \longrightarrow \xi_{\mathrm{N}}$. It can easily be shown that $(i_m, I_{\mathrm{A}_0\times\mathrm{A}_0})$ is a morphism from ω to $\xi_{\mathrm{F}(\omega)}$ and (α, I_{A}) is an A-module morphism from $(\mathrm{F}(\omega), \mathrm{A})$ to (N, A) satisfying $f = \alpha i_m$. Obviously, α is unique from its definition. Moreover, it can be shown that $(\mathrm{F}(\omega), \mathrm{A})$ with the morphism $(i_m, I_{\mathrm{A}_0\times\mathrm{A}_0})$ is unique up to isomorphism. □

The construction of the free module gives a functor F from $\mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A})$ to $\mathrm{ModAlg}(R)/\mathrm{A}$ defined as $F(\omega) = (\mathrm{F}(\omega), \mathrm{A})$ on objects and as $F(f, I_{\mathrm{A}_0\times\mathrm{A}_0}) = (Ff, I_{\mathrm{A}})$ on morphisms such that $Ff([aka']) = [a(fk)a']$ on generators.

Proposition 10 *The functor F is the left adjoint of the forgetful functor $U : \mathrm{ModAlg}(R)/\mathrm{A} \longrightarrow \mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A})$, which is defined as $U(\mathrm{N}, \mathrm{A}) = \xi_{\mathrm{N}}$ for each R-algebroid module (N, A) and is defined as $U(g, I_{\mathrm{A}}) = (Ug, I_{\mathrm{A}_0\times\mathrm{A}_0})$ on morphisms such that $(Ug)(n) = gn$ for all $n \in \mathrm{N}$.*

Proof We must find a natural equivalence

$$\Phi : \quad (\mathrm{ModAlg}(R)/\mathrm{A})(F(-), (-)) \quad \cong \quad (\mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A}))(-, U(-)),$$

which is required to give a map

$$\begin{array}{rrcl} \Phi : & Ob(\mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A})) \times Ob(\mathrm{ModAlg}(R)/\mathrm{A}) & \longrightarrow & \mathrm{Sets} \\ & (\omega : \mathrm{K} \longrightarrow \mathrm{A}_0 \times \mathrm{A}_0, (\mathrm{N}, \mathrm{A})) & \longmapsto & \Phi(\omega, (\mathrm{N}, \mathrm{A})) \end{array}$$

such that $\Phi(\omega, (\mathrm{N}, \mathrm{A}))$ is a bijection from $(\mathrm{ModAlg}(R)/\mathrm{A})(F(\omega), (\mathrm{N}, \mathrm{A}))$ to $(\mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A}))(\omega, U(\mathrm{N}, \mathrm{A}) = \xi_{\mathrm{N}})$ and is natural in both ω and (N, A) for all $\omega \in Ob(\mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A}))$ and $(\mathrm{N}, \mathrm{A}) \in Ob(\mathrm{ModAlg}(R)/\mathrm{A})$.

We abbreviate $\Phi(\omega, (\mathrm{N}, \mathrm{A}))$ as $\Phi(\omega, \mathrm{A})$ and define $\Phi(\omega, \mathrm{N})$ as $\Phi(\omega, \mathrm{N})(f, I_{\mathrm{A}}) = (\Phi(\omega, \mathrm{N})(f), I_{\mathrm{A}_0\times\mathrm{A}_0})$ such that

$$\begin{array}{rccl} \Phi(\omega, \mathrm{N})(f) : & \mathrm{K} & \longrightarrow & \mathrm{N} \\ & k & \longmapsto & \Phi(\omega, \mathrm{N})(f)(k) = f[1k1] \end{array}$$

for all $(f, I_{\mathrm{A}}) \in (\mathrm{ModAlg}(R)/\mathrm{A})((\mathrm{F}(\omega), \mathrm{A}), (\mathrm{N}, \mathrm{A}))$ where $\omega : K \to \mathrm{A}_0 \times \mathrm{A}_0$. Clearly, $\Phi(\omega, \mathrm{N})$ is well defined and 1-1. It is also onto since each morphism

$$(h,\ I_{\mathrm{A}_0\times\mathrm{A}_0}) : (\omega : \mathrm{K} \longrightarrow \mathrm{A}_0 \times \mathrm{A}_0) \longrightarrow (\xi_{\mathrm{N}} : \mathrm{N} \longrightarrow \mathrm{A}_0 \times \mathrm{A}_0)$$

is the image of the morphism (f, I_{A}) under $\Phi(\omega, \mathrm{N})$, where $f : \mathrm{F}(\omega) \longrightarrow \mathrm{N}$ is defined as $f[aka'] = {}^{a}(hk)^{a'}$ on generators.

Moreover, provided that $(-)^{\bullet}$ is a composition with $(-)$ from right, for all $(g, I_{\mathrm{A}_0\times\mathrm{A}_0}) : \omega \longrightarrow \omega'$, $(f, I_{\mathrm{A}}) : ((\mathrm{F}(\omega'), \mathrm{A}) \longrightarrow (\mathrm{N}, \mathrm{A}))$ and $k \in \mathrm{K}$

$$\begin{aligned}
\left(\Phi(\omega,\mathrm{N})(Fg)^{\bullet}\right)(f)(k) &= \left(\Phi(\omega,\mathrm{N})(Fg)^{\bullet}(f)\right)(k) = \left(\Phi(\omega,\mathrm{N})(f(Fg))\right)(k) \\
&= (f(Fg))[1k1] = f[1(gk)1] \\
&= (\Phi(\omega',\mathrm{N})(f))(gk) = ((\Phi(\omega',\mathrm{N})(f))g)(k) \\
&= (g^{\bullet}(\Phi(\omega',\mathrm{N})(f)))(k) = (g^{\bullet}\Phi(\omega',\mathrm{N}))(f)(k),
\end{aligned}$$

i.e. the diagram in Figure 4 is commutative and $\Phi(\omega,\mathrm{N})$ is natural in ω.

$$\begin{array}{ccc}
(\mathrm{ModAlg}(R)/\mathrm{A})((\mathrm{F}(\omega),\mathrm{A}),(\mathrm{N},\mathrm{A})) & \xrightarrow{\Phi(\omega,\mathrm{N})} & (\mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A}))(\omega,\xi_{\mathrm{N}}) \\
\uparrow{\scriptstyle (Fg)^{\bullet}} & & \uparrow{\scriptstyle g^{\bullet}} \\
(\mathrm{ModAlg}(R)/\mathrm{A})((\mathrm{F}(\omega'),\mathrm{A}),(\mathrm{N},\mathrm{A})) & \xrightarrow[\Phi(\omega',\mathrm{N})]{} & (\mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A}))(\omega',\xi_{\mathrm{N}})
\end{array}$$

Figure 4

A similar calculation shows that the diagram in Figure 5 is commutative for each $(g, I_{\mathrm{A}}) \in (\mathrm{ModAlg}(R)/\mathrm{A})((\mathrm{N},\mathrm{A}), (\mathrm{N}',\mathrm{A}))$, where $(-)_{\bullet}$ is composition with $(-)$ from left, and $\Phi(\omega,\mathrm{N})$ is natural in (N,A).

$$\begin{array}{ccc}
(\mathrm{ModAlg}(R)/\mathrm{A})((\mathrm{F}(\omega),\mathrm{A}),(\mathrm{N},\mathrm{A})) & \xrightarrow{\Phi(\omega,\mathrm{N})} & (\mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A}))(\omega,\xi_{\mathrm{N}}) \\
{\scriptstyle g_{\bullet}}\downarrow & & \downarrow{\scriptstyle (Ug)_{\bullet}} \\
(\mathrm{ModAlg}(R)/\mathrm{A})((\mathrm{F}(\omega),\mathrm{A}),(\mathrm{N}',\mathrm{A})) & \xrightarrow[\Phi(\omega,\mathrm{N}')]{} & (\mathrm{Sets}_0/(\mathrm{Alg}(R)/\mathrm{A}))(\omega,\xi_{\mathrm{N}'})
\end{array}$$

Figure 5

□

4. Free R-algebroid precrossed modules

The fact that if $\eta : \mathrm{N} \longrightarrow \mathrm{A}$ is a (pre)crossed module then there is a restricted function $\eta_m : Mor(\mathrm{N}) \longrightarrow \mathrm{A}$ as $\eta_m(n) = \eta n$ motivates us to form a category $\mathrm{Sets}/\mathrm{Alg}(R)$ whose objects are all functions $\omega : \mathrm{K} \longrightarrow \mathrm{A}$ where K is a set and A is an R-algebroid such that ωk is a morphism of A for all $k \in \mathrm{K}$ and whose morphisms are all pairs $(f,g) : \omega \longrightarrow \omega'$ where if $\omega' : \mathrm{K}' \longrightarrow \mathrm{B}$ then $f : \mathrm{K} \longrightarrow \mathrm{K}'$ is a function and $g : \mathrm{A} \longrightarrow \mathrm{B}$ is an R-algebroid morphism making the diagram in Figure 6 commutative.

$$\begin{array}{ccc}
\mathrm{K} & \xrightarrow{f} & \mathrm{K}' \\
{\scriptstyle \omega}\downarrow & & \downarrow{\scriptstyle \omega'} \\
\mathrm{A} & \xrightarrow[g]{} & \mathrm{B}
\end{array}$$

Figure 6

By fixing the R-algebroid A and taking g as I_{A}, we obtain a subcategory $\mathrm{Sets}/(\mathrm{Alg}(R)/\mathrm{A})$ of $\mathrm{Sets}/\mathrm{Alg}(R)$.

Note that, for each precrossed or crossed A-module $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{A})$, the function $\eta_m : Mor(\mathrm{N}) \longrightarrow \mathrm{A}$ is an object of $\mathrm{Sets}/\mathrm{Alg}(R)$.

Proposition 11 *For any object* $\omega : \mathrm{K} \longrightarrow \mathrm{A}$ *of* $\mathrm{Sets}/(\mathrm{Alg}(R)/\mathrm{A})$ *there exists an* R*-algebroid precrossed* A*-module* $F_P(\omega) = (\omega_P : \mathrm{F}_P(\omega) \longrightarrow \mathrm{A})$ *and a morphism* $(i_p, I_{\mathrm{A}}) : \omega \longrightarrow \omega_{Pm}$ *such that for all* R*-algebroid precrossed* A*-modules* $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{A})$ *and for all morphisms* $(f, I_{\mathrm{A}}) : \omega \longrightarrow \eta_m$ *there exists a unique precrossed* A*-module morphism* $(\alpha, I_{\mathrm{A}}) : F_P(\omega) \longrightarrow \mathcal{N}$ *satisfying* $f = \alpha i_p$*, which means the diagram in Figure 7 is commutative.*

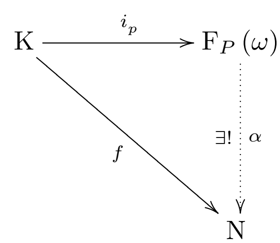

Figure 7

$F_P(\omega)$, with the morphism (i_p, I_{A}), is called the free R-algebroid precrossed A-module determined by ω. The free precrossed module is unique up to isomorphism.

Proof ω determines a function $\omega_{\mathrm{A}_0} : \mathrm{K} \longrightarrow \mathrm{A}_0 \times \mathrm{A}_0$ as $\omega_{\mathrm{A}_0}(k) = (s(\omega k), t(\omega k))$ and from the previous section there exists a free R-algebroid A-module $F(\omega_{\mathrm{A}_0})$ determined by ω_{A_0}, with an A-action defined as ${}^{a''}[aka'] = [(a''a)ka']$ and $[aka']^{a'''} = [ak(a'a''')]$ on generators with $ta'' = sa$ and $ta' = sa'''$. Now, taking $\mathrm{F}_P(\omega) = \mathrm{F}(\omega_{\mathrm{A}_0})$, define $\omega_P : \mathrm{F}_P(\omega) \longrightarrow \mathrm{A}$ as $\omega_P[aka'] = a(\omega k)a'$ on generators and $i_p : \mathrm{K} \longrightarrow \mathrm{F}_P(\omega)$ as $i_p k = [1k1]$ for all $k \in \mathrm{K}$. It can easily be checked that, by these definitions, $F_P(\omega) = (\omega_P : \mathrm{F}_P(\omega) \longrightarrow \mathrm{A})$ is a precrossed module and (i_p, I_{A}) is a morphism from ω to ω_{Pm}.

Defining $\alpha : \mathrm{F}_P(\omega) \longrightarrow \mathrm{N}$ as $\alpha[aka'] = {}^{a}(fk)^{a'}$ on generators, since the rest are detail, completes the proof. □

As in the case of free modules, the construction of free precrossed module gives a functor F_P : $\mathrm{Sets}/(\mathrm{Alg}(R)/\mathrm{A}) \longrightarrow \mathrm{PXAlg}(R)/\mathrm{A}$ defined as $F_P(\omega) = (\omega_P : \mathrm{F}_P(\omega) \longrightarrow \mathrm{A})$ on objects and as $F_P(f, I_{\mathrm{A}}) = (F_P f, I_{\mathrm{A}})$ on morphisms such that $F_P f[aka'] = [a(fk)a']$ on generators.

Proposition 12 *The functor* F_P *is the left adjoint of the forgetful functor* $U : \mathrm{PXAlg}(R)/\mathrm{A} \longrightarrow \mathrm{Sets}/(\mathrm{Alg}(R)/\mathrm{A})$*, which for a precrossed module* $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{A})$ *gives the function* η_m *and for a precrossed* A*-module morphism* $(f, I_{\mathrm{A}}) : \mathcal{N} \longrightarrow \mathcal{N}'$ *gives the morphism* $U(f, I_{\mathrm{A}}) = (Uf, I_{\mathrm{A}}) : \eta_m \longrightarrow \eta'_m$ *such that* $Uf(n) = fn$ *for all* $n \in \mathrm{N}$*.*

Proof We omit the proof, since the constructions are almost the same as those in the proof of Proposition 10. □

5. Peiffer ideal of a precrossed module

Since our aim in the next section is to obtain the free R-algebroid crossed modules, in this section we construct the Peiffer ideal for a precrossed module of R-algebroids to get a crossed module. The term 'Peiffer element' was first used by Brown and Huebschmann [9], and Baus and Conduché [6] gave a substantial theory of Peiffer commutator calculus. Brown et al. used the Peiffer subgroup to obtain crossed modules of groups in [7] and Shammu used Peiffer commutators to get crossed modules of algebras in [22].

Definition 13 *Let $\mathcal{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A})$ be a precrossed module of R-algebroids and let m, m' be two morphisms of* M *satisfying the condition $tm = sm'$. The Peiffer commutators of m and m' are defined as $[\![m, m']\!]_1 = m^{\mu m'} - mm'$ and $[\![m, m']\!]_2 = {}^{\mu m}m' - mm'$.*

If $\mathcal{M}$ is a crossed module then both of these commutators are zero. Conversely, a precrossed module in which all of these commutators are zero is a crossed module.

For all $x, y \in \mathrm{A}_0$, we denote the subgroup of $\mathrm{M}(x,y)$ generated by $[\![\mathrm{M},\mathrm{M}]\!]_{\mathrm{g}}(x,y) = \{[\![m,m']\!]_1, [\![m,m']\!]_2 : m, m' \in \mathrm{M},\ x = sm, tm' = y\}$, the set of all Peiffer commutators of $\mathrm{M}(x,y)$, by $[\![\mathrm{M},\mathrm{M}]\!](x,y)$. Since $\mathrm{M}(x,y)$ is abelian, $[\![\mathrm{M},\mathrm{M}]\!](x,y)$ is also abelian. By a direct calculation, it can be shown that $r \cdot [\![m,m']\!]_1 = [\![r \cdot m, m']\!]_1 = [\![m, r \cdot m']\!]_1$ and $r \cdot [\![m,m']\!]_2 = [\![r \cdot m, m']\!]_2 = [\![m, r \cdot m']\!]_2$ for all $[\![m,m']\!]_1$, $[\![m,m']\!]_2 \in [\![\mathrm{M},\mathrm{M}]\!](x,y)$ and for all $r \in R$, which means $[\![\mathrm{M},\mathrm{M}]\!](x,y)$ is closed under the action of R, and this results in that $[\![\mathrm{M},\mathrm{M}]\!](x,y)$ is an R-module, an R-submodule of $\mathrm{M}(x,y)$.

Proposition 14 (i) *The family $[\![\mathrm{M},\mathrm{M}]\!] = \{[\![\mathrm{M},\mathrm{M}]\!](x,y) : x, y \in \mathrm{A}_0\}$ is a two sided ideal of* M.

(ii) *$[\![\mathrm{M},\mathrm{M}]\!]$ is closed under the action of* A.

Proof For all $w, x, y, z \in \mathrm{A}_0$, $[\![m,m']\!]_1, [\![m,m']\!]_2 \in [\![\mathrm{M},\mathrm{M}]\!](x,y)$, $m'' \in \mathrm{M}(w,x)$, $m''' \in \mathrm{M}(y,z)$, $a \in \mathrm{A}(w,x)$, and $a' \in \mathrm{A}(y,z)$, a direct calculation gives that

$$
\begin{array}{lrcl}
(i) & m''[\![m,m']\!]_1 & = & [\![m''m, m']\!]_1 \in [\![\mathrm{M},\mathrm{M}]\!](w,y) \\
 & [\![m,m']\!]_1 m''' & = & [\![m, m'm''']\!]_1 - [\![m^{\mu m'}, m''']\!]_1 \in [\![\mathrm{M},\mathrm{M}]\!](x,z) \\
 & m''[\![m,m']\!]_2 & = & [\![m''m, m']\!]_2 - [\![m'', {}^{\mu m}m']\!]_2 \in [\![\mathrm{M},\mathrm{M}]\!](w,y) \\
 & [\![m,m']\!]_2 m''' & = & [\![m, m'm''']\!]_2 \in [\![\mathrm{M},\mathrm{M}]\!](x,z) \\
(ii) & {}^{a}[\![m,m']\!]_1 & = & [\![{}^{a}m, m']\!]_1 \in [\![\mathrm{M},\mathrm{M}]\!](w,y) \\
 & [\![m,m']\!]_1^{a'} & = & [\![m, (m')^{a'}]\!]_1 \in [\![\mathrm{M},\mathrm{M}]\!](x,z) \\
 & {}^{a}[\![m,m']\!]_2 & = & [\![{}^{a}m, m']\!]_2 \in [\![\mathrm{M},\mathrm{M}]\!](w,y) \\
 & [\![m,m']\!]_2^{a'} & = & [\![m, (m')^{a'}]\!]_2 \in [\![\mathrm{M},\mathrm{M}]\!](x,z).
\end{array}
$$

□

The ideal $[\![\mathrm{M},\mathrm{M}]\!]$ is called the '*Peiffer*' ideal of M.

Now construct the family

$$\frac{\mathrm{M}}{[\![\mathrm{M},\mathrm{M}]\!]} = \left\{ \frac{\mathrm{M}}{[\![\mathrm{M},\mathrm{M}]\!]}(x,y) = \frac{\mathrm{M}(x,y)}{[\![\mathrm{M},\mathrm{M}]\!](x,y)} : x, y \in \mathrm{A}_0 \right\}$$

of quotient R-modules. Clearly, $\frac{\mathrm{M}}{[\![\mathrm{M},\mathrm{M}]\!]}$ is a pre-R-algebroid which is an A-module thanks to the addition, multiplication, R-action and associative A-action induced by those defined on M.

We write M^{cr} instead of $\frac{\mathrm{M}}{[\![\mathrm{M},\mathrm{M}]\!]}$ and $\overline{m}$ instead of $m+[\![\mathrm{M},\mathrm{M}]\!](x,y)$ for all $m\in\mathrm{M}(x,y)$, to abbreviate. μ induces a map

$$\begin{array}{rccl}\mu^{\mathrm{cr}}: & \mathrm{M}^{\mathrm{cr}} & \longrightarrow & \mathrm{A}\\ & \overline{m} & \longmapsto & \mu^{\mathrm{cr}}\overline{m}=\mu m\end{array}$$

since μ maps $[\![\mathrm{M},\mathrm{M}]\!]$ to $0_{\mathrm{A}}=\left\{0_{\mathrm{A}(x,y)}:x,y\in\mathrm{A}_0\right\}$, where $0_{\mathrm{A}(x,y)}$ is the additive identity of $\mathrm{A}(x,y)$.

Proposition 15 *(i) If* $\mathcal{M}=(\mu:\mathrm{M}\longrightarrow\mathrm{A})$ *is a precrossed module of* R*-algebroids, then* $\mathcal{M}^{\mathrm{cr}}=(\mu^{\mathrm{cr}}:\mathrm{M}^{\mathrm{cr}}\longrightarrow\mathrm{A})$ *is a crossed module.*

(ii) Provided that $\phi:\mathrm{M}\longrightarrow\mathrm{M}^{\mathrm{cr}}$ *is the quotient morphism, for all crossed* A*-modules* $\mathcal{N}=(\eta:\mathrm{N}\longrightarrow\mathrm{A})$ *and for all precrossed* A*-module morphisms* $(\alpha,I_{\mathrm{A}}):\mathcal{M}\longrightarrow\mathcal{N}$*, there exists a unique crossed* A*-module morphism* $(\alpha',I_{\mathrm{A}}):\mathcal{M}^{\mathrm{cr}}\longrightarrow\mathcal{N}$ *satisfying* $\alpha=\alpha'\phi$*.*

Proof (i) It can easily be shown that μ^{cr} is a pre-R-algebroid morphism. We show that it satisfies the crossed module conditions: For all $m,m'\in\mathrm{M}$ and for all $a,a'\in\mathrm{A}$ with $ta=sm$, $tm=sm'=sa'$

$$\text{CM1)}\quad \mu^{\mathrm{cr}}\left({}^{a}\overline{m}\right)=\mu^{\mathrm{cr}}\left(\overline{{}^{a}m}\right)=\mu\left({}^{a}m\right)=a\left(\mu m\right)=a\left(\mu^{\mathrm{cr}}\overline{m}\right)$$

and similarly $\mu^{\mathrm{cr}}\left(\overline{m}^{a'}\right)=\left(\mu^{\mathrm{cr}}\overline{m}\right)a'$,

$$\begin{array}{rcl}\text{CM2)}\quad \overline{m}^{\mu^{\mathrm{cr}}\overline{m'}} & = & \overline{m}^{\mu m'}=\overline{m^{\mu m'}}=m^{\mu m'}+[\![\mathrm{M},\mathrm{M}]\!](sm,tm')\\ & = & m^{\mu m'}+(-[\![m,m']\!]_1+[\![\mathrm{M},\mathrm{M}]\!](sm,tm'))\\ & = & m^{\mu m'}+(-\left(m^{\mu m'}-mm'\right)+[\![\mathrm{M},\mathrm{M}]\!](sm,tm'))\\ & = & mm'+[\![\mathrm{M},\mathrm{M}]\!](sm,tm')=\overline{mm'}=\overline{m}\overline{m'}\end{array}$$

and similarly ${}^{\mu^{\mathrm{cr}}\overline{m}}\overline{m'}=\overline{m}\overline{m'}$.

(ii) Define $\alpha':\mathrm{M}^{\mathrm{cr}}\longrightarrow\mathrm{N}$ as $\alpha'\overline{m}=\alpha m$. Obviously, (α',I_{A}) is a crossed A-module morphism and for all $m\in\mathrm{M}$

$$(\alpha'\phi)(m)=\alpha'(\phi m)=\alpha'\overline{m}=\alpha m.$$

The uniqueness of α' comes from its definition. □

Thus, we get a functor $(-)^{\mathrm{cr}}:\mathrm{PXAlg}(R)\longrightarrow\mathrm{XAlg}(R)$, which gives a crossed module $\mathcal{M}^{\mathrm{cr}}$ for any precrossed module $\mathcal{M}$ and is defined as $(f,g)^{\mathrm{cr}}=(f^{\mathrm{cr}},g)$ on morphisms where if $(f,g):\mathcal{M}\longrightarrow\mathcal{M}'$ then $(f^{\mathrm{cr}},g):\mathcal{M}^{\mathrm{cr}}\longrightarrow\mathcal{M}'^{\mathrm{cr}}$ such that $f^{\mathrm{cr}}\overline{m}=\overline{fm}$ for all $m\in\mathrm{M}$.

Proposition 16 *The functor* $(-)^{\mathrm{cr}}:\mathrm{PXAlg}(R)\longrightarrow\mathrm{XAlg}(R)$ *is the left adjoint of the inclusion functor* I_n *:* $\mathrm{XAlg}(R)\longrightarrow\mathrm{PXAlg}(R)$.

Proof For all $\mathcal{M}\in Ob(\mathrm{PXAlg}(R))$, $\mathcal{N}\in Ob(\mathrm{XAlg}(R))$ and crossed module morphisms $g=(g_1,g_2):\mathcal{M}^{\mathrm{cr}}\longrightarrow\mathcal{N}$ the pair $h=(h_1,g_2):\mathcal{M}\longrightarrow\mathcal{N}$ with $h_1m=g_1\overline{m}$ for all $m\in\mathrm{M}$ is clearly a precrossed module morphism. Then the map $\Phi(\mathcal{M},\mathcal{N})$ defined as

$$\begin{array}{rrcl}\Phi(\mathcal{M},\mathcal{N}): & \mathrm{XAlg}(R)(\mathcal{M}^{\mathrm{cr}},\mathcal{N}) & \longrightarrow & \mathrm{PXAlg}(R)(\mathcal{M},\mathcal{N})\\ & g=(g_1,g_2) & \longmapsto & \Phi(\mathcal{M},\mathcal{N})(g)=h=(h_1,g_2)\end{array}$$

can be shown to be a bijection, which is natural in both $\mathcal{M}$ and $\mathcal{N}$, and this completes the proof. □

6. Free R-algebroid crossed modules

Proposition 17 *For any object* $\omega : \mathrm{K} \longrightarrow \mathrm{A}$ *of* $\mathrm{Sets}/(\mathrm{Alg}(R)/\mathrm{A})$ *there exists an* R*-algebroid crossed* A*-module* $F_X(\omega) = (\omega_X : \mathrm{F}_X(\omega) \longrightarrow \mathrm{A})$ *and a morphism* $(i_c, I_\mathrm{A}) : \omega \longrightarrow \omega_{Xm}$ *such that for all* R*-algebroid crossed* A*-modules* $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{A})$ *and for all morphisms* $(f, I_\mathrm{A}) : \omega \longrightarrow \eta_m$ *there exists a unique crossed* A*-module morphism* $(\alpha, I_\mathrm{A}) : F_X(\omega) \longrightarrow \mathcal{N}$ *such that* $f = \alpha i_c$*, i.e. the diagram in Figure 8 is commutative.*

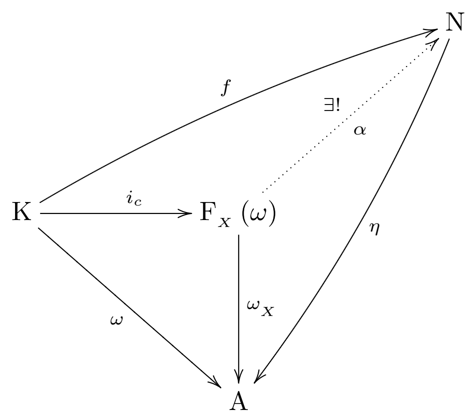

Figure 8

$F_X(\omega)$*, with the morphism* (i_c, I_A)*, is called the free* R*-algebroid crossed* A*-module determined by* ω*. The free crossed module is unique up to isomorphism.*

Proof In the fourth section we got the free R-algebroid precrossed A-module $F_P(\omega) = (\omega_P : \mathrm{F}_P(\omega) \longrightarrow \mathrm{A})$ determined by ω, with the morphism $(i_p, I_\mathrm{A}) : \omega \longrightarrow \omega_{Pm}$.

Then, taking $F_X(\omega) = (F_P(\omega))^{\mathrm{cr}}$, where $\omega_X = \omega_P^{\mathrm{cr}}$, and then defining $i_c : \mathrm{K} \longrightarrow \mathrm{F}_X(\omega)$ as $i_c k = \overline{[1k1]}$ for all $k \in \mathrm{K}$ and $\alpha : \mathrm{F}_X(\omega) \longrightarrow \mathrm{N}$ as $\alpha\overline{[aka']} = {}^{a}(fk)^{a'}$ on generators completes the proof. □

Composing the free precrossed module functor F_P and the functor $(-)^{\mathrm{cr}}$ we get a functor F_X : $\mathrm{Sets}/(\mathrm{Alg}(R)/\mathrm{A}) \longrightarrow \mathrm{XAlg}(R)/\mathrm{A}$ defined as $F_X(\omega) = (\omega_X : \mathrm{F}_X(\omega) \longrightarrow \mathrm{A})$ on objects and as $F_X(f, I_\mathrm{A}) = (F_X f, I_\mathrm{A})$ on morphisms where $(F_X f)\overline{[aka']} = \overline{[a(fk)a']}$ on generators.

Proposition 18 *If* $\omega : \mathrm{K} \longrightarrow \mathrm{A}$ *and* $(g, I_\mathrm{A}) : \omega \longrightarrow \omega'$ *in* $\mathrm{Sets}/(\mathrm{Alg}(R)/\mathrm{A})$ *then* $\omega'_{X_g} : (F_X g)(\mathrm{F}_X(\omega)) \to \mathrm{A}$ *where* ω'_{X_g} *is the restriction of* ω'_X *on* $(F_X g)(\mathrm{F}_X(\omega))$*, with* $i_g : g(\mathrm{K}) \longrightarrow (F_X g)(\mathrm{F}_X(\omega))$ *defined as* $i_g(gk) = \overline{[1gk1]}$*, is the free* R*-algebroid crossed* A*-module determined by* $\omega'_g : g(\mathrm{K}) \longrightarrow \mathrm{A}$ *where* ω'_g *is the restriction of* ω' *on* $g(\mathrm{K})$*.*

Proof For any R-algebroid crossed A-module $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{A})$ and for any morphism $(f, I_\mathrm{A}) : \omega'_g \longrightarrow \mathcal{N}$ the map $\alpha_g : (F_X g)(\mathrm{F}_X(\omega)) \longrightarrow \mathrm{N}$ defined as $\alpha_g\overline{[a(gk)a']} = {}^{a}(fgk)^{a'}$ on generators clearly forms a unique crossed module morphism with I_A and makes the universal diagram commutative, completing the proof. □

Proposition 19 *As in the case of free precrossed modules, the functor* F_X *is the left adjoint of the forgetful functor* $U : \mathrm{XAlg}(R)/\mathrm{A} \longrightarrow \mathrm{Sets}/(\mathrm{Alg}(R)/\mathrm{A})$*, which for a crossed module* $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{A})$ *gives the* *function* η_m *and for a crossed* A*-module morphism* $(f, I_{\mathrm{A}}) : \mathcal{N} \longrightarrow \mathcal{N}'$ *gives the morphism* $U(f, I_{\mathrm{A}}) = (Uf, I_{\mathrm{A}}) : \eta_m \longrightarrow \eta'_m$ *such that* $(Uf)(n) = fn$ *for all* $n \in \mathrm{N}$.

Proof For all $\mathcal{N} \in Ob\,(\mathrm{XAlg}(R)/\mathrm{A})$ and $\omega \in Ob\,(\mathrm{Sets}/\,(\mathrm{Alg}(R)/\mathrm{A}))$ we have bijections $(\mathrm{XAlg}(R)/\mathrm{A})\,(F_X(\omega), \mathcal{N}) \cong (\mathrm{PXAlg}(R)/\mathrm{A})\,(F_P(\omega), \mathcal{N})$ from Proposition 16 and $(\mathrm{PXAlg}(R)/\mathrm{A})\,(F_P(\omega), \mathcal{N}) \cong (\mathrm{Sets}/\,(\mathrm{Alg}(R)/\mathrm{A}))\,(\omega, \eta_m)$ from Proposition 12, and their composition gives the needed isomorphism which is natural in $\mathcal{N}$ and ω. □

Proposition 20 *i) There exists a natural transformation*

$$\delta = \{(\delta_\omega, I_{\mathrm{A}}) : \omega \in \mathrm{Sets}/\,(\mathrm{Alg}(R)/\mathrm{A})\} : I_{\mathrm{Sets}/(\mathrm{Alg}(R)/\mathrm{A})} \Longrightarrow UF_X$$

where $(\delta_\omega, I_{\mathrm{A}}) : \omega \longrightarrow (UF_X)(\omega)$ *is a morphism for all* $\omega \in \mathrm{Sets}/\,(\mathrm{Alg}(R)/\mathrm{A})$ *and* $I_{\mathrm{Sets}/(\mathrm{Alg}(R)/\mathrm{A})}$ *is the identity functor on* $\mathrm{Sets}/\,(\mathrm{Alg}(R)/\mathrm{A})$.

ii) For each $\omega \in \mathrm{Sets}/\,(\mathrm{Alg}(R)/\mathrm{A})$, $\mathcal{N} \in \mathrm{XAlg}\,(R)/\mathrm{A}$ *and morphism* $(g, I_{\mathrm{A}}) : \omega \longrightarrow U(\mathcal{N}) = \eta_m$ *there exists a unique crossed* A*-module morphism* $(f, I_{\mathrm{A}}) : F_X(\omega) \longrightarrow \mathcal{N}$ *such that* $g = (Uf)\,\delta_\omega$.

Proof *i)* If $\omega : \mathrm{K} \longrightarrow \mathrm{A}$, defining $\delta_\omega k = \overline{[1k1]}$ for all $k \in \mathrm{K}$ completes the proof since the rest are clear.

ii) Define $f\overline{[aka']} = {}^{a}(gk)^{a'}$ on generators. Then obviously (f, I_{A}) is a crossed A-module morphism and $gk = f\overline{[1k1]} = (Uf)\,\overline{[1k1]} = (Uf)\,\delta_\omega k$ for all $k \in \mathrm{K}$. Moreover, $(Uf)\,\delta_\omega = (Uf')\,\delta_\omega$ implies $gk = f\overline{[1k1]} = f'\overline{[1k1]}$ and $f\overline{[aka']} = f'\overline{[aka']}$ for all $k \in \mathrm{K}$ and for all generators $\overline{[aka']} \in F_X(\omega)$ and this ensures the uniqueness of f for fixed g. □

Proposition 21 *i) There exists a natural transformation*

$$\theta = \{(\theta_{\mathcal{N}}, I_{\mathrm{A}}) : \mathcal{N} \in \mathrm{XAlg}(R)/\mathrm{A}\} : F_X U \Longrightarrow I_{\mathrm{XAlg}(R)/\mathrm{A}}$$

where $(\theta_{\mathcal{N}}, I_{\mathrm{A}}) : (F_X U)(\mathcal{N}) \longrightarrow \mathcal{N}$ *is a crossed* A*-module morphism for all* $\mathcal{N} \in \mathrm{XAlg}(R)/\mathrm{A}$ *and* $I_{\mathrm{XAlg}(R)/\mathrm{A}}$ *is the identity functor on* $\mathrm{XAlg}(R)/\mathrm{A}$.

ii) For all $\omega \in \mathrm{Sets}/\,(\mathrm{Alg}(R)/\mathrm{A})$, $\mathcal{N} \in \mathrm{XAlg}(R)/\mathrm{A}$ *and crossed* A*-module morphism* $(f, I_{\mathrm{A}}) : F_X(\omega) \longrightarrow \mathcal{N}$ *there exists a unique morphism* $(g, I_{\mathrm{A}}) : \omega \longrightarrow U(\mathcal{N}) = \eta_m$ *such that* $f = \theta_{\mathcal{N}}\,(F_X g)$.

Proof *i)* For each $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{A})$, defining $\theta_{\mathcal{N}}\left(\overline{[ana']}\right) = {}^{a}n^{a'}$ on generators completes the proof since the rest are clear.

ii) Define $gk = f\overline{[1k1]}$. Then

$$f\overline{[aka']} = {}^{a}\left(f\overline{[1k1]}\right)^{a'} = {}^{a}(gk)^{a'} = \theta_{\mathcal{N}}\overline{[a\,(gk)\,a']} = \theta_{\mathcal{N}}\,(F_X g)\,\overline{[aka']}$$

for all generators $\overline{[aka']} \in F_X(\omega)$. Moreover, g is unique since if $(g', I_{\mathrm{A}}) : \omega \longrightarrow U(\mathcal{N})$ is another morphism with $f = \theta_{\mathcal{N}}\,(F_X g')$ then

$$gk = f\overline{[1k1]} = (\theta_{\mathcal{N}}\,(F_X g'))\left(\overline{[1k1]}\right) = \theta_{\mathcal{N}}\left(\overline{[1g'k1]}\right) = {}^{1}(g'k)^{1} = g'k$$

for all $k \in \mathrm{K}$. □

References

[1] Alp M. Pullback crossed modules of algebroids. Iran J Sci Tech; 2008; 32: 1-5.

[2] Alp M. Pushout crossed modules of algebroids. Iran J Sci Tech; 2008; 32: 175-181.

[3] Amgott SM. Separable categories. J Pure Appl Algebra 1986; 40: 1-14.

[4] Arvasī Z. Crossed modules of algebras. Mathematical and Computational Applications 2004; 9: 173-182.

[5] Arvasi Z, Porter T. Simplicial and crossed resolutions of commutative algebras. J Algebra 1996; 181: 426-448.

[6] Baues HJ, Conduché D. The central series for Peiffer commutators in groups with operators. J Algebra 1990; 133: 1-34.

[7] Brown R, Higgins PJ, Sivera R. Nonabelian Algebraic Topology. Helsinki, Finland: European Mathematical Society, EMS Tracts in Mathematics Vol. 15, 2011.

[8] Brown R. Homotopies and automorphisms of crossed modules of groupoids. Appl Categor Struct 2003; 11: 185-206.

[9] Brown R, Huebschmann J. Identities among relations. Low-dimensional topology, Bangor, 1979. Lond Math S 1982; 48: 153-202.

[10] Ellis GJ. Higher dimensional crossed modules of algebras. J Pure Appl Algebra 1988; 52: 277-282.

[11] Gerstenhaber M. On the deformation of rings and algebras: II. Ann Math 1966; 84: 1-19.

[12] Kassell C, Loday JL. Extensions centrale d'algèbres de Lie. Ann Inst Fourier 1982; 32: 119-142 (in French).

[13] Lichtenbaum S, Schlessinger M. The cotangent complex of a morphism. T Am Math Soc 1967; 128: 41-70.

[14] Lue AST. Non-abelian cohomology of associative algebras. Q J Math 1968; 19: 159-180.

[15] Mitchell B. Rings with several objects. Adv Math 1972; 8: 1-161.

[16] Mitchell B. Some applications of module theory to functor categories. B Am Math Soc 1978; 84: 867-885.

[17] Mitchell B. Separable Algebroids. Providence, RI, USA: AMS, 1985.

[18] Mosa GH. Higher dimensional algebroids and crossed complexes. PhD, University of Wales, Cardiff, UK, 1986.

[19] Porter T. Homology of commutative algebras and an invariant of Simis and Vasconcelos. J Algebra 1986; 99: 458-465.

[20] Porter T. Some categorical results in the theory of crossed modules in commutative algebras. J Algebra 1987; 109: 415-429.

[21] Porter T. The Crossed Menagerie: An introduction to crossed gadgetry and cohomology in algebra and topology. 17 April 2018. https://ncatlab.org/nlab/files/menagerie12a.pdf.

[22] Shammu NM. Algebraic and categorical structure of categories of crossed modules of algebras. PhD, University of Wales, Cardiff, UK, 1992.

[23] Whitehead JHC. On adding relations to homotopy groups. Ann Math 1941; 42: 409-428.

[24] Whitehead JHC. Note on a previous paper entitled "On adding relations to homotopy groups.". Ann Math 1946; 47: 806-810.