Q-Korselt numbers

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Abstract: Let \( \alpha = \frac{\alpha_1}{\alpha_2} \in \mathbb{Q} \setminus \{0\} \); a positive integer \( N \) is said to be an \( \alpha \)-Korselt number (\( K_\alpha \)-number, for short) if \( N \neq \alpha \) and \( \alpha_2 p - \alpha_1 \) divides \( \alpha_2 N - \alpha_1 \) for every prime divisor \( p \) of \( N \). In this paper we prove that for each squarefree composite number \( N \) there exist finitely many rational numbers \( \alpha \) such that \( N \) is a \( K_\alpha \)-number and if \( \alpha \leq 1 \) then \( N \) has at least three prime factors. Moreover, we prove that for each \( \alpha \in \mathbb{Q} \setminus \{0\} \) there exist only finitely many squarefree composite numbers \( N \) with two prime factors such that \( N \) is a \( K_\alpha \)-number.

Key words: Prime number, Carmichael number, Korselt number, squarefree composite number, Korselt set, Korselt weight

1. Introduction
A Carmichael number is a composite number \( N \) that divides \( a^N - a \) for all integers \( a \) \([2, 4]\). In 1899, Korselt gave a complete characterization of Carmichael numbers.

Theorem 1.1 (Korselt criterion \([8]\)) A composite integer \( N > 1 \) is a Carmichael number if and only if \( p - 1 \) divides \( N - 1 \) for all prime factors \( p \) of \( N \).

This criterion helped in the discovery of the existence of infinitely many Carmichael numbers in 1994 by Alford et al. (see \([1]\) for details). In the proof of the infinitude of Carmichael numbers the authors asked if this proof can be generalized to produce other kinds of pseudoprimes by writing the following:

“One can modify our proof to show that for any fixed nonzero integer \( a \), there are many squarefree, composite integers \( n \) such that \( p - a \) divides \( n - 1 \) for all primes \( p \) dividing \( n \). However, we have been unable to prove this for \( p - a \) dividing \( n - b \), for \( b \) other than 0 or 1.”

The query of Alford et al. inspired Bouallegue et al. to state in a recent paper a new kind of pseudoprimes called Korselt numbers (see \([3]\) for details). For \( \alpha \in \mathbb{Z} \setminus \{0\} \), a number \( N \) is called an \( \alpha \)-Korselt number if \( p - \alpha \mid N - \alpha \) for each prime divisor \( p \) of \( N \). By this definition, Carmichael numbers are exactly the squarefree composite 1-Korselt numbers. In this paper, we extend the definition of \( \alpha \)-Korselt numbers given in \([3]\) by allowing \( \alpha \) to be a rational number. We state the following definition.

Definition 1.2 Let \( N \in \mathbb{N} \setminus \{0, 1\} \) and \( \alpha = \frac{\alpha_1}{\alpha_2} \in \mathbb{Q} \setminus \{0\} \). \( N \) is said to be an \( \alpha \)-Korselt number (\( K_\alpha \)-number, for short) if \( N \neq \alpha \) and \( \alpha_2 p - \alpha_1 \) divides \( \alpha_2 N - \alpha_1 \) for every prime divisor \( p \) of \( N \).

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The set of all \( K_\alpha \)-numbers, where \( \alpha \in \mathbb{Q} \), is called the set of \( \mathbb{Q} \)-Korselt numbers.

For a fixed \( N \in \mathbb{N} \setminus \{0, 1\} \), we need to determine the set of all \( \alpha \in \mathbb{Q} \setminus \{0\} \) such that \( N \) is a \( K_\alpha \)-number. This leads to the following definition.

**Definition 1.3** Let \( N \) be a positive integer and \( \mathbb{H} \) be a nonempty subset of \( \mathbb{Q} \).

1. By the \( \mathbb{H} \)-Korselt set of \( N \), we mean the set \( \mathbb{H} - KS(N) \) of all \( \alpha \in \mathbb{H} \setminus \{0, N\} \) such that \( N \) is a \( K_\alpha \)-number.

2. The cardinality of \( \mathbb{H} - KS(N) \) will be called the \( \mathbb{H} \)-Korselt weight of \( N \); we denote it by \( \mathbb{H} - KW(N) \).

By this definition, the notion of \( \mathbb{Q} \)-Korselt numbers generalizes that given by Bouallegue et al. and thus Carmichael numbers. Among the most recent works in this area are the papers [3, 5–7], where the notion of Korselt numbers over \( \mathbb{Z} \) was studied and several related results were obtained. In this paper, our aim is to introduce the notion of \( \mathbb{Q} \)-Korselt numbers and to discuss generalizations of properties holding when \( \mathbb{Z} \).

Therefore, we proceed as follows:

- In Section 2, after giving some general results about \( \mathbb{Q} \)-Korselt numbers, we prove that for each squarefree composite number \( N \), there exist only finitely many rational numbers \( \alpha \) such that \( N \) is a \( K_\alpha \)-number.

- In section 3, we prove that for every rational number \( \alpha \leq 1 \), if a squarefree composite number \( N \) is a \( K_\alpha \)-number then \( N \) must have at least three prime factors. Furthermore, we show that for each rational number \( \alpha > 1 \), there exist only finitely many \( K_\alpha \)-numbers with two prime factors.

Throughout this paper and for \( \alpha = \frac{\alpha_1}{\alpha_2} \in \mathbb{Q} \), we will suppose without loss of generality that \( \alpha_2 > 0 \), \( \alpha_1 \in \mathbb{Z} \), and gcd\( (\alpha_1, \alpha_2) = 1 \). Moreover, in this work we are concerned only with squarefree composite numbers \( N \).

2. **\( \mathbb{Q} \)-Korselt set properties**

**Proposition 2.1** Let \( \alpha \in \mathbb{Q} \setminus \{0\} \) and \( N = p_1p_2 \ldots p_m \) be a \( K_\alpha \)-number such that \( p_1 < p_2 < \ldots < p_m \) and \( m \geq 2 \). Then the following inequalities hold:

\[
\frac{(m+2)p_1 - N}{m+1} \leq \alpha \leq \frac{N + mp_m}{m+1}.
\]

**Proof** \( \alpha \in \mathbb{Q} - KS(N) \) implies that \( N - \alpha = k_i(p_i - \alpha) \) with \( k_i \in \mathbb{Z} \) for each \( i = 1 \ldots m \). We consider two cases:

**Case 1:** Assume that \( \alpha < 0 \). First, let us show that \( k_m \geq 3 \).

Since \( N - \alpha > p_m - \alpha > 0 \), then \( k_m = \frac{N - \alpha}{p_m - \alpha} > 1 \).

Next, we show that \( k_m \neq 2 \). Suppose by contradiction that \( k_m = 2 \).

Then \( \alpha = 2p_m - N \in \mathbb{Z} \), but as \( \alpha \neq p_m \) and \( \alpha \neq 0 \), we get \( N \neq p_m \) and \( N \neq 2p_m \). Thus, there exists an integer \( N_1 \geq 3 \) such that \( N = N_1p_m \). Let \( p_s \) be a prime factor of \( N_1 \). Then

\[
p_s - \alpha = p_s + (N_1 - 2)p_m \mid N - \alpha = 2p_m(N_1 - 1).
\]

However, as gcd\( (p_m, p_s - \alpha) = 1 \), it follows that

\[
p_s - \alpha = p_s + (N_1 - 2)p_m \mid 2(N_1 - 1).
\]
and hence
\[ p_s + (N_1 - 2)p_m \leq 2(N_1 - 1). \]
Since \( 4 \leq p_s + 2 \leq p_m \), we get
\[ 2 + 4(N_1 - 2) \leq p_s + (N_1 - 2)p_m \leq 2(N_1 - 1). \]
Therefore, \( N_1 \leq 2 \), which contradicts \( N_1 \geq 3 \), so \( k_m \geq 3 \).

Now, as \( (p_i - \alpha)_{1 \leq i \leq m} \) is increasing and positive, then \( \left( k_i = \frac{N - \alpha}{p_i - \alpha} \right)_{1 \leq i \leq m} \) is decreasing. Hence, as \( k_m \geq 3 \), \( \frac{N - \alpha}{p_1 - \alpha} = k_1 \geq m + 2 \). Thus,
\[
\frac{(m + 2)p_1 - N}{m + 1} \leq \alpha.
\]

**Case 2:** Suppose that \( \alpha > 0 \). We claim that \( \alpha < N \). If not, then (as \( \alpha \neq N \)) we get \( p_m < N < \alpha \). This implies that \( 0 < \alpha - N < \alpha - p_m \), and hence \( 0 < \frac{\alpha - N}{\alpha - p_m} = k_m < 1 \), contradicting the fact that \( k_m \in \mathbb{Z} \).

Now let us prove that \( \alpha \leq \frac{N + mp_m}{m + 1} \).

- If \( \alpha \leq p_m \), it is immediate.

- Now suppose that \( p_m < \alpha < N \). Since \( (\alpha - p_i)_{1 \leq i \leq m} \) is decreasing and positive, then \( \left( |k_i| = \frac{N - \alpha}{\alpha - p_i} \right)_{1 \leq i \leq m} \) is increasing. Hence, \( |k_m| \geq m \) and consequently \( N - \alpha = |k_m| (\alpha - p_m) \geq m(\alpha - p_m) \). Thus,
\[
\alpha \leq \frac{N + mp_m}{m + 1}.
\]

Finally, combining the two cases, we get
\[
\frac{(m + 2)p_1 - N}{m + 1} \leq \alpha \leq \frac{N + mp_m}{m + 1}.
\]

By the following result, we provide a characterization of the \( \mathbb{Q} \)-Korselt set of a squarefree composite number \( N \).

**Proposition 2.2** Let \( N \) be a squarefree composite number with prime divisors \( p_i \), \( 1 \leq i \leq m \). If we let
\[
A_{ij} = \left\{ \frac{dp_j - \delta p_i}{d - \delta} ; d \neq \delta, \delta \mid (N - p_i), d \mid (N - p_j), \text{ and } (p_i - p_j) \mid (d - \delta) \right\},
\]
for \( 1 \leq i < j \leq m \), then
\[
\mathbb{Q} \text{-K} \mathcal{S}(N) = \bigcap_{1 \leq i < j \leq m} A_{ij}.
\]
**Proof** First note that for each $1 \leq i \leq m$, $N$ is a $K_{\alpha}$-number if and only if $\alpha_2p_i - \alpha_1 \mid \alpha_2N - \alpha_1$ or equivalently $\alpha_2p_i - \alpha_1 \mid N - p_i$.

Now let $\alpha \in \mathbb{Q}$-$\mathcal{KS}(N)$. Then for each $(i, j)$ with $1 \leq i < j \leq m$, we have

\[
\begin{aligned}
\alpha_2p_i - \alpha_1 &\mid N - p_i \\
\alpha_2p_j - \alpha_1 &\mid N - p_j.
\end{aligned}
\]

This implies that there are two distinct divisors $d$ and $\delta$ of $N - p_i$ and $N - p_j$, respectively, such that

\[
\begin{aligned}
\alpha_2p_i - \alpha_1 &= d \\
\alpha_2p_j - \alpha_1 &= \delta.
\end{aligned}
\]

Solving the system we get

\[
\begin{aligned}
\alpha_1 &= \frac{dp_j - \delta p_i}{p_i - p_j}, \\
\alpha_2 &= \frac{d - \delta}{p_i - p_j},
\end{aligned}
\]

and so $\alpha = \frac{dp_j - \delta p_i}{d - \delta}$. Since $\alpha_1$ and $\alpha_2$ are integers we conclude that $\alpha \in A_{ij}$ and hence

\[
\mathbb{Q}$-$\mathcal{KS}(N) \subseteq \bigcap_{1 \leq i < j \leq m} A_{ij}.
\]

Next let $\alpha \in \bigcap_{1 \leq i < j \leq m} A_{ij}$. Then $\alpha \in A_{ij}$, for each pair $(i, j)$ such that $1 \leq i < j \leq m$. This implies that $\alpha = \frac{dp_j - \delta p_i}{d - \delta}$, for some divisors $d$ and $\delta$ of $N - p_i$ and $N - p_j$, respectively, with $(p_i - p_j) \mid (d - \delta)$.

Setting $\alpha_1 = \frac{dp_j - \delta p_i}{p_i - p_j}$ and $\alpha_2 = \frac{d - \delta}{p_i - p_j}$, then $\alpha_1, \alpha_2 \in \mathbb{Z}$ and

\[
\alpha_2p_i - \alpha_1 = d \mid N - p_i \quad \text{for} \quad i = 1 \ldots m.
\]

Therefore, $\alpha = \frac{\alpha_1}{\alpha_2} \in \mathbb{Q}$-$\mathcal{KS}(N)$.

By the previous proposition, we immediately get the following result.

**Theorem 2.3** For any given squarefree composite number $N$, there are only finitely many rational numbers $\alpha$ for which $N$ is a $K_{\alpha}$-number.

By the characterization of the $\mathbb{Q}$-Korselt set of a squarefree composite number $N$, given in Proposition 2.2, and with a simple Maple program, we provide in Table 1 and Table 2 data representing some squarefree composite numbers and their $\mathbb{Q}$-Korselt sets as follows:

- Table 1 gives for each integer $2 \leq d \leq 8$ the $\mathbb{Q}$-Korselt set of the smallest $\mathbb{Q}$-Korselt number $N_d$ with $d$ prime factors.

- Table 2 gives for each integer $0 \leq k \leq 10$ the smallest squarefree composite number $N_k$ such that $\mathbb{Q}$-$\mathcal{KW}(N_k) = k$.

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Table 1. \(\mathbb{Q} \cdot \mathbb{K}S(N_d)\) where \(N_d\) is the smallest \(\mathbb{Q}\)-Korselt number with \(d\) prime factors.

<table>
<thead>
<tr>
<th>(d)</th>
<th>(N_d)</th>
<th>(\mathbb{Q} \cdot \mathbb{K}S(N_d))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6 = 2 \cdot 3</td>
<td>{3, 10, 14, 8, 5, 18, 12, 9}</td>
</tr>
<tr>
<td>3</td>
<td>30 = 2 \cdot 3 \cdot 5</td>
<td>{4, 15, 40, 5, 10, 15, 24}</td>
</tr>
<tr>
<td>4</td>
<td>210 = 2 \cdot 3 \cdot 5 \cdot 7</td>
<td>{6, 21}</td>
</tr>
<tr>
<td>5</td>
<td>2730 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13</td>
<td>{15}</td>
</tr>
<tr>
<td>6</td>
<td>255255 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17</td>
<td>{15}</td>
</tr>
<tr>
<td>7</td>
<td>8580495 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 23</td>
<td>{15}</td>
</tr>
<tr>
<td>8</td>
<td>294076965 = 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29</td>
<td>{21}</td>
</tr>
</tbody>
</table>

Table 2. The smallest squarefree composite number \(N_k\) such that \(\mathbb{Q} \cdot \mathbb{K}W(N_k) = k\).

<table>
<thead>
<tr>
<th>(k)</th>
<th>(N_k)</th>
<th>(\mathbb{Q} \cdot \mathbb{K}S(N_k))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>138 = 2 \cdot 3 \cdot 23</td>
<td>{12}</td>
</tr>
<tr>
<td>1</td>
<td>22 = 2 \cdot 11</td>
<td>{12}</td>
</tr>
<tr>
<td>2</td>
<td>102 = 2 \cdot 3 \cdot 17</td>
<td>{12, 17}</td>
</tr>
<tr>
<td>3</td>
<td>14 = 2 \cdot 7</td>
<td>{8, 6}</td>
</tr>
<tr>
<td>4</td>
<td>42 = 2 \cdot 3 \cdot 7</td>
<td>{6, 21, 28, 9}</td>
</tr>
<tr>
<td>5</td>
<td>10 = 2 \cdot 5</td>
<td>{4, 6, 10, 5, 14}</td>
</tr>
<tr>
<td>6</td>
<td>273 = 3 \cdot 7 \cdot 13</td>
<td>{-7, 8, 9, 78, 19, 21}</td>
</tr>
<tr>
<td>7</td>
<td>70 = 2 \cdot 5 \cdot 7</td>
<td>{4, 6, 5, 7, 56, 25, 48}</td>
</tr>
<tr>
<td>8</td>
<td>30 = 2 \cdot 3 \cdot 5</td>
<td>{4, 6, 15, 40, 5, 10, 15, 24}</td>
</tr>
<tr>
<td>9</td>
<td>6 = 2 \cdot 3</td>
<td>{4, 3, 10, 14, 8, 5, 18, 12, 9}</td>
</tr>
<tr>
<td>10</td>
<td>110 = 2 \cdot 5 \cdot 11</td>
<td>{8, 20, 44, 55, 88, 22, 31, 13, 35, 46}</td>
</tr>
</tbody>
</table>

3. \(\mathbb{Q}\)-Korselt numbers with two prime factors

In this section, we shall discuss the case where \(N\) is a squarefree composite number with two prime factors. Let \(p\) and \(q\) be two prime numbers such that \(p < q\), \(N = pq\) and \(\alpha = \frac{a_1}{a_2}\) be a rational number.

**Proposition 3.1** If \(N\) is a \(K_\alpha\)-number such that \(\gcd(\alpha, N) = 1\), then

\[q - p + 1 \leq \alpha \leq q + p - 1.\]
Proof Since $N$ is a $K_N$-number, then

$$\begin{align*}
(S_1) & \quad \left\{ \begin{array}{l}
\alpha_2 p - \alpha_1 | p(q - 1) \\
\alpha_2 q - \alpha_1 | q(p - 1).
\end{array} \right.
\end{align*}$$

As, in addition, $\gcd(\alpha_1, p) = \gcd(\alpha_1, q) = 1$, it follows that

$$\begin{align*}
(S_2) & \quad \left\{ \begin{array}{l}
\alpha_2 p - \alpha_1 | q - 1 \\
\alpha_2 q - \alpha_1 | p - 1.
\end{array} \right.
\end{align*}$$

Hence, by (3.2), we get

$$-p + 1 \leq \alpha_1 - \alpha_2 q \leq p - 1.$$  

Knowing that $\alpha_2 \geq 1$, we deduce that

$$q - p + 1 \leq q - \frac{p - 1}{\alpha_2} \leq \alpha = \frac{\alpha_1}{\alpha_2} \leq q + \frac{p - 1}{\alpha_2} \leq q + p - 1.$$  

\[\square\]

In order to establish the set of $\alpha = \frac{\alpha_1}{\alpha_2} \in \mathbb{Q}$ with $\gcd(\alpha_1, N) \neq 1$ and for which $N$ is a $K_N$-number, we need the next two results.

**Proposition 3.2** Let $N$ be a $K_N$-number such that $\alpha < q - p + 1$. Then the following assertions hold:

1) $q$ divides $\alpha_1$.

2) If $p$ divides $\alpha_1$ (i.e. $N$ divides $\alpha_1$ and so $\gcd(\alpha_1, N) = N$), then $\alpha_1 = N$ and $\alpha_2 = 2p - 1$.

Proof

1) Since $\alpha = \frac{\alpha_1}{\alpha_2} < q - p + 1$, we have $\alpha_2(p - 1) < \alpha_2 q - \alpha_1$.

If $\gcd(q, \alpha_1) = 1$, then by (3.2) it follows that

$$\alpha_2(p - 1) < \alpha_2 q - \alpha_1 \leq p - 1.$$  

Hence, $\alpha_2 < 1$, which contradicts $\alpha_2 \in \mathbb{N} \setminus \{0\}$. Thus, $q | \alpha_1$.

2) Let $\alpha_1 = \alpha'' p q$ with $\alpha'' \in \mathbb{N} \setminus \{0\}$. Then $(S_1)$ gives

$$\begin{align*}
(S_3) & \quad \left\{ \begin{array}{l}
\alpha_2 - \alpha'' q | q - 1 \\
\alpha_2 - \alpha'' p | p - 1.
\end{array} \right.
\end{align*}$$

Let us show that $\alpha_1 = N$ and $\alpha_2 = 2p - 1$.

As $\alpha = \frac{\alpha_1}{\alpha_2} < q - p + 1$, then

$$\alpha_2(p - 1) < \alpha_2 q - \alpha_1 = (\alpha_2 - \alpha'' p)q.$$
It follows by (3.4), that
\[ \alpha_2(p - 1) < q(\alpha_2 - \alpha''_1 p) \leq q(p - 1). \]
Hence, \( \alpha_2 < q \). Furthermore, since by (3.3), \( \alpha''_1 q - \alpha_2 < q - 1 \), it follows that \( \alpha''_1 q < \alpha_2 + q - 1 < 2q - 1 \), and this forces \( \alpha''_1 = 1 \). Therefore, \( \alpha_1 = pq = N \).

Now let us prove that \( \alpha_2 = 2p - 1 \). First, as \( pq \leq q < q + 1 \), then \( p < \alpha_2 \left( \frac{q - p + 1}{q} \right) < \alpha_2 \).

Consequently, as \( \alpha''_1 = 1 \) and \( \alpha_2 - p > 0 \), it follows by (3.4) that \( \alpha_2 - p = \frac{p - 1}{k} \) with \( k \in \mathbb{N} \setminus \{0\} \). We claim that \( k = 1 \). Indeed, suppose by contradiction that \( k \neq 1 \); then \( \alpha_2 - p \leq \frac{p - 1}{2} \) and hence
\[ \alpha_2 \leq \frac{3p - 1}{2}. \tag{3.5} \]

Furthermore, since by hypothesis \( \frac{pq}{\alpha_2} = \alpha < q - p + 1 \), it follows by (3.5) that \( pq < \alpha_2(q - p + 1) \leq \frac{3p - 1}{2}(q - p + 1) \). This is equivalent to \( q - 3p + 1 < p(q - 3p + 1) \) and hence
\[ 3p - 1 < q. \tag{3.6} \]

However, as in addition \( \alpha \neq N \), i.e. \( \alpha_2 \neq 1 \) and \( \alpha''_1 = 1 \), we get by (3.3) \( q - \alpha_2 \leq \frac{q - 1}{2} \). This yields by (3.5) \( q < 2\alpha_2 - 1 \leq 3p - 2 \), a contradiction with (3.6). Thus, \( k = 1 \) and so \( \alpha_2 = 2p - 1 \).

\[ \Box \]

Lemma 3.3 If \( N \) is a \( K_\alpha \)-number such that \( \gcd(\alpha_1, N) \neq 1 \) and \( q + p - 1 < \alpha \), then \( \alpha_1 = pq = N \).

Proof As \( q + p - 1 < \alpha = \frac{\alpha_1}{\alpha_2} \), then we have
\[ 0 < \alpha_2(q - 1) < \alpha_1 - \alpha_2 p \tag{3.7} \]
and
\[ 0 < \alpha_2(p - 1) < \alpha_1 - \alpha_2 q. \tag{3.8} \]

First we claim that \( \gcd(p, \alpha_1) \neq 1 \). Indeed, if not, then by combining (3.1) and (3.7), we get
\[ 0 < \alpha_2(q - 1) < \alpha_1 - \alpha_2 p \leq q - 1. \]

This implies that \( \alpha_2 < 1 \), which contradicts \( \alpha_2 \in \mathbb{N} \setminus \{0\} \). Thus, \( p \mid \alpha_1 \).

Similarly, by (3.2) and (3.8) we get \( q \mid \alpha_1 \). Hence, \( \alpha_1 = \alpha''_1 pq \) with \( \alpha''_1 \in \mathbb{N} \). Let us show that \( \alpha''_1 = 1 \). By (3.3) and (3.4), we get respectively
\[ \alpha''_1 q - \alpha_2 \leq q - 1 \tag{3.9} \]
and
\[ \alpha''_1 p - \alpha_2 \leq p - 1. \tag{3.10} \]

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Multiplying (3.9) by \( p \) and combining it with (3.7), we obtain

\[
\alpha_2(q - 1) < \alpha_1 - \alpha_2 p = p(\alpha'' q - \alpha_2) \leq p(q - 1),
\]
and hence

\[
\alpha_2 < p. \tag{3.11}
\]

Now, combining (3.10) and (3.11), we get

\[
(\alpha'' - 1)p < \alpha_1' p - \alpha_2 \leq p - 1.
\]

This implies that \( \alpha_1'' = 1 \), so \( \alpha_1 = pq = N \).

\[ \square \]

**Proposition 3.4** Suppose that \( N \) is a \( K_\alpha \)-number with gcd\( (\alpha_1, N) \neq 1 \). Then the following assertions hold:

1) If \( \alpha \in \mathbb{Z} \) (i.e. \( \alpha_2 = 1; \alpha = \alpha_1 \)), then \( q \nmid \alpha, \ p \mid \alpha \) and

\[
\alpha \in \left\{ \left\lfloor \frac{q}{p} \right\rfloor, \left\lceil \frac{q}{p} \right\rceil \right\}.
\]

2) If \( \alpha \in \mathbb{Q} \setminus \mathbb{Z} \), then \( \frac{q}{p} \leq \alpha \leq q + p - 1 \).

**Proof**

1) See [7, Corollary 3.6].

2) Let \( \alpha \in \mathbb{Q} \setminus \mathbb{Z} \) be such that gcd\( (\alpha_1, N) \neq 1 \). Let us show that \( \alpha \leq q + p - 1 \).

Assume that \( q + p - 1 < \alpha \). Then, by Lemma 3.3, (S1), and (3.11), we have \( 0 < q - \alpha_2 = \frac{q - 1}{k} \).

with \( k \in \mathbb{N} \). Since \( \alpha \neq N \) (i.e. \( \alpha_2 \neq 1 \)) and hence \( k \geq 2 \), it follows that \( q - \alpha_2 \leq \frac{q - 1}{2} \); therefore, \( \alpha_2 \geq \frac{q + 1}{2} > \frac{q}{2} \). As by Lemma 3.3, \( \alpha_1 = pq = N \), it yields that \( \alpha = \frac{pq}{\alpha_2} < \frac{2pq}{q} = 2p < p + q - 1 \), which contradicts the assumption \( \alpha > q + p - 1 \).

It remains to prove that \( \frac{q}{p} \leq \alpha \). First, since \( \frac{q}{p} < q - p + 1 \), we may suppose that \( \alpha < q - p + 1 \).

By Proposition 3.2, \( \alpha_1 = \alpha_1' q \) with \( \alpha_1' \in \mathbb{Z} \). Let us prove that \( \alpha_1' > 0 \). The result is immediate by Proposition 3.2 when \( p \mid \alpha_1 \). Now, if gcd\( (p, \alpha_1) \neq 1 \) and by (3.1) we have

\[
\alpha_2 p - \alpha_1' q = \alpha_2 p - \alpha_1 \leq q - 1,
\]

this implies that \( p < \alpha_2 p + 1 \leq q(1 + \alpha_1') \), which forces \( \alpha_1' > 0 \).

On the other hand, we have by (S1)

\[
(\alpha_2 - \alpha_1')q = \alpha_2 q - \alpha_1 \leq q(p - 1).
\]
Hence, \( \alpha_2 \leq \alpha'_1 + p - 1 \), so
\[
\alpha = \frac{\alpha_1}{\alpha_2} = \frac{\alpha'_1 q}{\alpha_2} \geq \frac{\alpha'_1 q}{\alpha_1 + p - 1}.
\]
Since, in addition, \( \frac{\alpha'_1}{\alpha_1 + p - 1} \) is minimum when \( \alpha'_1 = 1 \), it follows that \( \alpha \geq \frac{q}{p} \).

\( \Box \)

By Propositions 3.4 and 3.1, the next two results follow immediately.

**Corollary 3.5** Let \( \alpha \in \mathbb{Q} \setminus \{0\} \).

If \( N \) is a \( K_\alpha \)-number, then \( \frac{q}{p} \leq \alpha \leq q + p - 1 \).

**Theorem 3.6** Let \( \alpha \in \mathbb{Q} \setminus \{0\} \). If \( \alpha \leq 1 \), then each \( K_\alpha \)-number has at least three prime factors.

The next result shows that an \( \alpha > 1 \) can belong to only finitely many \( \mathbb{Q} \)-KS\((pq)\).

**Theorem 3.7** Let \( \alpha \in \mathbb{Q} \setminus \{0\} \) with \( \alpha > 1 \), and suppose that \( N \) is a \( K_\alpha \)-number. Then the following assertions hold:

(a) If \( \alpha \in \mathbb{Z} \), then \( p < q \leq 4\alpha - 3 \).

(b) If \( \alpha = \frac{\alpha_1}{\alpha_2} \in \mathbb{Q} \setminus \mathbb{Z} \), then \( p < q \leq \alpha_1 \).

**Proof**

(a) See [3, Theorem 1.10].

(b) First, if \( q \) divides \( \alpha_1 \), then the result is immediate.

Now assume that \( \gcd(q, \alpha_1) = 1 \). As \( N = pq \) is a \( K_\alpha \)-number, it follows by \((S_2)\) that \( \alpha_2 q - \alpha_1 \) divides \( p - 1 \). This implies that \( \alpha_2 q - \alpha_1 \leq p - 1 < q - 1 \). Thus, \( q < \frac{\alpha_1 - 1}{\alpha_2 - 1} < \alpha_1 \).

\( \Box \)

**Remark 3.8** In case (b) of Theorem 3.7, the upper bound can be reached when \( q = 3, p = 2 \), and \( \alpha = \frac{3}{2} \).

We obtain immediately from Theorem 3.7 the following result.

**Theorem 3.9** Let \( \alpha \in \mathbb{Q} \setminus \{0\} \). Then there are only finitely many \( K_\alpha \)-numbers with exactly two prime factors.

Now we ask: do there exist (and how many) rationals \( 1 < \alpha < C \), where \( C \) is a fixed rational number, for which there are no \( K_\alpha \)-numbers with two prime factors? Computationally, this problem can be solved by running a computer program with exhaustive research (see [3, Example 1.11]). However, for the case \( \alpha \in \mathbb{Q} \setminus \mathbb{Z} \), it seems to be more difficult computationally and theoretically to find such a solution. This does not prevent us from providing, by the next proposition, all rationals \( 1 < \alpha < 2 \) for which there are no \( K_\alpha \)-numbers with two prime factors.
Proposition 3.10 Let $\alpha \in \mathbb{Q}$ be such that $1 < \alpha < 2$. $N = pq$ is a $K_\alpha$-number if and only if $\alpha = \frac{q}{p}$ with $(p, q) \in \{(2, 3), (3, 5)\}$.

Proof Suppose that $\alpha \in \mathbb{Q}$-$KS(N)$. Since $\alpha < 2 \leq q - p + 1$, then by Proposition 3.2, $q$ divides $\alpha_1$. Hence, $\alpha_1 = \alpha_1'q$ with $\alpha_1' \in \mathbb{N}$.

First we claim that $\gcd(p, \alpha_1) = 1$. Suppose by contradiction that $p$ divides $\alpha_1'$ (i.e. $N | \alpha_1$). Then, by Proposition 3.2, $\alpha = \frac{pq}{2p - 1}$, but, as by hypothesis $\frac{pq}{2p - 1} = \alpha < 2$, we obtain $p(q - 4) < -2$. Hence, $q = 3$ and $p = 2$, and so $\alpha_1 = pq = 6$ and $\alpha_2 = 2p - 1 = 3$, which contradicts the fact that $\gcd(\alpha_1, \alpha_2) = 1$.

Now, as $\gcd(p, \alpha_1) = 1$, then (3.2) gives

\[
\begin{cases} \alpha_2 p - \alpha_1' q | q - 1 \\ \alpha_2 - \alpha_1' | p - 1 \end{cases}
\]

(3.12)\hspace{2cm} (3.13)

Since $\alpha = \frac{\alpha_1}{\alpha_2} < 2$, i.e. $\frac{\alpha_1'q}{2} = \frac{\alpha_1}{2} < \alpha_2$, we get by (3.12)

\[
\frac{\alpha_1'}{2} q p - \alpha_1' q \leq \alpha_2 p - \alpha_1' q \leq q - 1.
\]

Hence, $\alpha_1' q (\frac{p}{2} - 1) < q$, so $p = 2$ or ($\alpha_1' = 1$ and $p = 3$).

- If $p = 2$, then by (3.13), we get $\frac{\alpha_1' q}{2} - \alpha_1 < \alpha_2 - \alpha_1' \leq p - 1 = 1$. Hence, $\alpha_1' (q - 2) < 2$, and consequently $\alpha_1' = 1$, $q = 3$, and $\alpha = \frac{3}{2}$.

- Now assume that $p = 3$ and $\alpha_1' = 1$. As $\alpha_1 = q$ and $\alpha_2 > \frac{q}{2}$, then by (3.13), we get $\frac{q}{2} - 1 < \alpha_2 - \alpha_1' = \alpha_2 - 1 \leq p - 1 = 2$. Therefore, $q < 6$. However, as in addition $q > p = 3$, necessarily $q = 5$, and so $\alpha_2 = 3$ and $\alpha = \frac{5}{3}$

Conversely, we verify easily that $2 \times 3 = 6$ is a $K_{\frac{3}{2}}$-number and $3 \times 5 = 15$ is a $K_{\frac{5}{3}}$-number.

\[\square\]

By Proposition 3.10, we may say that for each $1 < \alpha < 2$ with $\alpha \neq \frac{3}{2}$ and $\alpha \neq \frac{5}{3}$, there is no squarefree composite number $N$ with two prime factors such that $N$ is a $K_\alpha$-number. The question about the infinitude of the $K_\alpha$-numbers for a given $\alpha \in \mathbb{Q}$ remains posed. This can not be easily solved with an idea inspired by the proof of the case $\alpha = 1$ given by Alford et al. in [1]. However, following the heuristic ideas of Erdos, we believe the following:

Conjecture 3.11 For any given $\alpha \in \mathbb{Q} \setminus \{0\}$ there exist infinitely many $K_\alpha$-numbers.

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