Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 1448 –1457

doi:10.3906/mat-1512-46

Arf numerical semigroups

Sedat İLHAN[1,*], **Halil İbrahim KARAKAŞ**[2]
[1]Department of Mathematics, Faculty of Science, Dicle University, Diyarbakır, Turkey
[2]Faculty of Commercial Sciences, Başkent University, Ankara, Turkey

Received: 10.12.2015 • **Accepted/Published Online:** 20.01.2017 • **Final Version:** 23.11.2017

Abstract: The aim of this work is to exhibit the relationship between the Arf closure of a numerical semigroup S and its Lipman semigroup $L(S)$. This relationship is then used to give direct proofs of some characterizations of Arf numerical semigroups through their Lipman sequences of semigroups. We also give an algorithmic construction of the Arf closure of a numerical semigroup via its Lipman sequence of semigroups.

Key words: Numerical semigroups, Arf numerical semigroups, Arf closure

1. Introduction

Let $\mathbb{N}_0$ denote the set of nonnegative integers. A subset $S \subseteq \mathbb{N}_0$ satisfying

$$(i)\ 0 \in S \qquad (ii)\ x, y \in S \Rightarrow x + y \in S \qquad (iii)\ |\mathbb{N}_0 \setminus S| < \infty$$

is called a *numerical semigroup*. It is well known (see, for instance, [2, 3, 5]) that the condition (iii) above is equivalent to saying that the greatest common divisor $gcd(S)$ of elements of S is 1.

If A is a subset of $\mathbb{N}_0$, we will denote by $\langle A \rangle$ the submonoid of $\mathbb{N}_0$ generated by A. The monoid $\langle A \rangle$ is a numerical semigroup if and only if $gcd(A) = 1$.

Every numerical semigroup S admits a unique *minimal system of generators* $\{a_1, a_2, \ldots, a_e\}$ with $a_1 < a_2 < \cdots < a_e$; that is, $S = \{\sum_{k=1}^{e} c_i a_i : c_1, \ldots, c_e \in \mathbb{N}_0\}$ and no proper subset of $\{a_1, a_2, \ldots, a_e\}$ generates S. Given such a system of generators, the integers a_1 and e are called *multiplicity* and *embedding dimension* of S, and they are denoted by $m(S)$ and $e(S)$, respectively. The multiplicity $m(S)$ is the smallest positive element of S.

For a numerical semigroup S, *the maximal ideal* of S is $M = S \setminus \{0\}$. The largest integer that is not in S is called the *Frobenius number* of S and it is denoted by $\mathsf{f}(S)$. Clearly, $\mathsf{f}(\mathbb{N}_0) = -1$ and $\mathsf{f}(S) \geq 1$ if and only if $S \neq \mathbb{N}_0$.

Let S be a numerical semigroup having Frobenius number $\mathsf{f} = \mathsf{f}(S)$. If $S \neq \mathbb{N}_0$, it is customary to denote the elements of S that are less than f by $s_0 = 0, s_1, \ldots, s_{n-1}$ with $s_{i-1} < s_i$ for $1 \leq i \leq n = n(S)$ and to write

$$S = \{s_0 = 0, s_1, \ldots, s_{n-1}, s_n = \mathsf{f} + 1, \rightarrow\},$$

*Correspondence: sedati@dicle.edu.tr
2010 *AMS Mathematics Subject Classification:* Primary: 20M14; Secondary: 20M99

where "→" means that all subsequent natural numbers belong to S. Note here that $n = n(S) = |S \cap \{0,1,\ldots,\mathsf{f}\}|$. The elements $s_0 = 0, s_1, \ldots, s_{n-1}$ are called *non-gaps* of S less than f. Note that the first nonzero non-gap $s_1 = m(S)$ is the multiplicity of S. Those positive integers that do not belong to S are called *gaps* of S.

Inspired by the paper [1] of Arf, Lipman introduced and studied Arf rings in his paper [4] where characterizations of those rings via their value semigroups yield Arf numerical semigroups. Besides their importance in algebraic geometry, Arf numerical semigroups have gained lately a particular interest due to their applications to algebraic geometric codes, in particular to one-point algebraic geometric codes when the Weierstrass semigroup of the point is an Arf numerical semigroup.

2. Arf numerical semigroups

A numerical semigroup S satisfying the additional condition

$$x, y, z \in S; x \geq y \geq z \Rightarrow x + y - z \in S (*)$$

is called an *Arf numerical semigroup.*

This is the original definition of an Arf numerical semigroup given by Arf in [1]. We will refer to the condition $(*)$ as the *Arf condition.* Fifteen conditions equivalent to the Arf condition are given in Theorem 1.3.4 of [2] where the definition of an Arf numerical semigroup is based on those equivalent conditions. We will discuss some conditions equivalent to the Arf condition in the last section.

It is easy to observe that the intersection of any family of Arf numerical semigroups is again an Arf numerical semigroup. Thus, since $\mathbb{N}_0$ is an Arf numerical semigroup, one can consider the smallest Arf numerical semigroup containing a given numerical semigroup. The smallest Arf numerical semigroup containing a numerical semigroup S is called the *Arf closure* of S, and it is denoted by $Arf(S)$. Taking the Arf closure preserves set inclusion; i.e. if S and T are numerical semigroups such that $S \subseteq T$, then $Arf(S) \subseteq Arf(T)$.

Given a numerical semigroup S, we define

$$A(S) = \{x + y - z : x, y, z \in S \text{ and } x \geq y \geq z\}.$$

It is clear that $A(S)$ is a numerical semigroup containing S and it has the same multiplicity as S; i.e. $m(A(S)) = m(S)$. Note that if S and T are numerical semigroups with $S \subseteq T$, then $A(S) \subseteq A(T)$. Note also $A(S) \subseteq Arf(S)$, and S is an Arf numerical semigroup if and only if $A(S) = S$. Moreover, we have the following lemma:

Lemma 2.1 *For any numerical semigroup S, $Arf(S) = Arf(A(S))$.*

If U and V are two subsets of $\mathbb{N}_0$, we define $U + V = \{u + v : u \in U, v \in V\}$; for any $a \in \mathbb{Z}$, we let $a + U = \{a\} + U$. We also define $U - V = \{z \in \mathbb{Z} : z + V \subseteq U\}$. For a positive integer k we put $kU = U + \cdots + U$ (k summands).

Lemma 2.2 *Let S be a numerical semigroup and let s be any element of S. Then $(s+S) \cup \{0\}$ is a numerical semigroup and we have $A((s + S) \cup \{0\}) = (s + A(S)) \cup \{0\}$.*

The proof follows from the fact that $(s+x)+(s+y)-(s+z) = s+(x+y-z) \in (s+A(S))\cup\{0\}$ where $s,x,y,z \in S$ with $x \geq y \geq z$.

Now let us consider the following sequence of numerical semigroups for a given numerical semigroup S:

$$A_0 = S \subseteq A_1 = A(S) \subseteq A_2 = A(A_1) \subseteq \cdots \subseteq A_k = A(A_{k-1}) \subseteq \cdots .$$

By Lemma 2.1, we have

$$Arf(S) = Arf(A_1) = Arf(A_2) = \cdots = Arf(A_k) = \cdots .$$

Since $\mathbb{N}_0 \setminus S$ is finite, the sequence

$$A_0 = \subseteq A_1 = A(S) \subseteq A_2 = A(A_1) \subseteq \ldots \subseteq A_k = A(A_{k-1}) \subseteq \ldots$$

is finite. Let $\alpha = \alpha(S)$ be the smallest index for which $A_\alpha = A_{\alpha+1}$. Then $A_{\alpha+1} = A(A_\alpha) = A_\alpha$ and in this case we know that A_α is an Arf numerical semigroup. Hence, $A_\alpha = Arf(A_\alpha) = Arf(S)$. The sequence

$$A_0 = \subseteq A_1 = A(S) \subset A_2 = A(A_1) \subset \ldots \subset A_\alpha = A(A_{\alpha-1}) = Arf(S)$$

is called the *Arf sequence of semigroups* of S, and $\alpha = \alpha(S)$ is called the *Arf index* of S. We know that $Arf(S)$ has the same multiplicity as S. Note also that S is Arf if and only if the Arf index of S is zero.

Lemma 2.3 *Let S be a numerical semigroup and consider any element $s \in S$. Then the Arf index of $(s+S)\cup\{0\}$ is the same as the Arf index of S and we have $Arf((s+S)\cup\{0\}) = (s+Arf(S))\cup\{0\}$.*

Proof Successive application of Lemma 2.2 gives

$$A^k((s+S)\cup\{0\}) = (s+A^k(S))\cup\{0\}$$

for any $k \geq 1$. If the Arf index of S is α, then

$$A^\alpha((s+S)\cup\{0\}) = (s+A^\alpha(S))\cup\{0\} = (s+Arf(S))\cup\{0\}.$$

It follows that

$$A^{\alpha+1}((s+S)\cup\{0\}) = A((s+Arf(S))\cup\{0\}) = (s+Arf(S))\cup\{0\} = A^\alpha((s+S)\cup\{0\}),$$

which proves that the Arf index of $(s+S)\cup\{0\}$ is the same as the Arf index α of S, and that $Arf((s+S)\cup\{0\}) = (s+Arf(S))\cup\{0\}$. □

Corollary 2.4 *Let S be a numerical semigroup and consider any element $s \in S$. Then S is an Arf numerical semigroup if and only if $(s+S)\cup\{0\}$ is an Arf numerical semigroup.*

3. Lipman sequence of a numerical semigroup

Let S be a numerical semigroup with the maximal ideal M. For each $k \geq 1$, we consider the set $kM - kM = \{x \in \mathbb{N}_0 : x + kM \subseteq kM\}$. We note that $kM - kM$ is a numerical semigroup, $kM - kM \subseteq (k+1)M - (k+1)M$ and $S \subseteq (kM - kM)$ since $S + kM \subseteq kM$ for each $k \geq 1$. Thus, $L(S) = \bigcup_{k \geq 1}(kM - kM)$ is a numerical semigroup containing S. $L(S)$ is called the *Lipman semigroup* of S or the semigroup of S obtained by *blowing up* M.

In what follows, we will give some results about Lipman semigroups that can be found in [2].

Lemma 3.1 *For a numerical semigroup S, the following are equivalent:*

(i) $L(S) = S$,

(ii) $S = M - M$,

(iii) $S = \mathbb{N}_0$.

There may be slight differences in the proofs, but the main reason we include them here is to give a self-contained exposition.

Now let us consider the sequence

$$L_0 = S \subseteq L_1 = L(S) \subseteq L_2 = L(L_1) \subseteq \cdots \subseteq L_k = L(L_{k-1}) \subseteq \cdots .$$

This sequence is called the *Lipman sequence of semigroups* of S. Since $\mathbb{N}_0 \setminus S$ is finite, there exists λ such that $L_\lambda = L_{\lambda+1}$; actually, $L_\lambda = \mathbb{N}_0$ by Lemma 3.1. If $S \neq \mathbb{N}_0$, we shall assume that $L_{\lambda-1} \neq \mathbb{N}_0$ and call λ the *Lipman index* of S.

Lemma 3.2 *Let S be a numerical semigroup with the maximal ideal M and the multiplicity m. Then there exists $h \geq 1$ such that $(h+1)M = m + hM$ and $L(S) = hM - hM$.*

Proof Let $\mathtt{C}_i = \{x \in M : x \equiv i(mod\ m)\}$ for each $i = 0, 1, \ldots, m-1$. No $\mathtt{C}_i$ is empty since M contains all positive integers from some point on. Let μ_i be the least element of $\mathtt{C}_i$, $0 \leq i \leq m-1$. Thus, $\mathtt{C}_i = \{\mu_i + km : k \geq 0\}$. For each $i = 0, 1, \ldots, m-1$, there exists $h_i \geq 1$ such that $\mu_i \in h_i M \setminus (h_i + 1)M$. Let $h = max\{h_i : 0 \leq i \leq m-1\}$. Then the smallest element of $\mathtt{C}_i$ in $hM \setminus (h+1)M$ is $\tilde{\mu}_i = \mu_i + (h - h_i)m$. We note that $hM \setminus (h+1)M = \{\tilde{\mu}_0, \tilde{\mu}_1, \ldots, \tilde{\mu}_{m-1}\}$. Since $m + hM \subseteq (h+1)M \subseteq hM$ and $|hM \setminus (m + hM)| = m$, we see that $(h+1)M = m + hM$. The last assertion follows from $(h+1)M = m + hM$ since in that case $(h+1)M - (h+1)M = hM - hM$. □

Lemma 3.3 *If S is a numerical semigroup with the maximal ideal M and the multiplicity m, then $L(S) = \{z - km : z \in kM, k \geq 1\} = \bigcup_{k \geq 1}(-km + kM)$.*

Proof Let h be as in Lemma 3.2 and consider an element of the form $z - km$ with $z \in kM$ and $k \geq 1$. One can easily see by induction on k that $(h+k)M = km + hM$. Thus,

$$(z - km) + (h+k)M = (z - km) + (km + hM) = z + hM \subseteq (h+k)M$$

and this shows that $z - km \in (h+k)M - (h+k)M \subseteq L(S)$. This proves the inclusion $\{z - km : z \in kM, k \geq 1\} \subseteq L(S)$. To prove the reverse inclusion, let $x \in L(S)$. Then $x \in kM - kM$ for some $k \geq 1$ and $z = x + km \in x + kM \subseteq kM$. Hence, $x = z - km$ with $z \in kM,\ k \geq 1$. The last expression as the union is clear. □

Corollary 3.4 *Let $S =< a_1 = m, a_2, \ldots, a_e >$ be a numerical semigroup with minimal system of generators $a_1 = m < a_2 < \ldots < a_e$ and the maximal ideal M. Then $L(S) =< m, a_2 - m, \ldots, a_e - m >$.*

Proof The inclusion $< m, a_2 - m, \ldots, a_e - m > \subseteq L(S)$ is clear by Lemma 3.3. Now let $x \in L(S)$. Then $x = z - km,\ z \in kM,\ k \geq 1$. We write $z = b_1 + b_2 + \cdots + b_k$ with $b_i \in M,\ 1 \leq i \leq k$. For each $i = 1, \ldots, k$, we can write $b_i = c_{i1}m + \sum_{j=2}^{e} c_{ij}a_j$ with $c_{ij} \in \mathbb{N}_0$ and at least one of the $c_{ij} \neq 0$ for $1 \leq j \leq e$. Hence,

$$x = \sum_{i=1}^{k}(c_{i1}m + \sum_{j=2}^{e} c_{ij}a_j) - km.$$

This sum can be arranged as

$$x = (\sum_{i=1}^{k}\sum_{j=1}^{e} c_{ij} - k)m + \sum_{i=1}^{k}\sum_{j=2}^{e} c_{ij}(a_j - m),$$

where $\sum_{i=1}^{k}\sum_{j=1}^{e} c_{ij} - k \geq 0$, proving that $x \in < m, a_2 - m, \ldots, a_e - m >$. □

Lemma 3.5 *Let S be a numerical semigroup with the maximal ideal M and the multiplicity m. The following are equivalent for any positive integer h:*

(i) $(h+1)M = m + hM$,

(ii) $L(S) = hM - hM$,

(iii) $L(S) = -hm + hM$,

(iv) $|hM \setminus (h+1)M| = m$.

Proof *(i)* ⇒ *(ii)* As in the proof of Lemma 3.3, this gives $(h+k)M = km + hM$ for all $k \geq 1$. It follows that $(h+k)M - (h+k)M = hM - hM$ and thus the increasing union $L(S) = \cup_{i \geq 1}(iM - iM)$ stabilizes at h.

(ii) ⇒ *(iii)* $-hm + hM \subseteq L(S)$ by Lemma 3.3. For the reverse inclusion, let $x \in L(S) = hM - hM$. Then $x + hm \in x + hM \subseteq hM$. Hence, $x \in -hm + hM$, proving $L(S) \subseteq -hm + hM$.

(iii) ⇒ *(i)* $-(h+1)m + (h+1)M \subseteq L(S)$ by Lemma 3.3. On the other hand, $L(S) = -hm + hM = -(h+1)m + m + hM \subseteq -(h+1)m + (h+1)M$. Therefore, $L(S) = -(h+1)m + (h+1)M$. Hence,

$$(h+1)M = (h+1)m + L(S) = m + (hm + L(S)) = m + hM.$$

(i) ⇒ *(iv)* $|hM \setminus (h+1)M| = |hM \setminus (m + hM)| = m$.

$(iv) \Rightarrow (i)$ Since $m+hM \subseteq (h+1)M \subseteq hM$ and $m = |hM \setminus (m+hM)|$, the equality $|hM \setminus (h+1)M| = m$ implies that $(h+1)M = m + hM$. □

4. Arf closure via the Lipman sequence

The following lemma will lead to a connection between the Arf closure of a numerical semigroup S and the Arf closure of its Lipman semigroup $L(S)$. This connection will be used to compute the Arf closure of a numerical semigroup via its Lipman sequence and also to characterize later Arf numerical semigroups in terms of their Lipman semigroups.

Lemma 4.1 *Let S be a numerical semigroup with the maximal ideal M and the multiplicity m. Then*

$$S \subseteq (m + L(S)) \cup \{0\} \subseteq Arf(S).$$

Proof Let $\{a_1, a_2, \ldots, a_e\}$ be a minimal system of generators of S. Then, by Corollary 3.4,

$$a_i = m + (a_i - m) \in (m + L(S)) \cup \{0\}$$

for each $i = 1, \ldots, e$. Hence, $S \subseteq (m + L(S)) \cup \{0\}$. As for the other inclusion, by Lemma 3.3, any element $u \in L(S)$ can be expressed as $u = z_k - km$ for some $z_k \in kM$ and $k \geq 1$. We prove that $m + u = z_k - (k-1)m \in Arf(S)$. We proceed by induction on k. The assertion is obvious for $k = 1$. Let $k > 1$ and assume that the assertion holds for all positive integers less than k. Then we have

$$m + u = z_k - (k-1)m = z_{k-1} - (k-2)m + z_1 - m$$

for some $z_{k-1} \in (k-1)M$ and $z_1 \in M$. Here $z_{k-1} - (k-2)m \in Arf(S)$ by induction assumption and we have $z_{k-1} - (k-2)m \geq m,\ z_1 \geq m$. Thus,

$$m + u = z_k - (k-1)m = (z_{k-1} - (k-2)m) + z_1 - m \in Arf(S).$$

This completes the proof. □

Theorem 4.2 *For any numerical semigroup S with the multiplicity m,*

$$Arf(S) = (m + Arf(L(S))) \cup \{0\}.$$

Proof We have

$$S \subseteq (m + L(S)) \cup \{0\} \subseteq Arf(S)$$

by Lemma 4.1. Consider the Arf closure of each semigroup above. Since

$$Arf(m + L(S)) \cup \{0\} = (m + Arf(L(S))) \cup \{0\}$$

by Lemma 2.3, we get $Arf(S) = (m + Arf(L(S))) \cup \{0\}$. □

Corollary 4.3 *For any numerical semigroup S with the multiplicity m,*

$$f(Arf(S)) = m + f(Arf(L(S))).$$

Proof Obviously, $m + f(Arf(L(S))) \notin (m + Arf(L(S))) \cup \{0\} = Arf(S)$ and $m + f(Arf(L(S))) + k \in (m + Arf(L(S))) \cup \{0\} = Arf(S)$ for each $k \geq 1$. □

Corollary 4.4 *Let S be a numerical semigroup and let $m_k = m(L_k)$ where L_k is the kth term of the Lipman sequence of semigroups of S for each $k \geq 0$. Then $Arf(L_k) = (m_k + (Arf(L_{k+1}))) \cup \{0\}$ for each $k \geq 0$.*

Proof Since $L_{k+1} = L(L_k)$, we have $Arf(L_k) = (m_k + (Arf(L_{k+1}))) \cup \{0\}$ for each $k \geq 0$ by Theorem 4.2. □

Corollary 4.5 *Let S be a numerical semigroup and let $m_k = m(L_k)$ where L_k is the kth term of the Lipman sequence of semigroups of S for each $k \geq 0$. Let $f(Arf(S)) = f^{(a)}$, $n(Arf(S)) = n^{(a)}$. Then $n^{(a)} = \lambda(S) = \lambda$ and we have*

$$Arf(S) = \{s_0^{(a)}, s_1^{(a)}, \dots, s_{\lambda-1}^{(a)}, s_\lambda^{(a)} = f^{(a)} + 1, \rightarrow\},$$

where:

$$s_1^{(a)} = m_0 = m(S),$$

$$s_2^{(a)} = m_0 + m_1,$$

$$\dots$$

$$s_{\lambda-1}^{(a)} = m_0 + m_1 + \cdots + m_{\lambda-2},$$

$$s_\lambda^{(a)} = m_0 + m_1 + \cdots + m_{\lambda-2} + m_{\lambda-1}$$

and

$$f^{(a)} = m_0 + m_1 + \cdots + m_{\lambda-2} + m_{\lambda-1} - 1.$$

Proof Let $\lambda(S) = \lambda$, the Lipman index of S, and let us note that

$$Arf(S) = (m + Arf(L(S))) \cup \{0\} = (m_0 + Arf(L_1)) \cup \{0\},$$

$$Arf(L_1) = (m_1 + Arf(L_2)) \cup \{0\},$$

$$Arf(L_2) = (m_2 + Arf(L_3)) \cup \{0\},$$

$$\dots$$

$$Arf(L_{\lambda-2}) = (m_{\lambda-2} + Arf(L_{\lambda-1})) \cup \{0\},$$

$$Arf(L_{\lambda-1}) = (m_{\lambda-1} + Arf(L_\lambda)) \cup \{0\} = (m_{\lambda-1} + \mathbb{N}_0) \cup \{0\}$$

□

by Theorem 4.2 and Corollary 4.4. Thus, the first nonzero non-gap of $Arf(S)$ is $s_1^{(a)} = m = m_0$; the second nonzero non-gap of $Arf(S)$ is the sum of m_0 and the first nonzero non-gap of $Arf(L_1)$ (which is the multiplicity of L_1) so that $s_2^{(a)} = m_0 + m_1$. Continuing in this way, we see that the kth nonzero non-gap of $Arf(S)$ is $s_k^{(a)} = m_0 + m_1 + \cdots + m_{k-1}$ for each $k = 2, \ldots, \lambda$. We also see that $s_\lambda^{(a)} = m_0 + m_1 + \cdots + m_{\lambda-2} + m_{\lambda-1}$ and any integer following it belongs to $Arf(S)$. Therefore, $n^{(a)} = \lambda(S) = \lambda$ and $\mathsf{f}^{(a)} = m_0 + m_1 + \cdots + m_{\lambda-2} + m_{\lambda-1} - 1$.

In the rest of this section, we will present an algorithmic procedure for computing the Arf closure of a given numerical semigroup. Combining Corollary 3.4 and Corollary 4.5, we obtain a practical procedure for computing nonzero non-gaps of $Arf(S)$ for any numerical semigroup S. This procedure, which is given in terms of multiplicities of the Lipman sequence, is very similar to the procedure given in [6].

If S is a numerical semigroup with minimal system of generators $\{a_1 = m, a_2, \ldots, a_e\}$, then $m = m_0$ is the multiplicity of $S = L_0$ and we see by Lemma 3.4 that $L_1 = L(S) =< m, a_2 - m, \ldots, a_e - m >$. The multiplicity of L_1 is $min\{m, a_2 - m\}$. Obviously we can compute the multiplicity m_k of L_k by using the minimal system of generators of L_{k-1} for each $k \geq 2$. We proceed until we find the multiplicity $m_\lambda = 1$ of $L_\lambda = \mathbb{N}_0$.

Example 4.6 *Let us determine the Arf closure* $Arf(S)$ *for* $S =< 4, 10, 25 >$. *We have* $m = m_0 = 4$ *and*

$$L_0 = S =< 4, 10, 25 >, \quad m_0 = 4$$
$$L_1 = L(S) =< 4, 6, 21 >, \quad m_1 = 4$$
$$L_2 = L(L_1) =< 4, 2, 17 >=< 2, 17 >, \quad m_2 = 2$$
$$L_3 = L(L_2) =< 2, 15 >, \quad m_3 = 2$$
$$L_4 = L(L_3) =< 2, 13 >, \quad m_4 = 2$$
$$L_5 = L(L_4) =< 2, 11 >, \quad m_5 = 2$$
$$L_6 = L(L_5) =< 2, 9 >, \quad m_6 = 2$$
$$L_7 = L(L_6) =< 2, 7 >, \quad m_7 = 2$$
$$L_8 = L(L_7) =< 2, 5 >, \quad m_8 = 2$$
$$L_9 = L(L_8) =< 2, 3 >, \quad m_9 = 2$$
$$L_{10} = L(L_9) =< 2, 1 >=< 1 >= \mathbb{N}_0, \quad m_{10} = 1, \lambda = 10.$$

It follows that

$$Arf(S) = \{0, 4, 8, 10, 12, 14, 16, 18, 20, 22, 24, 25, \rightarrow\}.$$

Here $\mathsf{f}(Arf(S)) = 23$, $n(Arf(S)) = \lambda(S) = \lambda(Arf(S)) = 10$. The number of gaps of $Arf(S)$ is 14.

5. Conditions equivalent to the Arf condition

In this section we will give alternative characterizations of Arf numerical semigroups in the context of the previous sections. Some of the characterizations that we give here coincide with those given in [2]. We first present some notations.

Let $S = \{s_0 = 0, s_1 = m, s_2, \ldots, s_{n-1}, s_n = \mathsf{f}+1, \rightarrow\}$ be a numerical semigroup as before, where m is the multiplicity, n is the number of nonzero non-gaps, and f is the Frobenius number of S. Denote the maximal ideal of S by M. Now for each $i = 0, \dots, n$ we define

$$S_i = \{x \in S : x \geq s_i\}$$

and

$$S(i) = S - S_i.$$

Thus, $S_0 = S$, $S(0) = S$, $S_1 = M$, and $S(1) = M - M$. Let us note that $S(i) = S_i - S_i$ for each $i = 1, \dots, n$ and

$$S \subset S(1) \subset S(2) \subset \cdots \subset S(n-1) \subset S(n) = \mathbb{N}_0.$$

We have $S(1) = \mathbb{N}_0$ if and only if $S = \mathbb{N}_0$ (see Lemma 3.1).

We use a different notation for $S(1)$; namely, we write $B(S) = M - M$ and thus we obtain the sequence of numerical semigroups

$$B_0(S) = S \subseteq B_1(S) = B(S) \subseteq \cdots \subseteq B_k(S) = B(B_{k-1}(S)) \subseteq \cdots .$$

Since $\mathbb{N}_0 \setminus S$ is finite, the above sequence is stationary at $\mathbb{N}_0$ (see Lemma 3.1). We let β be the smallest nonnegative integer such that $B_\beta(S) = \mathbb{N}_0$.

Lemma 5.1 *For a numerical semigroup $S = \{s_0, s_1, \dots, s_{n-1}, s_n = \mathsf{f}+1, \rightarrow\}$, the following are equivalent:*

(i) S is Arf.

(ii) $L(S)$ is Arf and $L(S) = B(S)$.

(iii) $\lambda(S) = \beta(S)$ and $L_k(S) = B_k(S)$ for all $k = 0, \dots, \lambda = \lambda(S)$.

(iv) $\lambda(S) = n(S)$ and $L_k(S) = -s_k + S_k$ for all $k = 1, \dots, \lambda = \lambda(S)$.

(v) $\lambda(S) = n(S)$ and $L_k(S) = S(k)$ for all $k = 1, \dots, \lambda = \lambda(S)$.

(vi) $S(k) = -s_k + S_k$ for all $k = 1, \dots, n = n(S)$.

(vii) $-s_k + 2S_k = S_k$ for all $k = 1, \dots, n = n(S)$.

Proof As usual, let M_k and m_k denote the maximal ideal and the multiplicity, respectively, of the kth term $L_k = L_k(S)$ of the Lipman sequence of S. We will use the fact that the maximal ideal of $L_0 = S$ is $M_0 = M = S_1$ and its multiplicity is $m_0 = m = s_1$.

(i) $\Rightarrow$ *(ii)* $S \subseteq (m + L(S)) \cup \{0\} \subseteq Arf(S)$ by Lemma 4.1. Therefore, $S = (m + L(S)) \cup \{0\}$ and $L(S)$ is Arf by Corollary 2.4. Here $M = m + L(S)$ and thus $B(S) = M - M = L(S) - L(S) = L(S)$.

(ii) $\Rightarrow$ *(iii)* By the first part of the proof, $L_2(S) = L(L(S))$ is Arf and we have $L_2(S) = L(L(S)) = B(L(S)) = B(B(S)) = B_2(S)$. Repeated application of this gives $\lambda(S) = \beta(S)$ and $L_k(S) = B_k(S)$ for all $k = 0, \dots, \lambda = \lambda(S)$.

$(iii) \Rightarrow (iv)$ $L_1(S) = L(S) = B(S) = M - M = M_0 - M_0 \Rightarrow L_1(S) = -m_0 + M_0$ by Lemma 3.5. Hence, $L_1(S) = -s_1 + S_1$, $m_1 = s_2 - s_1$ and $M_1 = -s_1 + S_2$. Now $L_2(S) = B_2(S) = B(B_1(S)) = B(L_1(S)) = M_1 - M_1$ and this implies $L_2(S)) = -m_1 + M_1 = -(s_2 - s_1) + (-s_1 + S_2) = -s_2 + S_2$; $M_2 = -s_2 + S_3$. By induction on k, we get $L_k(S) = -s_k + S_k$ for all $k \geq 1$. Moreover, $L_n(S) = -s_n + S_n = -(\mathsf{f} + 1) + \{f + 1, \rightarrow\} = \mathbb{N}_0$ and $-s_k + S_k \neq \mathbb{N}_0$ for $k < n$, showing that $n(S) = \lambda(S)$.

$(iv) \Rightarrow (v)$ Since $L_k(S) = -s_k + S_k$, we have

$$S(k) = S_k - S_k = L_k(S) - L_k(S) = L_k(S)$$

for each $k = 1, \ldots, n$.

$(v) \Rightarrow (vi)$ $L_1(S) = L(S) = S(1) = M - M \Rightarrow L_1(S) = S(1) = -m + M$ by Lemma 3.5. Thus, $S(1) = -s_1 + S_1$, $M_1 = -s_1 + S_2$ and $m_1 = s_2 - s_1$. Now $L_2(S) = S(2) = S_2 - S_2 = M_1 - M_1$ $\Rightarrow S(2) = -m_1 + M_1$ again by Lemma 3.5. Thus, $S(2) = -(s_2 - s_1) + (-s_2 + S_2) = -s_2 + S_2$. We have $M_2 = -s_2 + S_3$ and $m_2 = s_3 - s_2$. By induction, $S(k) = -s_k + S_k$ for each $k = 1, \ldots, n = n(S)$.

$(vi) \Rightarrow (vii)$ Clearly, $s_k + S_k \subseteq 2S_k$ and thus $S_k \subseteq -s_k + 2S_k$ for each $k = 1, \ldots, n$. Let $x, y \in S_k$ so that $x + y \in 2S_k$. Since $S(k) = -s_k + S_k$, $-s_k + x \in S(k)$ and therefore $(-s_k + x) + y = -s_k + (x + y) \in S_k$. This proves that $-s_k + 2S_k \subseteq S_k$. Hence, we have the equality $-s_k + 2S_k = S_k$.

$(vii) \Rightarrow (i)$ Take $x, y, z \in S$ with $x \geq y \geq z$. If $z > \mathsf{f}$, then $x + y - z > \mathsf{f}$ and thus $x + y - z \in S$. Therefore, we may assume that $z = s_k$ for some $k \leq n$. Then x and y belong to S_k and we have $x + y - z \in -s_k + 2S_k = S_k \subseteq S$. □

References

[1] Arf C. Une interprétation algébrique de la suite de multiplicité d'une branche algébrique. Proc London Math Soc 1949; 20: 256-287 (in French).

[2] Barucci V, Dobbs DE, Fontana M. Maximality properties in numerical semi-groups and applications to one-dimensional analytically irreducible local domains. Mem Am Math Soc 1997; 125: 1-77.

[3] Fröberg R, Gotlieb C, Häggkvist R. On numerical Semigroups. Semigroup Forum 1987; 35: 63-83.

[4] Lipman J. Stable ideals and Arf rings. Am J Math 1971; 93: 649-685.

[5] Rosales JC, Garcia-Sánchez PA. Numerical Semigroups. New York, NY, USA: Springer, 2009.

[6] Rosales JC, Garcia-Sánchez PA, Garcia-Garcia JI, Branco MB. Arf numeri- cal semigroups. J Algebra 2004; 276: 3-12.