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The Ptolemaean inequality in the closure of complex hyperbolic planes

Ioannis D. PLATIS[1], **Nilgün SÖNMEZ**[2,*]
[1]Department of Mathematics and Applied Mathematics, University of Crete, Heraklion, Crete, Greece
[2]Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey

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Abstract: We prove the Ptolemaean inequality and Ptolemaeus' theorem in the closure of complex hyperbolic planes endowed with the Cygan metric.

Key words: Complex hyperbolic plane, Cygan metric, Ptolemaean inequality, Ptolemaeus' theorem

1. Introduction

Let (S,ρ) be a metric space. The metric d is called Ptolemaean if for any quadruple of points the Ptolemaean Inequality holds; that is, for any distinct points p_1, p_2, p_3, and p_4

$$\rho(p_1,p_3)\cdot\rho(p_2,p_4) \le \rho(p_1,p_2)\cdot\rho(p_3,p_4) + \rho(p_2,p_3)\cdot\rho(p_4,p_1).$$

A subset σ of S is called a Ptolemaean circle if for any four distinct points p_1, p_2, p_3, and p_4 in σ such that p_1 and p_3 separate p_2 and p_4 we have

$$\rho(p_1,p_3)\cdot\rho(p_2,p_4) = \rho(p_1,p_2)\cdot\rho(p_3,p_4) + \rho(p_2,p_3)\cdot\rho(p_4,p_1).$$

Then we say that σ satisfies the theorem of Ptolemaeus. The prototype of course is the Euclidean plane case, which was proved by the ancient Greek mathematician Claudius Ptolemaeus (Ptolemy) of Alexandria almost 1800 years ago: the inequality holds for any four points of the Euclidean plane and Ptolemaean circles are Euclidean circles. From the times of antiquity it was realized that even in the simple Euclidean case, the Ptolemaean inequality has an intrinsic importance of its own and various generalizations have been given by a variety of researchers since then. In particular, generalizations to much more abstract spaces have appeared in the last 70 years. Illustratively, we refer to [1, 4] for the case of CAT(0) and CAT(−1) cases, respectively, as well as [3] for the case of spaces of nonpositive curvature, [6] for the case of Hilbert geometries, [10] for normed spaces, and [2] for more abstract spaces.

In this paper we investigate the Ptolemaean inequality and the theorem of Ptolemaeus in the metric space $(\overline{\mathbf{H}}^2_{\mathbb{C}},\rho)$, where $\overline{\mathbf{H}}^2_{\mathbb{C}}$ is the compactified complex hyperbolic plane and ρ is the Cygan metric; see Section 2 for the definitions. Working with a much more general concept, the first author showed in [8] that both the Ptolemaean inequality and the theorem of Ptolemaeus hold in the boundary of complex hyperbolic plane $\partial\mathbf{H}^2_{\mathbb{C}}$ when the latter is endowed with the Korányi–Cygan metric $d_{\mathfrak{H}}$; see Section 2.2. The boundary of a complex

*Correspondence: nceylan@aku.edu.tr
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hyperbolic plane is a sphere and the Cygan metric ρ is the natural extension of $d_{\mathfrak{H}}$ to the whole ball. For the proof we use metric cross-ratios and their properties; see Section 3.1. Using cross-ratios, both the Ptolemaean inequality and the theorem of Ptolemaeus can be expressed in quite a neat way.

This paper is organized as follows: in the preliminary Section 2 we state well-known facts about the complex hyperbolic plane, its boundary, horospherical coordinates, and the Cygan metric. In Section 3 we prove the Ptolemaean inequality (Theorem 3.1), and finally in Section 4 we prove Ptolemaeus' theorem. Both proofs are carried out via the metric cross-ratio in $(\overline{\mathbf{H}^2_{\mathbb{C}}}, \rho)$ and its invariance properties.

2. Preliminaries

The material in this section is well known; for further details we refer to [5, 7]. In Section 2.1 we define the complex hyperbolic plane and describe its isometries with respect to the Bergman metric. Section 2.2 is devoted to the boundary of the complex hyperbolic plane; in particular, we define the Heisenberg group and the Heisenberg (Korányi–Cygan) metric. Horospherical coordinates for the complex hyperbolic plane are described in Section 2.3 and finally the Cygan metric is given in Section 2.4.

2.1. Complex hyperbolic plane

Let $\mathbb{C}^{2,1}$ be the vector space $\mathbb{C}^3$ with the Hermitian form of signature $(2,1)$ given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J \mathbf{z} = \overline{w}_3 z_1 + \overline{w}_2 z_2 + \overline{w}_1 z_3,$$

where J is the matrix of the Hermitian form:

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We consider the following subspaces of $\mathbb{C}^{2,1}$:

$$\begin{aligned} V_- &= \Big\{ \mathbf{z} \in \mathbb{C}^{2,1} \ : \ \langle \mathbf{z}, \mathbf{z} \rangle < 0 \Big\}, \\ V_0 &= \Big\{ \mathbf{z} \in \mathbb{C}^{2,1} \setminus \{\mathbf{0}\} \ : \ \langle \mathbf{z}, \mathbf{z} \rangle = 0 \Big\}. \end{aligned}$$

Let $\mathbb{P} : \mathbb{C}^{2,1} \setminus \{\mathbf{0}\} \longrightarrow \mathbb{C}P^2$ be the canonical projection onto the two-dimensional complex projective space. Then complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ is defined to be $\mathbb{P}V_-$ and its boundary $\partial \mathbf{H}^2_{\mathbb{C}}$ is $\mathbb{P}V_0$. Specifically, $\mathbb{C}^{2,1} \setminus \{\mathbf{0}\}$ may be covered with three charts H_1, H_2, H_3 where H_j comprises those points in $\mathbb{C}^{2,1} \setminus \{\mathbf{0}\}$ for which $z_j \neq 0$. It is clear that V_- is contained in H_3. The canonical projection from H_3 to $\mathbb{C}^n$ is given by $\mathbb{P}(\mathbf{z}) = (z_1 z_3^{-1}, \, z_2 z_3^{-1}) = z$. Therefore, we can write $\mathbf{H}^n_{\mathbb{C}} = \mathbb{P}(V_-)$ as

$$\mathbf{H}^2_{\mathbb{C}} = \left\{ (z_1, \, z_2, \, z_3) \in \mathbb{C}^3 \ : \ 2\Re(z_1) + |z_2|^2 < 0 \right\},$$

which is called the Siegel domain model for $\mathbf{H}^2_{\mathbb{C}}$. There are distinguished points in V_0, which we denote by $\mathbf{o}$ and ∞:

$$\mathbf{o} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then $V_0 \setminus \{\infty\}$ is contained in H_3 and $V_0 \setminus \{\mathbf{o}\}$ (in particular ∞) is contained in H_1. Let $\mathbb{P}\mathbf{o} = o$ and $\mathbb{P}\infty = \infty$. Then we can write $\partial \mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_0)$ as

$$\partial \mathbf{H}^2_{\mathbb{C}} \setminus \{\infty\} = \left\{(z_1,\, z_2) \in \mathbb{C}^2 \;:\; 2\Re(z_1) + |z_2|^2 = 0\right\}.$$

In particular, $o = (0,0) \in \mathbb{C}^2$.

Conversely, given a point z of $\mathbb{C}^2 = \mathbb{P}(H_3) \subset \mathbb{C}P^2$ we may lift $z = (z_1, z_2)$ to a point $\mathbf{z}$ in $H_3 \subset \mathbb{C}^{2,1}$, called the *standard lift* of z, by writing $\mathbf{z}$ in nonhomogeneous coordinates as

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}.$$

The Riemannian metric on $\mathbf{H}^2_{\mathbb{C}}$ is defined by the distance function ρ given by the formula

$$\cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{\langle \mathbf{z},\mathbf{w}\rangle \langle \mathbf{w},\mathbf{z}\rangle}{\langle \mathbf{z},\mathbf{z}\rangle \langle \mathbf{w},\mathbf{w}\rangle} = \frac{\left|\langle \mathbf{z},\mathbf{w}\rangle\right|^2}{|\mathbf{z}|^2|\mathbf{w}|^2},$$

where $\mathbf{z}$ and $\mathbf{w}$ in V_- are the standard lifts of z and w in $\mathbf{H}^2_{\mathbb{C}}$ and $|\mathbf{z}| = \sqrt{-\langle \mathbf{z},\mathbf{z}\rangle}$. Alternatively,

$$ds^2 = -\frac{4}{\langle \mathbf{z},\mathbf{z}\rangle^2} \det \begin{bmatrix} \langle \mathbf{z},\mathbf{z}\rangle & \langle d\mathbf{z},\mathbf{z}\rangle \\ \langle \mathbf{z},d\mathbf{z}\rangle & \langle d\mathbf{z},d\mathbf{z}\rangle \end{bmatrix}.$$

The sectional curvature of $\mathbf{H}^2_{\mathbb{C}}$ is pinched between -1 and $-1/4$. Also, $\mathbf{H}^2_{\mathbb{C}}$ is a complex manifold, the metric is Kähler (in fact, it is the Bergman metric), and the holomorphic sectional curvature is equal to -1.

Denote by $\mathrm{U}(2,1)$ the group of unitary matrices for the Hermitian form $\langle\cdot,\cdot\rangle$. Each matrix $A \in \mathrm{U}(2,1)$ satisfies the relation $A^{-1} = JA^*J$ where A^* is the Hermitian transpose of A. The isometry group of the complex hyperbolic plane is the projective group $\mathrm{PU}(2,1)$. Instead, we may use $\mathrm{SU}(2,1)$, which is a three-fold cover of $\mathrm{PU}(2,1)$.

Two kinds of subspaces of $\mathbf{H}^2_{\mathbb{C}}$ are of particular interest: $\mathbb{C}$-lines and mainly $\mathbb{R}$-planes.

A $\mathbb{C}$-line is an isometric image of the embedding of $\mathbf{H}^1_{\mathbb{C}} = \{z \in \mathbb{C} \mid \Re(z) < 0\}$ into $\mathbf{H}^2_{\mathbb{C}}$. We may assume that the embedding is the standard one:

$$z \mapsto (z, 0).$$

The isometries preserving a $\mathbb{C}$-line are a subgroup isomorphic to $\mathrm{PU}(1,1)$.

An $\mathbb{R}$-plane (or Lagrangian plane) $\mathcal{R}$ is a real 2-dimensional subspace of $\mathbf{H}^2_{\mathbb{C}}$ characterized by $\langle \mathbf{v},\mathbf{w}\rangle \in \mathbb{R}$ for all $\mathbf{v},\mathbf{w} \in \mathcal{R}$. Any real plane $\mathcal{R}$ is the isometric image of an embedded copy of $\mathbf{H}^2_{\mathbb{R}} = \{(x_1,x_2) \in \mathbb{R}^2 \;:\; 2x_1 + x_2^2 < 0\}$ into $\mathbf{H}^2_{\mathbb{C}}$; here, we may assume that the embedding is the standard one:

$$(x_1,x_2) \mapsto (x_1,x_2,0).$$

The isometries preserving the $\mathbb{R}$-plane above are a subgroup isomorphic to $\mathrm{PO}(2,1)$.

2.2. The boundary: Heisenberg group

A finite point z is in the boundary of the Siegel domain if its standard lift to $\mathbb{C}^{2,1}$ is

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}, \quad \text{where} \quad 2\Re(z_1) + |z_2|^2 = 0.$$

We set

$$\zeta = z_2/\sqrt{2}, \quad \zeta = z_2 \in \mathbb{C}.$$

Therefore,

$$\mathbf{z} = \begin{bmatrix} -|\zeta|^2 + iv \\ \sqrt{2}\zeta \\ 1 \end{bmatrix}.$$

In this way, the boundary of the Siegel domain is identified to the one point compactification of $\mathbb{C} \times \mathbb{R}$, that is, the sphere S^3. The action of the stabilizer of infinity $\mathrm{Stab}(\infty)$ gives this set the structure of a group, which we shall denote by $\mathfrak{H}$. The multiplication for $\mathfrak{H}$ is

$$(\zeta, v) * (\zeta', v') = \left(\zeta + \zeta', v + v' + 2\Im(\overline{\zeta'}\zeta)\right).$$

We call $\mathfrak{H}$ the Heisenberg group; $\partial\mathbf{H}^2_{\mathbb{C}} = \mathfrak{H} \cup \{\infty\}$. The Korányi gauge $|\cdot|_{\mathfrak{H}}$ defined on $\mathfrak{H}$ is given by

$$|(\zeta, v)|_{\mathfrak{H}} = \left|-|\zeta|^2 + iv\right|^{1/2},$$

where on the right-hand side we have the Euclidean norm. Notice that $|\cdot|_{\mathfrak{H}}$ is not a norm in the usual sense; however, from this gauge we obtain a metric on $\mathfrak{H}$, the Heisenberg metric $d_{\mathfrak{H}}$, which is defined by the relation

$$d_{\mathfrak{H}}\left((\zeta, v), ((\zeta', v')\right) = \left|((\zeta, v)^{-1} * ((\zeta', v')\right|_{\mathfrak{H}}.$$

We remark that the Heisenberg metric is not a path metric. By taking the standard lift of points on $\partial\mathbf{H}^2_{\mathbb{C}} \setminus \{\infty\}$ to $\mathbb{C}^{2,1}$ we can write the metric $d_{\mathfrak{H}}$ as:

$$d_{\mathfrak{H}}\left((\zeta, v), (\zeta', v')\right) = \left|\left\langle \begin{bmatrix} -|\zeta|^2 + iv \\ \sqrt{2}\zeta \\ 1 \end{bmatrix}, \begin{bmatrix} -|\zeta'^2 + iv' \\ \sqrt{2}\zeta' \\ 1 \end{bmatrix} \right\rangle\right|^{1/2}.$$

The metric $d_{\mathfrak{H}}$ is invariant under left translations and rotations about the axis $\mathcal{V} = \{0\} \times \mathbb{R}$. Left translations are essentially the left action of $\mathfrak{H}$ on itself: given a point $(\zeta', v') \in \mathfrak{H}$ we define

$$T_{(\zeta', v')}(\zeta, v) = (\zeta', v') * (\zeta, v).$$

Rotations come from the action of $\mathrm{U}(1)$ on $\mathbb{C}$: given a $\theta \in \mathbb{R}$ we define

$$R_\theta(\zeta, v) = (e^{i\theta}\zeta, v).$$

The metric is also invariant by conjugation J:

$$J(\zeta, v) = (\overline{\zeta}, -v).$$

These actions form the group $\mathrm{Isom}(\mathfrak{H}, d_{\mathfrak{H}})$ of $d_{\mathfrak{H}}$-isometries; this acts transitively on $\mathfrak{H}$. Note also that the stabilizer of 0 consists of rotations. All the above transformations are extended naturally (and uniquely) on the boundary $\partial\mathbf{H}^2_{\mathbb{C}}$, by requiring the extended transformations to map ∞ to itself.

Now, if $\delta \in \mathbb{R}^+_*$, we define

$$D_\delta(\zeta, v) = (\delta\zeta, \delta^2 v), \quad D_\delta(\infty) = \infty.$$

We call the map D_δ a dilation. It is clear that for every $(\zeta, v), (\zeta', v') \in \partial\mathbf{H}^2_{\mathbb{C}}$ we have

$$d_{\mathfrak{H}}\left(D_\delta(\zeta, v), D_\delta(\zeta', v')\right) = \delta\, d_{\mathfrak{H}}\left((\zeta, v), (\zeta', v')\right).$$

Therefore, $d_{\mathfrak{H}}$ is scaled up to multiplicative constants by the action of dilations. The similarity group $\mathrm{Sim}(\mathfrak{H}, d_{\mathfrak{H}}))$ is the group comprising Heisenberg isometries and dilations. Finally, inversion I is given by

$$I(\zeta, v) = \left(\zeta(-|\zeta|^2 + iv)^{-1}\ ,\ v\left|-|\zeta|^2 + iv\right|^{-2}\right), \ \text{ if } (\zeta, v) \neq o, \infty, \qquad I(o) = \infty,\ I(\infty) = o.$$

Inversion I is an involution of $\partial\mathbf{H}^2_{\mathbb{C}}$. Moreover, for all $p = (\zeta, v), p' = (\zeta', v') \in \mathfrak{H} \setminus \{o\}$ we have

$$d_{\mathfrak{H}}(I(p), o) = \frac{1}{d_{\mathfrak{H}}(p, o)}, \quad d_{\mathfrak{H}}(I(p), I(p')) = \frac{d_{\mathfrak{H}}(p, p')}{d_{\mathfrak{H}}(p, o) \cdot d_{\mathfrak{H}}(o, p')}.$$

The group generated by similarities and inversion is isomorphic to the group $\mathrm{PU}(2,1)$ of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$ with respect to the Bergman metric; each holomorphic isometry can be written as a composition of similarities and inversion. Given two distinct points on the boundary, we can find an element of $\mathrm{PU}(2,1)$ mapping those points to 0 and ∞, respectively; in particular, $\mathrm{PU}(2,1)$ acts doubly transitively on the boundary.

2.3. Horospherical coordinates

For a fixed $u \in \mathbb{R}^+$ consider all those points $z \in \mathbf{H}^2_{\mathbb{C}}$ for which the standard lift $\mathbf{z}$ satisfies $\langle \mathbf{z}, \mathbf{z} \rangle = -2u$. Equivalently,

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}, \quad \text{where } 2\Re(z_1) + |z_2|^2 = -2u.$$

By writing again $z_2 = 2\zeta$ we have $z_1 = -|\zeta|^2 - u + iv$. Thus, z corresponds to a point $(\zeta, v, u) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}^+$:

$$\mathbf{z} = \begin{bmatrix} -|\zeta|^2 - u + iv \\ \sqrt{2}\zeta \\ 1 \end{bmatrix}.$$

Let H_u denote the set of points in $\mathbf{H}^2_{\mathbb{C}}$ with $\langle \mathbf{z}, \mathbf{z} \rangle = -2u$. This set is called the horosphere of height u. Clearly H_u carries the structure of the Heisenberg group: a point z in the Siegel domain is consequently canonically identified to $(\zeta, v, u) \in \mathfrak{H} \times \mathbb{R}^+$; we call (ζ, v, u) the horospherical coordinates of z. The set of the finite boundary points is the horosphere of height zero: $H_0 = \partial\mathbf{H}^2_{\mathbb{C}} \setminus \{\infty\} \simeq \mathfrak{H}$, where the isomorphism is

$$\mathfrak{H} \ni (\zeta, v) \mapsto (\zeta, v, 0) \in H_0.$$

Explicitly, we have the following transformations of H_u:

1. Left Heisenberg translation by (ζ', v') is given by

$$T^u_{(\zeta',v')} : (\zeta, v, u) \mapsto (\zeta + \zeta', v + v' + 2\Im(\zeta'\overline{\zeta}), u). \tag{2.1}$$

2. Rotation in an angle θ is given by

$$R^u_\theta : (\zeta, v, u) \mapsto (e^{i\theta}\zeta, v, u). \tag{2.2}$$

3. Dilation by $\delta > 0$ is given by

$$D^u_\delta : (\zeta, v, u) \mapsto (\delta\zeta, \delta^2 v, u). \tag{2.3}$$

4. Inversion I^u is given by

$$I^u : (\zeta, v, u) \mapsto (I(\zeta, v), u)\,. \tag{2.4}$$

5. Conjugation J^u is given by

$$J^u : (\zeta, v, u) \mapsto (J(\zeta, v), u)\,. \tag{2.5}$$

The group G_u comprising compositions of transformations (2.1)–(2.4) is thus a group isomorphic to $\mathrm{PU}(2,1)$. Any two points $p, q \in H_u$ may be mapped to $\infty, (0,0,u)$, respectively, by an element of G_u; therefore, G_u acts doubly transitively on H_u. The group $\overline{G_u}$ comprises elements of G_u followed by conjugation J^u and is a group isomorphic to $\overline{\mathrm{PU}(2,1)}$, that is, the group comprising elements of $\mathrm{PU}(2,1)$ followed by J.

Remark 2.1 *Two horospheres of strictly positive height u and u' may be mapped onto one another via an element of $\mathrm{PU}(2,1)$: if with no loss of generality we suppose that $u < u'$, then $D_\delta(H_u) = (H_{u'})$, where $\delta = (u')^{1/2}$. Of course, there is no element of $\mathrm{PU}(2,1)$ mapping a horosphere of positive height H_u to H_0, and $\mathrm{PU}(2,1)$ preserves V_- and V_0.*

The horoball U_u of height u is the union of all horospheres of height $t > u$; it is an open topological ball of dimension 4. The complex hyperbolic plane is thus a horoball itself, that is, the horoball U_0 of height 0.

2.4. The Cygan metric

The *Cygan metric* on $\overline{\mathbf{H}^2_{\mathbb{C}}} \setminus \{\infty\}$ is an extension of the Heisenberg metric $d_{\mathfrak{H}}$ on $\mathfrak{H}$ to an incomplete metric ρ; this is done by defining

$$\rho\left((\zeta_1, v_1, u_1), (\zeta_2, v_2, u_2)\right) = \left||\zeta_1 - \zeta_2|^2 + |u_1 - u_2| - iv_1 + iv_2 - 2i\Im(\zeta_1\overline{\zeta_2})\right|^{1/2}.$$

We stress here that this agrees with $\langle \mathbf{z}_1, \mathbf{z}_2 \rangle$ if and only if one (or both) of $\mathbf{z}_1$ or $\mathbf{z}_2$ is a null vector; that is, it corresponds to a point at the boundary. The Cygan metric ρ satisfies the following: if $p = (q, u) = (\zeta, v, u)$ and $p' = (q', u') = (\zeta', v', u')$, then:

1. $\rho(p, p') \geq 0$ and $\rho(p, p') = 0$ if and only if $p = p'$;

2. $\rho(p,p') = \rho(p',p)$.

Note also that if p, p' lie in the same horosphere H_u if and only if

$$\rho(p,p') = d_{\mathfrak{H}}\left((\zeta,v),(\zeta',v')\right),$$

it follows that when we consider the restriction ρ_u of ρ on a horosphere H_u, then the orientation-preserving isometries of ρ_u are compositions of transformations (2.1) and (2.2), and the orientation-preserving similarities are compositions of transformations (2.1), (2.2), and (2.3) and the full group of similarities comprising orientation-preserving similarities followed by conjugation (2.5).

It remains to show that ρ satisfies the triangle inequality; by showing this we shall also have a proof that the Heisenberg metric $d_{\mathfrak{H}}$ on $\mathfrak{H}$ is also a metric.

Proposition 2.2 *The function $\rho : \overline{\mathbf{H}^2_{\mathbb{C}}} \setminus \{\infty\} \times \overline{\mathbf{H}^2_{\mathbb{C}}} \setminus \{\infty\} \to \mathbb{R}^+$ satisfies the triangle inequality.*

Proof We remark first that if $p = (\zeta, v, u)$ and $q = (\zeta', v', u')$, then

$$\rho(p,q) = \rho\left((T_{(-\zeta',-v')}(\zeta,u),v),(0,0,u')\right).$$

According to the above remark it suffices to consider the points

$$p_1 = (\zeta_1, v_1, u_1), \quad p_3 = (0,0,u_3), \quad p_2 = (\zeta_2, v_2, u_2)$$

and show that

$$\rho(p_1,p_2) \leq \rho(p_1,p_3) + \rho(p_3,p_2).$$

For this, we indeed have

$$\begin{aligned}
\rho^2(p_1,p_2) &= \left||\zeta_1-\zeta_2|^2 + |u_1-u_2| - iv_1 + iv_2 - 2i\Im(\zeta_1\overline{\zeta_2})\right| \\
&\leq \left||\zeta_1|^2 + |\zeta_2|^2 - 2\Re(\zeta_1\overline{\zeta_2}) + |u_1-u_3| + |u_2-u_3| - iv_1 + iv_2 - 2i\Im(\zeta_1\overline{\zeta_2})\right| \\
&\leq \left||\zeta_1|^2 + |u_1-u_3| - iv_1\right| + 2|\zeta_1||\zeta_2| + \left||\zeta_2|^2 + |u_2-u_3| - iv_2\right| \\
&\leq \left||\zeta_1|^2 + |u_1-u_3| - iv_1\right| + 2\left||\zeta_1|^2 + |u_1-u_3| - iv_1\right|^{1/2}\left||\zeta_2|^2 + |u_2-u_3| - iv_2\right|^{1/2} \\
&\quad + \left||\zeta_2|^2 + |u_2-u_3| - iv_2\right| \\
&\leq \left(\rho(p_1,p_3) + \rho(p_2,p_3)\right)^2.
\end{aligned}$$

□

Remark 2.3 *It is quite useful to remark that in the above case the triangle inequality holds as an equality if and only if*

$$u_1 = u_2 = u_3, \quad v_1 = v_2 = 0, \quad \zeta_1, \zeta_2 \in \mathbb{R},\ \zeta_1 \cdot \zeta_2 \leq 0.$$

The group of orientation-preserving similarities Sim^+_ρ of the Cygan metric is identified to the subgroup of $\mathrm{PU}(2,1)$ comprising elements that stabilize ∞. Explicitly, the orientation-preserving Cygan isometries comprise Heisenberg isometries, that is translations

$$T_{(\zeta',v')}(\zeta,v,u) = \left(T_{(\zeta',v')}(\zeta,v),u\right),$$

rotations

$$R_\theta(\zeta, v, u) = (R_\theta(\zeta, v), u)$$

and orientation-preserving conjugation

$$J(\zeta, v, u) = (\overline{\zeta}, -v, u).$$

Notice that all the above transformations preserve horospheres and their restrictions on each horosphere H_u are T^u, R^u_θ, and J^u, respectively. Now dilations D_δ, $\delta > 0$, are given by

$$D_\delta(\zeta, v, u) = \left(D_\delta(\zeta, v), \delta^2 \cdot u\right).$$

The full group of Cygan similarities comprise orientation-preserving similarities followed by orientation-reversing conjugation:

$$(\zeta, v, u) \mapsto (\overline{\zeta}, -v, -u).$$

Again, the restriction of J on an arbitrary horosphere H_u is J^u. We finally discuss inversion I; this is given by

$$I(\zeta, v, u) = \left(\frac{\zeta}{-|\zeta|^2 + iv - u}, -\frac{v}{\left|-|\zeta|^2 + iv - u\right|^2}, \frac{u}{\left|-|\zeta|^2 + iv - u\right|^2}\right).$$

Inversion I is an involution of $\overline{\mathbf{H}^2_{\mathbb{C}}}$; notice that $I(\infty) = o$. Moreover, for all $p = (\zeta, v, u)$, $p' = (\zeta', v', u') \in \overline{\mathbf{H}^2_{\mathbb{C}}} \setminus \{o\}$ we have

$$\rho(I(p), o) = \frac{1}{\rho(p, o)}, \quad \rho(I(p), I(p')) = \frac{\rho(p, p')}{\rho(p, o)\, \rho(o, p')}.$$

The above formula can be proved by carrying out straightforward calculations; however, it follows directly by a more general statement, see for instance Prop. 2.7 in [9]. Note further that $I = I^0$ and the restriction of I onto any other horosphere H_u, $u > 0$, is not equal to I^u. Finally, we remark that inversion I fixes the unit Cygan sphere $S(0,1)$ centered at o:

$$S(0,1) = \{z = (\zeta, v, u) \ : \ \rho(z, o) = 1\}.$$

3. Ptolemaean inequality

In this section we prove the Ptolemaean inequality for the compactified complex hyperbolic plane $\overline{\mathbf{H}^2_{\mathbb{C}}}$ endowed with the Cygan metric. For the proof, we use the ρ-metric cross-ratio defined in Section 3.1. The Ptolemaean inequality is then stated and proved (Theorem 3.1) in Section 3.2.

3.1. The metric cross-ratio

Let $\overline{\mathbf{H}^2_{\mathbb{C}}}$ be the compactified hyperbolic plane; that is,

$$\overline{\mathbf{H}^2_{\mathbb{C}}} = \mathbf{H}^2_{\mathbb{C}} \cup \mathfrak{H} \cup \{\infty\}.$$

We extend the Cygan metric ρ in $\overline{\mathbf{H}^2_{\mathbb{C}}} \setminus \{\infty\}$ into a metric in $\overline{\mathbf{H}^2_{\mathbb{C}}}$, which we will again denote by ρ, by requiring

$$\rho(p, \infty) = +\infty, \ \text{ if } \ p \neq \infty, \quad \rho(\infty, \infty) = 0.$$

Denote by $\mathfrak{C}(\overline{\mathbf{H}^2_{\mathbb{C}}})$ the set of quadruples of pairwise distinct points of $\overline{\mathbf{H}^2_{\mathbb{C}}}$; that is,

$$\mathfrak{C}(\overline{\mathbf{H}^2_{\mathbb{C}}}) = \left(\overline{\mathbf{H}^2_{\mathbb{C}}}\right)^4 \setminus \{\text{diagonals}\},$$

and let $\mathfrak{p} = (p_1, p_2, \ p_3, p_4) \in \mathfrak{C}(\overline{\mathbf{H}^2_{\mathbb{C}}})$ be arbitrary. There are six distances in $(0, +\infty]$ involved:

$$\rho(p_i, p_j), \quad i, j = 1, \dots, 4, \ \ i \neq j.$$

We adopt the convention: $(+\infty) : (+\infty) = 1$, and to $\mathfrak{p}$ we associate the cross-ratio $\mathbb{X}^\rho(\mathfrak{p})$ defined by

$$\mathbb{X}^\rho(\mathfrak{p}) = \frac{\rho(p_2, p_4)}{\rho(p_2, p_3)} \cdot \frac{\rho(p_1, p_3)}{\rho(p_1, p_4)}.$$

From the discussion in Section 2.4 it follows that the cross-ratio $\mathbb{X}^\rho$ remains invariant under the diagonal action of $\mathrm{PU}(2,1)$ in $\mathfrak{C}(\overline{\mathbf{H}^2_{\mathbb{C}}})$. As a corollary we have that if p_i lie on the same horosphere H_u, then $\mathbb{X}^\rho$ is invariant under the action of G_u.

For every $i, j, k, l = 1, \dots, 4$, such that $p_i, p_j, p_k, p_l \in \overline{\mathbf{H}^2_{\mathbb{C}}}$ are pairwise disjoint, the following symmetry conditions are clearly satisfied:

(S1)

$$\mathbb{X}^\rho(p_i, p_j, p_k, p_l) = \mathbb{X}^\rho(p_j, p_i, p_l, p_k) = \mathbb{X}^\rho(p_k, p_l, p_i, p_j)$$

(notice that all the above are also equal to $\mathbb{X}^\rho(p_l, p_k, p_j, p_i)$),

(S2)

$$\mathbb{X}^\rho(p_i, p_j, p_k, p_l) \cdot \mathbb{X}^\rho(p_i, p_j, p_l, p_k) = 1,$$

(S3)

$$\mathbb{X}^\rho(p_i, p_j, p_k, p_l) \cdot \mathbb{X}^\rho(p_i, p_l, p_j, p_k) = \mathbb{X}^\rho(p_i, p_k, p_j, p_l).$$

Now let $\mathfrak{p} = (p_1, p_2, p_3, p_4) \in \mathfrak{C}(\overline{\mathbf{H}^2_{\mathbb{C}}})$ and set

$$\mathbb{X}^\rho_1(\mathfrak{p}) = \mathbb{X}^\rho(p_1, p_2, p_3, p_4), \quad \mathbb{X}^\rho_2(\mathfrak{p}) = \mathbb{X}^\rho(p_1, p_3, p_2, p_4).$$

Then due to properties (S1), (S2), and (S3), the cross-ratios of all possible permutations of points of $\mathfrak{p}$ are functions of $\mathbb{X}^\rho_1(\mathfrak{p})$ and $\mathbb{X}^\rho_2(\mathfrak{p})$.

3.2. Ptolemaean inequality

The Ptolemaean inequality for the metric space $(\overline{\mathbf{H}^2_{\mathbb{C}}}, \rho)$ can be stated as follows:

Theorem 3.1 *Let $\mathfrak{p} = (p_1, p_2, p_3, p_4) \in \mathfrak{C}(\overline{\mathbf{H}^2_{\mathbb{C}}})$ and consider the cross-ratios $\mathbb{X}_i(\mathfrak{p})$, $i = 1, 2$. Then the following inequalities hold:*

$$\mathbb{X}_1(\mathfrak{p}) + \mathbb{X}_2(\mathfrak{p}) \geq 1 \quad \textit{and} \quad |\mathbb{X}_1(\mathfrak{p}) - \mathbb{X}_2(\mathfrak{p})| \leq 1. \tag{3.1}$$

For the proof we need the following lemmas:

Lemma 3.2 *If* $\mathfrak{p} = (p_1, p_2, p_3, p_4) \in \mathfrak{C}(\mathbf{H}^2_{\mathbb{C}})$*, then there exist points* $p'_1, p'_2, p'_3 \in \overline{\mathbf{H}^2_{\mathbb{C}}}$ *such that if* $\mathfrak{p} = (p_1, p_2, p_3, p_4)$ *and* $\mathfrak{p}' = (p'_1, p'_2, p'_3, o)$*, and then*

$$\mathbb{X}^{\rho}_i(\mathfrak{p}) = \mathbb{X}^{\rho}_i(\mathfrak{p}'), \quad i = 1, 2.$$

Proof We write

$$p_i = (\zeta_1, v_i, u_i), \quad i = 1, \dots, 4.$$

With no loss of generality we may assume that $u_4 = \min\{u_i,\ i = 1, \dots, 4\}$. Then we set

$$p'_i = \left(T_{(-\zeta_4, -v_4)}(\zeta_i, v_i), u_i - u_4\right), \quad i = 1, \dots, 4.$$

Clearly, $p'_4 = o$. Now,

$$\rho(p'_i, p'_j) = \rho(p_i, p_j)$$

for all $i, j = 1, \dots 4$, $i \neq j$, and hence the lemma is proved. □

Lemma 3.3 *If* $\mathfrak{p} = (p_1, p_2, p_3, p_4) \in \mathfrak{C}(\mathbf{H}^2_{\mathbb{C}})$*, there exists a* $\mathfrak{q} = (p, q, r, \infty) \in \mathfrak{C}(\overline{\mathbf{H}^2_{\mathbb{C}}})$ *such that*

$$\mathbb{X}^{\rho}_i(\mathfrak{p}) = \mathbb{X}^{\rho}_i(\mathfrak{q}), \quad i = 1, 2.$$

Proof Given the quadruple $\mathfrak{p} = (p_1, p_2, p_3, p_4) \in \mathfrak{C}(\mathbf{H}^2_{\mathbb{C}})$, we track down the quadruple $\mathfrak{p}' = (p'_1,\ p'_2, p'_3, o)$, which we may obtain from Lemma 3.2. Applying the inversion I to points of $\mathfrak{p}'$, we have a quadruple $\mathfrak{q} = (p, q, r, \infty) \in \mathfrak{C}(\overline{\mathbf{H}^2_{\mathbb{C}}})$. □

Proof of Theorem 3.1 In the first place we consider quadruples of the form $\mathfrak{p} = (p, q, r, \infty) \in \mathfrak{C}(\overline{\mathbf{H}^2_{\mathbb{C}}})$ and we will show that Inequalities (3.1) hold for these quadruples. Notice that we do not assume any specific conditions for the points other than infinity. Now we have

$$\mathbb{X}^{\rho}_1(\mathfrak{p}) = \mathbb{X}(p, q, r, \infty) = \frac{\rho(r, p)}{\rho(r, q)}, \quad \mathbb{X}^{\rho}_2(\mathfrak{p}) = \mathbb{X}(p, r, q, \infty) = \frac{\rho(q, p)}{\rho(q, r)},$$

and the result follows from the triangle inequality.

Next, we consider an arbitrary $\mathfrak{p} = (p_1, p_2, p_3, p_4) \in \mathfrak{C}(\overline{\mathbf{H}^2_{\mathbb{C}}})$. If one or more of the points of $\mathfrak{p}$ lie on the boundary, then by applying a Heisenberg translation and inversion if necessary, we obtain a quadruple of the form $\mathfrak{p}' = (p, q, r, \infty)$ such that

$$\mathbb{X}^{\rho}_i(\mathfrak{p}) = \mathbb{X}^{\rho}_i(\mathfrak{q}), \quad i = 1, 2.$$

If none of the points of $\mathfrak{p}$ belong to the boundary, then from Lemma 3.3 there exists a quadruple of the form $\mathfrak{p}' = (p, q, r, \infty)$ such that

$$\mathbb{X}^{\rho}_i(\mathfrak{p}) = \mathbb{X}^{\rho}_i(\mathfrak{q}), \quad i = 1, 2.$$

The proof is complete. □

4. Ptolemaeus' theorem

We prove Ptolemaeus' theorem (4.3) in Section 4.2. An introductory discussion of $\mathbb{R}$-circles is given in Section 4.1.

4.1. $\mathbb{R}$-circles

An $\mathbb{R}$-circle $\mathcal{R}$ of height u is the intersection of a Lagrangian plane with a horosphere H_u, $u \geq 0$. We consider two particular $\mathbb{R}$-circles, namely the standard $\mathbb{R}$-circle of height 0 (passing through 0 and ∞),

$$\mathcal{R}^0_{\mathbb{R}} = \{(x,0,0) \in \mathfrak{H} \mid x \in \mathbb{R}\},$$

and the standard $\mathbb{R}$-circle of height 1 (passing through 0 and ∞),

$$\mathcal{R}^1_{\mathbb{R}} = \{(x,0,1) \in H_1 \mid x \in \mathbb{R}\}.$$

Infinite $\mathbb{R}$-circles, that is, $\mathbb{R}$-circles passing through infinity, are straight lines; on the other hand, finite $\mathbb{R}$-circles are more complicated curves, see for instance [5, 7]. An $\mathbb{R}$-circle $\mathcal{R}$ is homeomorphic to S^1; given four distinct points p_1, p_2, p_3, and p_4 in R, we say that a pair of these points separates the remaining pair if the elements of the latter lie in different components of the set comprising $\mathcal{R}$ minus the initial pair, e.g., p_1, p_3 separate the points p_2, p_4 if p_2 and p_4 lie in different components of $\mathcal{R} \setminus \{p_1, p_3\}$. See also Section 2.3 in [2].

Proposition 4.1 *Any $\mathbb{R}$-circle $\mathcal{R}$ may be mapped onto $\mathcal{R}^0_{\mathbb{R}}$ or $\mathcal{R}^1_{\mathbb{R}}$ by a map g in a manner so that if $\mathfrak{p} = (p_1, p_2, p_3, p_4) \in \mathfrak{C}(\mathcal{R})$ and $\mathfrak{p}' = (g(p_1), g(p_2), g(p_3), g(p_4))$, then*

$$\mathbb{X}^\rho(\mathfrak{p}) = \mathbb{X}^\rho(\mathfrak{p}').$$

Proof Suppose that $\mathcal{R}$ is an $\mathbb{R}$-circle of arbitrary height u. If it passes through ∞, then applying an element of G_u we may map it on the standard $\mathbb{R}$-circle of height u. If $u \neq 0$, by applying the dilation $D_{1/u}$ we have a map from $\mathcal{R}$ to $\mathcal{R}^1_{\mathbb{R}}$. Suppose now that our initial $\mathcal{R}$ does not pass through infinity. By applying a Heisenberg translation, we map it onto an $\mathbb{R}$-circle of height u that passes through $(0,0,u)$. Applying inversion I^u and a Heisenberg translation and a rotation R^u if necessary, we map the latter onto the standard $\mathbb{R}$-circle of height u. All the above transformations preserve the metric cross-ratio $\mathbb{X}^\rho$. The proof is complete. □

Proposition 4.2 *All $\mathbb{R}$-circles are Ptolemaean circles.*

Proof According to the previous proposition it suffices to show that the standard $\mathbb{R}$-circle of height u, where $u = 0$ or 1, is a Ptolemaean circle. Let p_i, $i = 1, 2, 3, 4$ points in the standard $\mathbb{R}$-circle of height u. We suppose first that p_1 and p_3 separate p_2 and p_4; we may assume that

$$p_1 = \infty, \quad p_2 = (x_2, 0, u), \quad p_3 = (x_3, 0, u), \quad p_4 = (0, 0, u),$$

where $x_2 > x_3 > 0$. Let $\mathfrak{p} = (p_1, p_2, p_3, p_4)$; then

$$\mathbb{X}^\rho_1(\mathfrak{p}) = \frac{x_2}{x_2 - x_3}, \quad \mathbb{X}^\rho_2(\mathfrak{p}) = \frac{x_3}{x_2 - x_3}.$$

Hence, $\mathbb{X}^\rho_1(\mathfrak{p}) - \mathbb{X}^\rho_2(\mathfrak{p}) = 1$. The cases where p_1 and p_2 separate p_3 and p_4 and p_1 and p_4 separate p_2 and p_3 are proved in an analogous manner. □

4.2. Proof of Ptolemaeus' theorem

We now state and prove Ptolemaeus' theorem.

Theorem 4.3 *A curve σ in $(\overline{\mathbf{H}^2_{\mathbb{C}}},\rho)$ is a Ptolemaean circle if and only if it is an $\mathbb{R}$-circle. Explicitly, let $\mathfrak{p}=(p_1,p_2,p_3,p_4)$ be a quadruple of pairwise distinct points lying on a Ptolemaean circle. Then:*

1. $\mathbb{X}_1^\rho(\mathfrak{p})-\mathbb{X}_2^\rho(\mathfrak{p})=1$ *if p_1 and p_3 separate p_2 and p_4;*
2. $\mathbb{X}_2^\rho(\mathfrak{p})-\mathbb{X}_1^\rho(\mathfrak{p})=1$ *if p_1 and p_2 separate p_3 and p_4;*
3. $\mathbb{X}_1^\rho(\mathfrak{p})+\mathbb{X}_2^\rho(\mathfrak{p})=1$ *if p_1 and p_4 separate p_2 and p_3.*

Proof According to Proposition 4.2 we only have to prove that if for a given quadruple $\mathfrak{p}=(p_1,p_2,p_3,p_4)\in\mathfrak{C}(\overline{\mathbf{H}^2_{\mathbb{C}}})$ we have equality in one of the Inequalities (3.1), then all points of $\mathfrak{p}$ lie on an $\mathbb{R}$-circle. For this, with no loss of generality we may suppose that the equation in question is

$$\mathbb{X}_1^\rho(\mathfrak{p})-\mathbb{X}_2^\rho(\mathfrak{p})=1,$$

and we shall distinguish two cases. First, one of the points of $\mathfrak{p}$ lies on the boundary, and second, no point of $\mathfrak{p}$ lies on the boundary. In the first case, we may assume after a Heisenberg translation and inversion if necessary that $\mathfrak{p}$ is the quadruple:

$$p_1=\infty,\quad p_2=(\zeta_2,v_2,u_2),\quad p_3=(\zeta_3,v_3,u_3),\quad p_4=(0,0,u_4).$$

Then $\mathbb{X}_1^\rho(\mathfrak{p})-\mathbb{X}_2^\rho(\mathfrak{p})=1$ reads as

$$\rho(p_4,p_2)=\rho(p_4,p_3)+\rho(p_3,p_2).$$

As in Remark 2.3, one shows that this implies

$$u_2=u_3=u_4=u,\quad v_2=v_3=0,\quad \zeta_2,\zeta_3\in\mathbb{R},\ \zeta_2\cdot\zeta_3>0,$$

and therefore p_i belong in the standard $\mathbb{R}$-circle of height u; moreover, p_1 and p_3 separate p_2 and p_4.

In the case where no point of $\mathfrak{p}$ belongs to the boundary, we may normalize so that

$$p_1=(0,0,u_1),\quad p_2=(\zeta_2,v_2,u_2),\quad p_3=(\zeta_3,v_3,u_3),\quad p_4=(\zeta_4,v_4,u_4).$$

In the case where $u_i=u$, $i=1,\dots,4$, that is, all p_i belong to the same horosphere of height u, we obtain the result by applying inversion I^u. We now claim that the possibility that p_i do not lie on the same horosphere cannot exist. Assuming the contrary, with no loss of generality we suppose that $u_1=\min\{u_i,\ i=1,\dots,4\}$ is the strict minimum of u_i; i.e. there exists at least one u_j, $j=2,3,4$, $j\neq 1$ with $u_1<u_j$. Consider then the auxiliary points

$$p_1'=o,\quad p_2'=(\zeta_2,v_2,u_2-u_1),\quad p_3'=(\zeta_3,v_3,u_3-u_1),\quad p_4'=(\zeta_4,v_4,u_4-u_1).$$

Applying inversion I, we obtain the points

$$\infty,\quad I(p_2'),\quad I(p_3'),\quad I(p_4').$$

If $I(p'_4) = (\zeta, v, u)$, applying the Heisenberg translation $T_{(-\zeta,-v)}$ we have the points

$$\infty, \quad q_2, \quad q_3, \quad (0,0,u).$$

Since the cross-ratios have remained invariant, we have that the latter points belong to the standard $\mathbb{R}$-circle of height u and $q_2 = (x_2, 0, u)$, $q_3 = (x_3, 0, u)$, and $x_2 \cdot x_3 > 0$. Moving backwards we have

$$I(p'_2) = (\zeta + x_2, v, u), \quad I(p'_2) = (\zeta + x_3, v, u), \quad I(p'_4) = (\zeta, v, u),$$

but this is possible only if $I = I^u$, a contradiction. This completes the proof. □

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