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Structural stability analysis of solutions to the initial boundary value problem for a nonlinear strongly damped wave equation

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Abstract: In this paper the initial-boundary value problem for a nonlinear strongly damped wave equation is considered. We analyze the structural stability of solutions of the nonlinear strongly damped wave equation with coefficients from $H^1(\Omega)$.

Key words: Structural stability, continuous dependence, strongly damped, nonlinear wave equation

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of $\mathbb{R}^n$ whose boundary $\partial\Omega$ is assumed to be class C^2. The model considered here is given as the following initial-boundary value problem:

$$u_{tt} - \Delta u + \beta \left|u_t\right|^2 u_t = \alpha \Delta u_t, \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega \tag{1.2}$$

$$u = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.3}$$

where α and β are positive constants.

Basically, in such a style of models, continuous dependence of solutions on the given coefficients reflects the effect of small changes on the solutions that eventually yields the structural stability result [4].

The term $\alpha\Delta u_t$ indicates that the stress is proportional not only to the strain, as with Hooke's law, but also to the strain rate as in a linearized Kelvin material [9].

Many works on strongly damped nonlinear wave equations have been carried out at different levels. In 1980, Webb [14] considered the following problem:

$$w_{tt} - \alpha\Delta w_t - \Delta w = f(w), \quad t > 0, \tag{1.4}$$

$$w(x,0) = \phi(x), \quad x \in \Omega \tag{1.5}$$

$$w_t(x,0) = \psi(x), \quad x \in \Omega \tag{1.6}$$

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$$w(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0. \tag{1.7}$$

He firstly established the existence of a unique strong global solution to (1.4) and then he analyzed the behavior of this solution as $t \to \infty$.

In [2], Massatt investigated both the existence and the limiting behavior for the equation $u_{tt} + Au_t + Au = f(t, u, u_t)$, where A is a sectoral operator and f satisfies certain regularity and growth assumptions, being periodic in t.

In [3], the authors considered the long-time behavior of a strongly damped nonlinear wave equation and showed that the initial boundary value problem has a global solution and that there exists a compact global attractor with finite dimension.

In [4], the authors investigated the existence and uniqueness of solutions of the following equation of hyperbolic type with a strong dissipation:

$$\begin{array}{l} u_{tt}(t,x) - \left(\alpha + \beta\left(\int\limits_{\Omega} |\nabla u(t,y)|^2 dy\right)^{y}\right)\Delta u(t,x) - \\ \lambda \Delta u_t(t,x) + \mu |u(t,x)|^{q-1} u(t,x) = 0, \quad x \in \Omega, \quad t \geq 0 \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \quad u|_{\partial\Omega} = 0. \end{array} \tag{1.8}$$

Then, in [6], Çelebi and Uğurlu considered the existence of a wide collection of finite sets of functionals on the phase space $H^2(0,1) \cap H_0^1(0,1)$ that completely determine the asymptotic behavior of solutions to the strongly damped nonlinear wave equation. They also showed that the asymptotic behavior of solutions is determined by the values of two sufficiently close points in the interval [0, 1].

Different conclusions were obtained in many other articles [2, 7, 10, 13, 15, 16]. References [1, 3–5, 12] can be given for more information on the structural stability result for interested readers.

In this study, our main aim is to analyze the global behavior of solutions to (1.1)–(1.3) and the structural stability of these solutions on coefficients α and β. The proof relies on energy-type a priori estimates.

Throughout this article we denote by $\|.\|$ the norm in $L^2(\Omega)$.

2. Essential inequality

Theorem 1 *For every* $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, *the solution* (u, u_t) *to (1.1)-(1.3) satisfies the following inequality:*

$$\|\nabla u_t\|^2 \leq D_1. \tag{2.1}$$

Here D_1 *is a positive constant, depending on the initial values of (1.1).*

Proof We first multiply (1.1) by $-\Delta u_t$ and integrate over Ω. Then we have

$$\frac{d}{dt}\left[\frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\Delta u\|^2\right] \leq 0. \tag{2.2}$$

It follows from (2.2) that

$$E_1(t) = \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 \leq E_1(0). \tag{2.3}$$

Hence, (2.1) follows from (2.3). □

3. Continuous dependence on coefficient α

We consider the following problems.

$$u_{tt} - \Delta u + \beta |u_t|^2 u_t = \alpha_1 \Delta u_t,\ x \in \Omega, t > 0 \tag{3.1}$$

$$u(x,0) = 0, u_t(x,0) = 0,\ x \in \Omega \tag{3.2}$$

$$u|_{\partial\Omega} = 0,\ x \in \partial\Omega, t > 0 \tag{3.3}$$

$$v_{tt} - \Delta v + \beta |v_t|^2 v_t = \alpha_2 \Delta v_t,\ x \in \Omega, t > 0 \tag{3.4}$$

$$v(x,0) = 0, v_t(x,0) = 0,\ x \in \Omega \tag{3.5}$$

$$v|_{\partial\Omega} = 0,\ x \in \partial\Omega, t > 0 \tag{3.6}$$

Assume that u is a solution of (3.1)–(3.3) and v is a solution of (3.4)–(3.6). We define the variables w and α by $w = u - v$ and $\alpha = \alpha_1 - \alpha_2$. It is easy to see that w satisfies the following initial boundary value problem:

$$w_{tt} - \Delta w + \beta \left(|u_t|^2 u_t - |v_t|^2 v_t \right) = \alpha_1 \Delta w_t + \alpha \Delta v_t,\ x \in \Omega, t > 0; \tag{3.7}$$

$$w(x,0) = 0, w_t(x,0) = 0,\ x \in \Omega; \tag{3.8}$$

$$w|_{\partial\Omega} = 0,\ x \in \partial\Omega, t > 0. \tag{3.9}$$

Theorem 2 *Let w be the solution to (3.7)–(3.9). Then w satisfies the estimate*

$$\|w_t\|^2 + \|\nabla w\|^2 \le M_1(\alpha_1 - \alpha_2)^2 t, \forall t > 0,$$

where M_1 is a positive constant and depends on the initial data and the parameters of (1.1).

Proof If we multiply (3.7) by w_t and integrate over Ω we get the relation

$$\frac{d}{dt}\left[\frac{1}{2}\|w_t\|^2 + \frac{1}{2}\|\nabla w\|^2\right] + \alpha_1 \|\nabla w_t\|^2 + \beta \int_\Omega \left(|u_t|^2 u_t - |v_t|^2 v_t \right) w_t dx +$$

$$\alpha \int_\Omega \nabla w_t \nabla v_t dx = 0. \tag{3.10}$$

It can be easily shown that

$$\left(|u_t|^2 u_t - |v_t|^2 v_t \right) w_t \ge 0. \tag{3.11}$$

Indeed,

$$\begin{aligned}\left(|u_t|^2u_t-|v_t|^2v_t\right)w_t &= \tfrac{1}{2}|u_t|^2\left(u_t-v_t+v_t\right)w_t-\tfrac{1}{2}|v_t|^2v_tw_t+\\ &\tfrac{1}{2}|u_t|^2u_tw_t+\tfrac{1}{2}|v_t|^2\left(u_t-v_t-u_t\right)w_t\\ &= \tfrac{1}{2}|u_t|^2w_tw_t+\tfrac{1}{2}|v_t|^2w_tw_t+\tfrac{1}{2}v_tw_t\left(|u_t|^2-|v_t|^2\right)+\tfrac{1}{2}u_tw_t\left(|u_t|^2-|v_t|^2\right)\\ &= \tfrac{1}{2}w_t^2\left(|u_t|^2+|v_t|^2\right)+\tfrac{1}{2}(|u_t|+|v_t|)^2(|u_t|-|v_t|)^2.\end{aligned}$$

Now using Cauchy–Schwarz and ε-Young inequalities with (3.11) we obtain from (3.10) that

$$\frac{d}{dt}E_2(t)+(\alpha_1-\frac{\varepsilon_1}{2})\left\|\nabla w_t\right\|^2\le\frac{\alpha^2}{2\varepsilon_1}\left\|\nabla v_t\right\|^2, \tag{3.12}$$

where

$$E_2(t)=\frac{1}{2}\left\|w_t\right\|^2+\frac{1}{2}\left\|\nabla w\right\|^2.$$

Taking into account (2.1) we get

$$\frac{d}{dt}E_2(t)\le\frac{|\alpha|^2}{\alpha_1}D_1, \tag{3.13}$$

which gives

$$E_2(t)\le\frac{|\alpha|^2}{\alpha_1}D_1t.$$

Hence, the statement of the theorem holds. □

4. Continuous dependence on coefficient β

We consider the following problems:

$$u_{tt}-\Delta u+\beta_1\left|u_t\right|^2u_t=\alpha\Delta u_t,\ x\in\Omega,t>0; \tag{4.1}$$

$$u(x,0)=0,u_t(x,0)=0,\ x\in\Omega; \tag{4.2}$$

$$u|_{\partial\Omega}=0,\ x\in\partial\Omega,t>0; \tag{4.3}$$

$$v_{tt}-\Delta v+\beta_2\left|v_t\right|^2v_t=\alpha\Delta v_t,\ x\in\Omega,t>0; \tag{4.4}$$

$$v(x,0)=0,v_t(x,0)=0,\ x\in\Omega; \tag{4.5}$$

$$v|_{\partial\Omega}=0,\ x\in\partial\Omega,t>0. \tag{4.6}$$

Let u be a solution of (4.1)–(4.3) and v be a solution of (4.4)–(4.6). Similar to the argument followed in the previous section, we define the variables w and β as $w=u-v$ and $\beta=\beta_1-\beta_2$. Then w satisfies the following initial boundary value problem:

$$w_{tt}-\Delta w+\beta_1\left(\left|u_t\right|^2u_t-\left|v_t\right|^2v_t\right)+\beta\left|v_t\right|^2v_t=\alpha\Delta w_t,\ x\in\Omega,t>0; \tag{4.7}$$

$$w(x,0)=0, w_t(x,0)=0,\ x\in\Omega; \tag{4.8}$$

$$w|_{\partial\Omega}=0,\ x\in\partial\Omega, t>0. \tag{4.9}$$

Now the following theorem establishes continuous dependence of the solution to (1.1)–(1.3) on the coefficient β in $H^1(\Omega)$.

Theorem 3 *Let w be the solution to (4.7)–(4.9). Then w satisfies the estimate*

$$\|w_t\|^2+\|\nabla w\|^2\le M_2(e^t-1)(\beta_1-\beta_2)^2, \forall t>0$$

where M_2 is a positive constant, depending on the parameters of (1.1).

Proof Multiplying (4.7) by w_t and integrating over Ω, we obtain

$$\frac{d}{dt}E_2(t)+\alpha\|\nabla w_t\|-2+\beta_1\int_\Omega\left(|u_t|^2u_t-|v_t|^2v_t\right)w_tdx+\beta\int_\Omega|v_t|^3w_tdx=0. \tag{4.10}$$

Using (3.11) in (4.10) we obtain

$$\frac{d}{dt}E_2(t)\le|\beta|\int_\Omega|v_t|^3|w_t|\,dx. \tag{4.11}$$

Using the Cauchy–Schwarz and the Cauchy inequalities we can estimate the term $|\beta|\int_\Omega|v_t|^3|w_t|\,dx$ as follows:

$$\begin{aligned}|\beta|\int_\Omega|v_t|^3|w_t|\,dx&\le|\beta|\left(\int_\Omega|v_t|^6dx\right)^{\frac12}\left(\int_\Omega|w_t|^2dx\right)^{\frac12}\\&\le\frac{|\beta|^2}{2}\int_\Omega|v_t|^6dx+\frac12\int_\Omega|w_t|^2dx=\frac{|\beta|^2}{2}\|v_t\|_6^6+\frac12\|w_t\|^2.\end{aligned} \tag{4.12}$$

Taking into account (4.12) in (4.11) we get

$$\frac{d}{dt}E_2(t)\le E_2(t)+\frac{|\beta|^2}{2}\|v_t\|_6^6. \tag{4.13}$$

If we use the Sobolev inequality for the second term of (4.13) and consider (2.1) we have from (4.13) that

$$\|v_t\|_6^6\le c\|\nabla v_t\|_2^6\le cD_1^3=c_1, \tag{4.14}$$

since

$$\frac{d}{dt}E_2(t)\le E_2(t)+\beta^2c_2 \tag{4.15}$$

where $c_2=\frac{c_1}{2}$. Solving the first-order differential inequality (4.15), we obtain

$$E_2(t)\le c_2(e^t-1)\beta^2,$$

which gives that $\|\nabla w\|\to0$ as $\beta\to0$, $t>0$ and hence the proof is completed. □

5. Conclusion

In this article, by using the multiplier method, we conclude that the solution of the problem (1.1)–(1.3) describing a strongly damped nonlinear wave equation is continuously dependent on the coefficients α and β.

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