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Some properties of concave operators

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Abstract: A bounded linear operator T on a Hilbert space $\mathcal{H}$ is concave if, for each $x \in \mathcal{H}$, $\|T^2x\|^2 - 2\|Tx\|^2 + \|x\|^2 \leq 0$. In this paper, it is shown that if T is a concave operator then so is every power of T. Moreover, we investigate the concavity of shift operators. Furthermore, we obtain necessary and sufficient conditions for N-supercyclicity of co-concave operators. Finally, we establish necessary and sufficient conditions for the left and right multiplications to be concave on the Hilbert–Schmidt class.

Key words: Concave operators, weighted shifts, N-supercyclicity

1. Introduction and preliminaries

Recall that a real valued function f on an interval I is *concave* if

$$f((1-t)a + tb) \geq (1-t)f(a) + tf(b)$$

whenever $a, b \in I$ and $0 \leq t \leq 1$. Clearly, f is *convex* if and only if $-f$ is concave. Moreover, a sequence $(a_n)_n$ in $\mathbb{R}$ is said to be concave if

$$a_{n+2} - 2a_{n+1} + a_n \leq 0 \quad (n = 0, 1, 2, \cdots).$$

If I is an open interval it is known that every concave function on I is continuous. Besides, every continuous function f satisfying

$$f(\frac{a+b}{2}) \geq \frac{1}{2}[f(a) + f(b)] \quad a, b \in I,$$

is concave [14]. Some more facts on concave functions run as follows:

(i) A sequence $(a_n)_n$ is concave if and only if the function $f(t)$ defined on $[0, \infty)$, which is linear on each interval $[n, n+1]$ and such that $f(n) = a_n$ $(n = 0, 1, 2, \cdots)$, is concave.

(ii) If $f(t)$ is a concave function on $[0, \infty)$, then so is the function $f(kt)$ for every $k = 1, 2, \cdots$.

(iii) A nonnegative concave function $f(t)$ on $[0, \infty)$ is nondecreasing and $\lim_{t\to\infty} f(t)^{1/t} = 1$.

(iv) A nonnegative concave function $f(t)$ on $(-\infty, \infty)$ is constant.

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Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, and let $B(\mathcal{H})$ be the space of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be *concave* if, for all $x \in \mathcal{H}$,

$$\|T^2x\|^2 - 2\|Tx\|^2 + \|x\|^2 \leq 0.$$

We remark that an operator T is concave if and only if the sequence $(\|T^nx\|^2)_{n=0}^{\infty}$ forms a concave sequence for every $x \in \mathcal{H}$. Thus, (i) and (iii) imply that for every nonzero x in $\mathcal{H}$, $\lim_{n\to\infty} \|T^nx\|^{1/n} = 1$.

The class of concave operators is closely related to the study of Brownian operators with respect to which the stochastic integral of a process with values in a separable Hilbert space has been defined. Indeed, Theorem B of [11] states that T is a concave operator with $||T||^2 \leq 2$ if and only if it extends to a Brownian operator.

It is obvious that every isometry is a concave operator. As another class of concave operators, we may consider a class of composition operators defined on a discrete measure space. Suppose that $X = \{(n,\ m) : n,\ m \in \mathbb{Z} \text{ such that } n \leq m\}$ and $(a_n)_{n=-\infty}^{\infty}$ is a sequence of positive real numbers. Let μ be the measure on the power set of X given by $\mu((n,\ n)) = 1$ for $n \in \mathbb{Z}$ and $\mu((n,\ m)) = a_n$ for $n < m$. Consider the measurable function $\varphi : X \to X$ given by $\varphi((n,\ n)) = (n-1,\ n-1)$ for $n \in \mathbb{Z}$ and $\varphi((n,\ m)) = (n,\ m-1)$ for $n < m$. Define the composition operator C_φ in $L^2(X,\ \mu)$ by $C_\varphi f = f \circ \varphi$. Then C_φ is a bounded linear operator on $L^2(X,\ \mu)$ if and only if $(a_n)_{n=-\infty}^{\infty}$ is a bounded sequence. Moreover, C_φ is concave if and only if $a_{n+1} \leq a_n$ for all integers n. Furthermore, C_φ is not unitarily equivalent to any orthogonal sum of weighted shifts or isometries; see [10, Example 4.4 and Remark 4.5]. Another class of concave operators consists of the Cauchy dual of the Bergman type operators. Note that an operator S in $B(\mathcal{H})$ is said to be of Bergman type if

$$||Sx + y||^2 \leq 2(||x||^2 + ||Sy||^2) \quad (x, y \in \mathcal{H})$$

and the operator $T = S(S^*S)^{-1}$ is called the Cauchy dual of S (see the proof of Theorem 3.6 of [13]).

In this paper, we show that if T is a concave operator then so is all of its nonnegative powers. Moreover, we give necessary and sufficient conditions under which a forward unilateral weighted shift is concave. We also show that the only concave bilateral weighted shifts are isometries.

The linear dynamics of operators is a branch of operator theory that appeared during the study of the famous invariant subset (subspace) problem. The interest in studying supercyclicity dates back to 1974 [9]. N-supercyclicity first originated in the work of Feldman [6]. Recall that for a subset E of a Hilbert space $\mathcal{H}$ and for $T \in B(\mathcal{H})$, the orbit of E under T, denoted by $orb(T, E)$, is the set $\{T^nx : n \geq 0, x \in E\}$. For any integer $n \geq 1$, the operator T is N-supercyclic if $\mathcal{H}$ has an N-dimensional subspace whose orbit under T is dense in $\mathcal{H}$. A one-supercyclic operator is called a supercyclic operator. Also, if the set E has only one element and $orb(T, E)$ is dense in $\mathcal{H}$ then T is called a hypercyclic operator. Clearly every hypercyclic operator is supercyclic and every supercyclic operator is an N-supercyclic operator, but the converses are not true [6]. Some good sources on the dynamics of operators include [1] and [8]. In this paper, we show that every concave operator is not N-supercyclic. Moreover, we obtain necessary and sufficient conditions for left and right multiplications to be concave on the Hilbert–Schmidt class of operators.

Throughout this paper, T is assumed to be a bounded linear operator on a Hilbert space $\mathcal{H}$. We begin with some easy observations. In the following result, $\mathbb{D}$ denotes the open unit disc. Also, $\sigma(T)$ and $\sigma_{ap}(T)$ are, respectively, the spectrum and the approximate point spectrum of T.

Proposition 1 *The approximate point spectrum of a concave operator T lies on the unit circle. Thus, $\sigma(T) \subset \partial\mathbb{D}$ or $\sigma(T) = \overline{\mathbb{D}}$.*

Proof Take $\lambda \in \sigma_{ap}(T)$ and suppose that $(x_n)_n$ is a sequence in $\mathcal{H}$ with $\|x_n\| = 1$ for each $n \in \mathbb{N}$ and

$$(T - \lambda I)(x_n) \to 0 \quad as\ n \to \infty.$$

Therefore,

$$\begin{aligned} |\ \|T^2 x_n\| - |\lambda^2|\ | &\leq \|T^2 x_n - \lambda^2 x_n\| \\ &\leq \|T\|\|(T-\lambda)x_n\| + |\lambda|\|(T-\lambda)x_n\| \to 0 \end{aligned}$$

as $n \to \infty$, which implies that

$$(|\lambda|^2 - 1)^2 = \lim_{n\to\infty} [\|T^2 x_n\|^2 - 2\|T x_n\|^2 + \|x_n\|^2] \leq 0.$$

Hence, $|\lambda| = 1$. Since $\partial\sigma(T) \subseteq \sigma_{ap}(T)$, we conclude that $\sigma(T) \subseteq \partial\mathbb{D}$ or $\sigma(T) = \overline{\mathbb{D}}$. □

Corollary 1 *The spectral radius of a concave operator is one.*

Corollary 2 *Concave operators are not compact.*

Proof Suppose that T is a concave operator. Since it is compact, $0 \in \sigma(T)$ and so $\overline{\mathbb{D}} \subseteq \sigma(T)$. However, this contradicts the fact that the spectrum of a compact operator is at most countable. □

2. Basic properties

Taking $\Delta_T = T^*T - I$, it is easily seen that T is a concave operator if and only if

$$T^*\Delta_T T \leq \Delta_T. \tag{1}$$

To prove that each power of every concave operator is concave, we need the following lemma. For simplicity we use Δ_n instead of Δ_{T^n} for every $n \geq 1$.

Lemma 1 *If T is a concave operator then the following inequalities hold:*

$$(T^{k+1})^*\Delta_1 T^{k+1} \leq (T^k)^*\Delta_1 T^k \quad (k = 0, 1, \cdots), \tag{2}$$

and for $n = 2, 3, \cdots$

$$(T^{n+k})^*\Delta_n T^{n+k} \leq \Delta_n \quad (k = 0, 1, \cdots). \tag{3}$$

Proof Note that (2) follows immediately from (1). Suppose that (3) holds for some n. Since $\Delta_{n+1} = T^*\Delta_n T + \Delta_1$ we can see from (3) and (2) that

$$\begin{aligned} (T^{n+1+k})^*\Delta_{n+1}T^{n+1+k} &= T^*\{(T^{n+k+1})^*\Delta_n T^{n+k+1}\}T + (T^{n+k+1})^*\Delta_1 T^{n+k+1} \\ &\leq T^*\Delta_n T + \Delta_1 = \Delta_{n+1}, \end{aligned}$$

completing induction. □

Theorem 1 *If T is concave then $\Delta_T \geq 0$; that is, $\|Tx\| \geq \|x\|$ for every $x \in \mathcal{H}$. Furthermore, T^n is concave for all $n \geq 2$.*

Proof It follows from (2) that

$$n\Delta_T \geq \sum_{k=1}^{n}(T^k)^*\Delta_T T^k = (T^{n+1})^*T^{n+1} - T^*T \geq -T^*T \quad (n = 1, 2, \cdots).$$

Hence,

$$\Delta_T \geq \lim_{n\to\infty} \frac{-1}{n} T^*T = 0.$$

Finally, (3) with $k = 0$ means that T^n is concave. □

Theorem 2 *A concave operator T with $\ker(T^*) = \{0\}$ is unitary.*

Proof The assumption $\ker(T^*) = \{0\}$ means that $\mathrm{ran}(T)$ is dense in $\mathcal{H}$. This coupled with the property $\|Tx\| \geq \|x\|$ $(x \in \mathcal{H})$ implies that T is invertible. Then, since

$$\Delta_{T^{-1}} - (T^{-1})^*\Delta_{T^{-1}}T^{-1} = (T^{-2})^*\big\{\Delta_T - T^*\Delta_T T\big\}T^{-2} \geq 0, \tag{4}$$

we can conclude that T^{-1} is concave, and hence $\|T^{-1}x\| \geq \|x\|$ $(x \in \mathcal{H})$. Combined with the property that $\|Tx\| \geq \|x\|$ $(x \in \mathcal{H})$ we conclude that T is unitary.

□

Corollary 3 *Every concave operator on a finite-dimensional Hilbert space is unitary.*

Proof By finite dimensionality and Theorem 1, $\ker T^* = \ker T = \{0\}$. □

Recall that an operator T is called co-concave if T^* is concave.

Corollary 4 *A concave operator T is unitary if T is co-concave or T is normal.*

Proof If T^* is concave, $\ker T^* = \{0\}$. If T is normal, $\ker T^* = \ker T = \{0\}$. □

Theorem 3 *Suppose that T is a concave operator and $\mathcal{M}$ is a closed T-invariant subspace. Then the restriction $T|_{\mathcal{M}}$ is concave. Furthermore, if $\dim(\mathcal{M}) < \infty$, then $\mathcal{M}$ reduces T.*

Proof The first assertion is trivial. Write

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

according to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Then, by concavity of T,

$$0 \leq \Delta_T = \begin{pmatrix} T_{11}^*T_{11} - I_{\mathcal{M}} & T_{11}^*T_{12} \\ T_{12}^*T_{11} & T_{12}^*T_{12} + T_{22}^*T_{22} - I_{\mathcal{M}^\perp} \end{pmatrix}.$$

When $\dim\mathcal{M} < \infty$, by Corollary 3, T_{11} is unitary and consequently

$$0 \leq \begin{pmatrix} 0 & T_{11}^* T_{12} \\ T_{12}^* T_{11} & T_{12}^* T_{12} + T_{22}^* T_{22} - I_{\mathcal{M}^\perp} \end{pmatrix}.$$

Positivity of this block matrix implies that

$$\langle (T_{12}^* T_{12} + T_{22}^* T_{22} - I_{\mathcal{M}^\perp})g, g\rangle \ \geq -2Re\langle T_{12}^* T_{11} h, g\rangle$$

for all $h, g \in \mathcal{H}$. Thus, $T_{11}^* T_{12} = 0$ and hence $T_{12} = 0$. This means that $\mathcal{M}$ reduces T.

□

To prove the next result, we use the Berberian construction [2] [15].

Proposition 2 (Lemma 2.7 of [15]) *Let $\mathcal{H}$ be a complex Hilbert space. Then there exists a Hilbert space $\mathcal{R} \supseteq \mathcal{H}$ and a unital linear map $\Pi : B(\mathcal{H}) \to B(\mathcal{R})$ such that:* (a) $\Pi(ST) = \Pi(S)\Pi(T)$, $\Pi(T^*) = (\Pi(T))^*$, $\|\Pi(T)\| = \|T\|$;
(b) $S \leq T \Longrightarrow \Pi(S) \leq \Pi(T)$;
(c) $\sigma(\Pi(T)) = \sigma(T)$, $\sigma_{ap}(\Pi(T)) = \sigma_{ap}(T) = \sigma_p(\Pi(T))$.

Corollary 5 *For a concave operator T the following statements hold.*

(a) *Every eigenvalue of T is a normal eigenvalue; that is, $Ta = \zeta a$ implies $T^* a = \overline{\zeta} a$.*

(b) *If $\zeta \in \sigma_{ap}(T)$ then $\overline{\zeta} \in \sigma_{ap}(T^*)$.*

Proof (a) Since $\mathcal{M} = \mathbb{C}a$ is a one-dimensional invariant subspace of T, by Theorem 3 it reduces T, which implies that $T^* a = \overline{\zeta} a$.
(b) Suppose that $\zeta \in \sigma_{ap}(T) = \sigma_p(\Pi(T))$. Since $\Pi(T)$ is a concave operator, by applying (a), we see that $\overline{\zeta} \in \sigma_p((\Pi(T))^*) = \sigma_p(\Pi((T^*))) = \sigma_{ap}(T^*)$. □

3. The concavity of shifts operators

An operator $T \in B(\mathcal{H})$ is called a forward unilateral (bilateral) weighted shift if there is an orthonormal basis $\{e_n : n \geq 0\}\big(\{e_n : n \in \mathbb{Z}\}\big)$ and a sequence of bounded complex numbers $\{w_n : n \geq 0\}\big(\{w_n : n \in \mathbb{Z}\}\big)$ such that $Te_n = w_n e_{n+1}$ for all $n \geq 0$ $(n \in \mathbb{Z})$. It is known that a weighted shift operator T is unitarily equivalent to a weighted shift operator with a nonnegative weight sequence. We can assume that $w_n \geq 0$ for all n (see [5], page 53). In addition, T is injective if and only if $w_n > 0$ for every n. Recall that the adjoint of T is called a backward unilateral (bilateral) shift. It is also known that T is an isometry if and only if $w_n = 1$ for all n.

Let $w_n = \sqrt{\frac{2^n+1}{2^n}}$ and $Te_n = w_n e_{n+1}$ for every $n \geq 0$. Then T is a concave forward weighted shift operator, due to

$$\|T^2 e_n\|^2 - 2\|Te_n\|^2 + 1 = \frac{1 - 2^n}{2^{2n} + 1} \leq 0.$$

As another example of such operators, take $w_o = \sqrt{2}$ and $w_n = 1$ for $n \geq 1$.

In spite of the above examples, the only concave bilateral weighted shifts are unitaries. Thanks to the fact that the kernel of such an operator is $\{0\}$, all weights are positive, which in turn implies that the kernel of its adjoint is $\{0\}$.

In the next result, we give a necessary and sufficient condition for a unilateral forward weighted shift to be concave.

Proposition 3 *A unilateral forward weighted shift with weight sequence $(w_n)_n$ is a concave operator if and only if*

$$1 \leq w_0 \;\; \text{and} \;\; 1 \leq w_{n+1} \leq \sqrt{2 - w_n^{-2}} \quad (n = 0, 1, 2, \cdots). \tag{5}$$

Moreover, in this case $(w_n)_n$ is decreasing and converges to 1.

Proof Let T be a unilateral forward weighted shift with weight sequence $(w_n)_n$. If T is concave then $w_n = \|Te_n\| \geq 1$ for all $n \geq 0$. Now the proof follows from the equality

$$\begin{aligned} \|T^2 e_n\|^2 - 2\|Te_n\|^2 + \|e_n\|^2 &= w_n^2 w_{n+1}^2 - 2w_n^2 + 1 \\ &= w_n^2 (w_{n+1}^2 - (2 - w_n^{-2})). \end{aligned}$$

On the other hand, since $\sqrt{2 - w_n^{-2}} \leq w_n$, the sequence w_n is decreasing; thus, (5) implies that $\lim_{n\to\infty} w_n = 1$. □

Note that since every concave operator is injective, there is not any concave backward unilateral weighted shift operator.

4. N-Supercyclicity of concave operators

Proposition 4 *No concave operator is N-supercyclic.*

Proof Take a concave operator $T \in B(\mathcal{H})$. Assume, on the contrary, that there exists a subspace E of $\mathcal{H}$, of dimension N, such that $\overline{orb(T, E)} = \mathcal{H}$. The subspace E has an empty interior because $E \neq \mathcal{H}$. Moreover,

$$\mathcal{H} = \overline{orb(T, E)} = \overline{E} \cup (\overline{\cup_{n=1}^{\infty} T^n E}),$$

which implies that $\mathcal{H} = \overline{\cup_{n=1}^{\infty} T^n E}$. Hence, T must have a dense range. Then the operator T is invertible. Thus, applying (4), we see that T^{-1} is also a concave operator. Therefore, $\|T^{-1}x\| \geq \|x\|$ for all $x \in \mathcal{H}$. Thus,

$$\|Tx\| \geq \|x\| = \|T^{-1}Tx\| \geq \|Tx\|.$$

Hence, T is a unitary operator that cannot be N-supercyclic (see Theorem 4.9 of [6], and see also [3]). □

Theorem 4 *Suppose that $T \in B(\mathcal{H})$ is a co-concave, N-supercyclic operator. Then $\cap_{n\geq 0} T^{*^n} \mathcal{H} = (0)$.*

Proof Put $M = \cap_{n\geq 0} T^{*^n} \mathcal{H}$. Clearly M is an invariant subspace of T^* and also of T. Indeed, since T^* is bounded below, TT^* is invertible. Let $S = (TT^*)^{-1}T$. Thus, for every $x \in \mathcal{H}$,

$$\|T^*Sx\|^2 = \langle S^*TT^*Sx, x\rangle = \langle S^*Tx, x\rangle = \langle x, T^*Sx\rangle \leq \|x\|\|T^*Sx\|,$$

which implies that $\|T^*Sx\| \le \|x\|$. However, since $\|Sx\| \le \|T^*Sx\|$, we conclude that $\|Sx\| \le \|x\|$ for all $x \in \mathcal{H}$. Moreover, for every nonnegative integer n, each $x \in M$ can be written as $x = {T^*}^n x_n$ for some $x_n \in \mathcal{H}$ and so $Sx = {T^*}^{n-1} x_n \in M$. Hence, $SM \subseteq M$.

On the other hand, if $x \in M$, then $x = {T^*}^2 y$ for some $y \in \mathcal{H}$. Thus,

$$\|S^2x\|^2 - 2\|Sx\|^2 + \|x\|^2 = \|y\|^2 - 2\|T^*y\|^2 + \|{T^*}^2 y\|^2 \le 0,$$

which states that the operator $S : M \longrightarrow M$ is a concave operator. Thus, if $x \in M$ then $\|Sx\| \ge \|x\|$; hence, $\|Sx\| = \|x\|$. Moreover, since $ST^* = I$, the operator $S : M \to M$ is onto, and it is also injective, so $ST^*x = T^*Sx = x$. Furthermore,

$$\|x\| = \|ST^*x\| = \|T^*x\|,$$

which implies that $TT^*x = x$. Consequently, $Tx = T(T^*Sx) = Sx \in M$; i.e. $TM \subseteq M$. Moreover, we deduce that $T : M \to M$ is in fact a unitary operator.

In continuation, we argue by contradiction and we assume that the subspace M is nonzero. We also suppose that there exists an N-dimensional subspace E of $\mathcal{H}$ such that $orb(T, E)$ is dense in $\mathcal{H}$. Let $(h_1, ..., h_N)$ be a basis of E and suppose that $h_i = g_i \oplus k_i$, $1 \le i \le N$ where $g_i \in M$ and $k_i \in M^\perp$. If $g_i = 0$ for all i then $\mathcal{H} = M^\perp$, which is impossible, so $g_i \ne 0$ for some i. Take $f \in M$, and let $\epsilon > 0$ be arbitrary. Then there are $n \ge 0$ and $\alpha_1, ..., \alpha_N$ in $\mathbb{C}$ such that

$$\|\sum_{i=1}^{N} \alpha_i T^n g_i - f\| \le \|\sum_{i=1}^{N} \alpha_i T^n (g_i \oplus k_i) - f \oplus 0\| < \epsilon.$$

Thus, taking $F = span\{g_1, ..., g_N\}$, we see that $\overline{orb(T|_M, F)} = M$. Therefore, $T|_M$ is an N-supercyclic unitary operator and this is absurd. □

As can be derived from the proof of Theorem 4, for a co-concave operator T, if $M := \cap_{n\ge0} {T^*}^n \mathcal{H}$, then $T : M \longrightarrow M$ is an isometry. Considering the fact that isometries have nontrivial invariant subspaces [7], we obtain the following corollary.

Corollary 6 *Suppose that $T \in B(\mathcal{H})$ is a co-concave operator such that $\cap_{n\ge0} {T^*}^n \mathcal{H} \ne (0)$. Then T has a nontrivial invariant subspace.*

To prove the next theorem, we need the supercyclicity criterion due to Salas [12].

Theorem 5 (Supercyclicity criterion.) *Suppose that X is a separable Banach space and T is a bounded operator on X. If there is an increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ and two dense sets $D_1, D_2 \subseteq X$ such that*

(1) *there exists a function $S : D_2 \to D_2$ satisfying $TSx = x$ for all $x \in D_2$,*
(2) *$\|T^{n_k}x\|.\|S^{n_k}y\| \to 0$ for every $x \in D_1$ and $y \in D_2$,*
then T is supercyclic.

Theorem 6 *Suppose that T is a co-concave operator such that $\cap_{n\ge0} {T^*}^n \mathcal{H} = (0)$; then T satisfies the supercyclicity criterion.*

Proof Since T^* is bounded below it is left invertible and so T is right invertible. Therefore, it admits a complete set of eigenvectors. Thus, if for every positive real number r, we denote $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, then $\mathcal{H} = \bigvee_{\mu \in \mathbb{D}_r} ker(T-\mu)$ (see [4], part (A) of the lemma). Let $S = T^*(TT^*)^{-1}$ and choose $r > 0$ so that $r < \frac{1}{\|S\|}$, and take

$$D_1 = D_2 = span\{ker(T-\mu) : \mu \in \mathbb{D}_r\}.$$

Now, if $x \in D_1 = D_2$, then

$$\|T^n x\| \|S^n x\| \leq |\mu|^n \|S\|^n \|x\| \leq (r\|S\|)^n \|x\| \to 0$$

as $n \to \infty$. Finally, $T^n S^n x = x$ for every $x \in \mathcal{H}$ and every $n \geq 0$. Hence, the operator T satisfies the supercyclicity criterion. □

Two direct consequences of the above theorem run as follows:

Corollary 7 *If T is a co-concave operator in $B(\mathcal{H})$ then T is supercyclic if and only if $\cap_{n\geq 0} T^{*^n}\mathcal{H} = (0)$.*

Corollary 8 *A co-concave operator is supercyclic if and only if it is N-supercyclic.*

Now, as an application of the above result, we present an example. Recall that the Dirichlet space $\mathcal{D}$ is the set of all functions analytic on the open unit disc $\mathbb{D}$ for which

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty,$$

where $dA(z)$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. The inner product on $\mathcal{D}$, which makes it into a Hilbert space, is defined by

$$\langle f, g \rangle = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} dA(z).$$

Thus, the associated norm of a function f in $\mathcal{D}$ is given by

$$||f||_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z).$$

Example 1 *Let M_z be the multiplication operator by the independent variable z on Dirichlet space $\mathcal{D}$ defined by $(M_z f)(\zeta) = \zeta f(\zeta)$, $\zeta \in \mathbb{D}$. If $f_n(\zeta) = \zeta^n$, $n = 0, 1, 2, \ldots$ then it is easily seen that*

$$||M_z^2 f_n||_{\mathcal{D}}^2 - 2||M_z f_n||_{\mathcal{D}}^2 + ||f_n||_{\mathcal{D}}^2 = 0,$$

which implies that M_z is a concave operator on $\mathcal{D}$. Moreover, $\cap_{n\geq 0} M_z^n \mathcal{D} = (0)$. Hence, $T = M_z^*$ is supercyclic on $\mathcal{D}$.

5. Concave operators on the Hilbert–Schmidt class

The Hilbert–Schmidt class, $C_2(\mathcal{H})$, is the class of all bounded operators S defined on a Hilbert space $\mathcal{H}$, satisfying

$$\|S\|_2^2 = \sum_{n=1}^{\infty} \|Se_n\|^2 < \infty,$$

where $\|.\|$ is the norm on $\mathcal{H}$ induced by its inner product. We recall that $C_2(\mathcal{H})$ is a Hilbert space equipped with the inner product defined by $\langle S, T\rangle = tr(T^*S)$ in which $tr(T^*S)$ denotes the trace of T^*S. Furthermore, $C_2(\mathcal{H})$ is an ideal of the algebra of all bounded operators on $\mathcal{H}$. Besides, the Hilbert–Schmidt class contains the finite rank operators as a dense linear manifold [5].

For any bounded operator T on a Hilbert space $\mathcal{H}$, the left multiplication operator L_T and the right multiplication operator R_T on $C_2(\mathcal{H})$ are defined by $L_T(S) = TS$ and $R_T(S) = ST$ for every $S \in C_2(\mathcal{H})$. Moreover, $L_T^* = L_{T^*}$ and $R_T^* = R_{T^*}$. In the next theorem, we see the relation between concavity of the operators T, L_T, and R_T.

Theorem 7 *Suppose that $\mathcal{H}$ is a Hilbert space and $T \in B(\mathcal{H})$. Then the following statements are equivalent:*

(a) T *is concave.*

(b) L_T *is concave.*

(c) R_T *is co-concave.*

Proof Observe that (a) and (b) are equivalent, thanks to the fact that $T \longmapsto L_T$ is a C^*-(into) isomorphism and $T \geq 0$ iff $L_T \geq 0$. Indeed,

$$\Delta_{L_T} - L_T^*\Delta_{L_T}L_T = L_{\Delta_T - T^*\Delta_T T}.$$

Now, suppose that L_T is concave. Taking into account that $(R_T)^* = R_{T^*}$, we will show that the operator R_{T^*} is concave. Let $S \in C_2(\mathcal{H})$. Then

$$\|R_{T^*}^2(S)\|_2 = \|T^2S^*\|_2 = \|L_T^2S^*\|_2.$$

Similarly, $\|R_{T^*}(S)\|_2 = \|L_TS^*\|_2$. Hence:

$$\|R_{T^*}^2(S)\|_2^2 - 2\|R_{T^*}(S)\|_2^2 + \|S\|_2^2 = \|L_T^2(S^*)\|_2^2 - 2\|L_T(S^*)\|_2^2 + \|S^*\|_2^2 \leq 0.$$

Thus, R_{T^*} is concave.

At last, suppose that R_T is co-concave. Taking $S \in C_2(\mathcal{H})$, we observe that

$$\begin{aligned}\sum_{n=1}^{\infty}[\|T^2S^*e_n\|^2 - 2\|TS^*e_n\|^2 + \|S^*e_n\|^2] &= \|T^2S^*\|_2^2 - 2\|TS^*\|_2^2 + \|S^*\|_2^2 \\ &= \|R_{T^*}^2(S)\|_2^2 - 2\|R_{T^*}(S)\|_2^2 + \|S\|_2^2 \leq 0.\end{aligned}$$

Now, for $h \in \mathcal{H}$, let S_k be the rank one operator defined by

$$S_kf = \langle f, h\rangle e_k.$$

Then

$$\|T^2h\|^2 - 2\|Th\|^2 + \|h\|^2 = \sum_{n=1}^{\infty}[\|T^2S_k^*e_n\|^2 - 2\|T^2S_k^*e_n\|^2 + \|S_k^*e_n\|^2] \leq 0,$$

which implies that T is a concave operator. □

It follows from Theorem 7 and Proposition 4 that the left multiplication operator of a concave operator is not N-supercyclic. However, as we are going to see in the next example, its right multiplication operator may be supercyclic.

Example 2 *Let T be a concave unilateral weighted shift operator defined by $Te_n = w_ne_{n+1}$, $n \geq 1$. If $S \in \cap_{n\geq 0}(R_T^*)^n(C_2(\mathcal{H}))$ then there is a sequence $(S_n)_{n\geq 1}$ of operators in $C_2(\mathcal{H})$ such that $S = S_nT^{*^n}$ for each $n \in \mathbb{N}$, but $T^{*^n}e_n = 0$ for all $n \geq 1$ and so $S \equiv 0$. Now Corollary 7 implies that R_T is supercyclic.*

We remark that if T is a concave bilateral weighted shift then $Te_n = e_{n+1}$ for each $n \in \mathbb{Z}$; thus, if $S \in C_2(\mathcal{H})$, then

$$\|R_TS\|_2^2 = \sum_{n\in\mathbb{Z}}\|STe_n\|^2 = \sum_{n\in\mathbb{Z}}\|Se_n\|^2 = \|S\|_2^2.$$

Similarly, $\|R_T^*S\|_2 = \|S\|_2$. Hence, R_T is a unitary operator that is not N-supercyclic.

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