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Coefficient estimates for general subclasses of m-fold symmetric analytic bi-univalent functions

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Abstract: In this work, we introduce and investigate two new subclasses of the bi-univalent functions in which both f and f^{-1} are m-fold symmetric analytic functions. For functions in each of the subclasses introduced in this paper, we obtain the coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$.

Key words: Analytic functions, univalent functions, bi-univalent functions, m-fold symmetric bi-univalent functions

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ that are univalent in $\mathbb{U}$.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

The inverse function $g = f^{-1}$ is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). The class of analytic bi-univalent functions was first

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introduced and studied by Lewin [16], where it was proved that $|a_2| < 1.51$. Brannan and Clunie [2] improved Lewin's result to $|a_2| \leq \sqrt{2}$ and later Netanyahu [18] proved that $|a_2| \leq 4/3$. Brannan and Taha [3] and Taha [24] also investigated certain subclasses of bi-univalent functions and found nonsharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. For a brief history and interesting examples of functions in the class Σ, see [21] (see also [3]). The aforecited work of Srivastava et al. [21] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Frasin and Aouf [12], Xu et al. [25, 26], Hayami and Owa [15], and others (see, for example, [1, 4–10, 13, 17, 19, 20]).

Let $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$. A domain E is said to be m-fold symmetric if a rotation of E about the origin through an angle $2\pi/m$ carries E on itself. It follows that a function $f(z)$ analytic in $\mathbb{U}$ is said to be m-fold symmetric $(m \in \mathbb{N})$ if

$$f\left(e^{2\pi i/m} z\right) = e^{2\pi i/m} f(z).$$

In particular, every $f(z)$ is 1-fold symmetric and every odd $f(z)$ is 2-fold symmetric. We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$.

A simple argument shows that $f \in \mathcal{S}_m$ is characterized by having a power series of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in \mathbb{U},\ m \in \mathbb{N}). \tag{3}$$

Srivastava et al. [22] defined m-fold symmetric bi-univalent functions, analogues to the concept of m-fold symmetric univalent functions. For the normalized form of f given by (3), they obtained the series expansion for f^{-1} as follows:

$$\begin{aligned} g(w) &= f^{-1}(w) = w - a_{m+1} w^{m+1} + \left[(m+1) a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ &\quad - \left[\frac{1}{2}(m+1)(3m+2) a_{m+1}^3 - (3m+2) a_{m+1} a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots. \end{aligned} \tag{4}$$

We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$. For $m = 1$, the formula (4) coincides with the formula (2) of the class Σ. For some examples of m-fold symmetric bi-univalent functions, see [22].

The object of the present paper is to introduce two new general subclasses of bi-univalent functions in which both f and f^{-1} are m-fold symmetric analytic functions and to obtain coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

In order to establish our main results, we shall require the following lemma.

Lemma 1 *[11] If $p \in \mathcal{P}$, then $|c_k| \leq 2$ $(k \in \mathbb{N})$, where the Carathéodory class $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which*

$$\Re(p(z)) > 0, \quad p(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

2. Coefficient estimates for the function class $\mathcal{N}^{\mu}_{\Sigma,m}(\alpha,\lambda)$

Definition 2 *For $\lambda \geq 1$ and $\mu \geq 0$, a function $f \in \Sigma_m$ given by (3) is said to be in the class $\mathcal{N}^{\mu}_{\Sigma,m}(\alpha,\lambda)$ if the following conditions are satisfied:*

$$\left|\arg\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)\right| < \frac{\alpha\pi}{2} \tag{5}$$

and

$$\left|\arg\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right)\right| < \frac{\alpha\pi}{2} \tag{6}$$

where $0 < \alpha \leq 1$; $m \in \mathbb{N}$; $z, w \in \mathbb{U}$; and $g = f^{-1}$ is defined by (4).

Remark 3 *In the following special cases of Definition 2, we show how the class of analytic bi-univalent functions $\mathcal{N}^{\mu}_{\Sigma,m}(\alpha,\lambda)$ for suitable choices of λ, μ, and m lead to certain new as well as known classes of analytic bi-univalent functions studied earlier in the literature.*

(i) *For $\mu = 1$, we obtain the m-fold symmetric bi-univalent function class*

$$\mathcal{N}^{1}_{\Sigma,m}(\alpha,\lambda) = \mathcal{A}^{\alpha,\lambda}_{\Sigma,m}$$

introduced by Sümer Eker [23]. In addition, for $m = 1$ we have the bi-univalent function class

$$\mathcal{N}^{1}_{\Sigma,1}(\alpha,\lambda) = \mathcal{B}_{\Sigma}(\alpha,\lambda)$$

introduced by Frasin and Aouf [12].

(ii) *For $\mu = 1$ and $\lambda = 1$, we have the m-fold symmetric bi-univalent function class*

$$\mathcal{N}^{1}_{\Sigma,m}(\alpha,1) = \mathcal{H}^{\alpha}_{\Sigma,m}$$

introduced by Srivastava et al. [22]. In addition, for $m = 1$ we have the bi-univalent function class

$$\mathcal{N}^{1}_{\Sigma,1}(\alpha,1) = \mathcal{H}^{\alpha}_{\Sigma}$$

introduced by Srivastava et al. [21].

(iii) *For $\mu = 0$ and $\lambda = 1$, we get a new class*

$$\mathcal{N}^{0}_{\Sigma,m}(\alpha,1) = \mathcal{S}^{\alpha}_{\Sigma,m}$$

of m-fold symmetric strongly bi-starlike functions of order α. In addition, for $m = 1$ we have the strongly bi-starlike function class

$$\mathcal{N}^{0}_{\Sigma,1}(\alpha,1) = \mathcal{S}^{*}_{\Sigma}[\alpha]$$

introduced by Brannan and Taha [3].

(iv) *For $\lambda = 1$, we have a new class*

$$\mathcal{N}^{\mu}_{\Sigma,m}(\alpha,1) = \mathcal{P}_{\Sigma,m}(\alpha,\mu),$$

which consists of m*-fold symmetric bi-Bazilevič functions.*
(v) *For* $m=1,$ *we have the bi-univalent function class*

$$\mathcal{N}_{\Sigma,1}^{\mu}\left(\alpha,\lambda\right)=\mathcal{N}_{\Sigma}^{\mu}\left(\alpha,\lambda\right)$$

introduced by Çağlar et al. [8].

Theorem 4 *Let the function* $f\left(z\right)$ *given by (3) be in the class* $\mathcal{N}_{\Sigma,m}^{\mu}\left(\alpha,\lambda\right).$ *Then*

$$\left|a_{m+1}\right|\leq\begin{cases}\frac{2\alpha}{\sqrt{(\mu+m\lambda)^{2}+m\alpha(\mu+2m\lambda-m\lambda^{2})}} & , \quad 1\leq\lambda<1+\sqrt{\frac{\mu+m}{m}}\\ \frac{2\alpha}{\mu+m\lambda} & , \quad \lambda\geq1+\sqrt{\frac{\mu+m}{m}}\end{cases} \tag{7}$$

and

$$\left|a_{2m+1}\right|\leq\frac{2\left(m+1\right)\alpha^{2}}{\left(\mu+m\lambda\right)^{2}}+\frac{2\alpha}{\mu+2m\lambda}. \tag{8}$$

Proof It follows from (5) and (6) that

$$\left(1-\lambda\right)\left(\frac{f\left(z\right)}{z}\right)^{\mu}+\lambda f'\left(z\right)\left(\frac{f\left(z\right)}{z}\right)^{\mu-1}=\left[p\left(z\right)\right]^{\alpha} \tag{9}$$

and

$$\left(1-\lambda\right)\left(\frac{g\left(w\right)}{w}\right)^{\mu}+\lambda g'\left(w\right)\left(\frac{g\left(w\right)}{w}\right)^{\mu-1}=\left[q\left(w\right)\right]^{\alpha}, \tag{10}$$

where

$$p\left(z\right)=1+p_{m}z^{m}+p_{2m}z^{2m}+p_{3m}z^{3m}+\cdots\in\mathcal{P} \tag{11}$$

and

$$q\left(w\right)=1+q_{m}w^{m}+q_{2m}w^{2m}+q_{3m}w^{3m}+\cdots\in\mathcal{P}. \tag{12}$$

Now, equating the coefficients in (9) and (10), we have

$$\left(\mu+m\lambda\right)a_{m+1} = \alpha p_{m} \tag{13}$$

$$\left(\mu+2m\lambda\right)\left[\frac{\mu-1}{2}a_{m+1}^{2}+a_{2m+1}\right] = \alpha p_{2m}+\frac{\alpha\left(\alpha-1\right)}{2}p_{m}^{2} \tag{14}$$

$$-\left(\mu+m\lambda\right)a_{m+1} = \alpha q_{m} \tag{15}$$

$$\left(\mu+2m\lambda\right)\left[\left(m+\frac{\mu+1}{2}\right)a_{m+1}^{2}-a_{2m+1}\right] = \alpha q_{2m}+\frac{\alpha\left(\alpha-1\right)}{2}q_{m}^{2}. \tag{16}$$

From (13) and (15), we obtain

$$p_{m}=-q_{m} \tag{17}$$

and

$$2\left(\mu+m\lambda\right)^{2}a_{m+1}^{2}=\alpha^{2}\left(p_{m}^{2}+q_{m}^{2}\right). \tag{18}$$

Also, from (14), (16), and (18), we find that

$$a_{m+1}^2 = \frac{\alpha^2 (p_{2m} + q_{2m})}{(\mu + m\lambda)^2 + m\alpha(\mu + 2m\lambda - m\lambda^2)}. \tag{19}$$

Applying Lemma 1 for (13) and (19), we get the desired estimate on the coefficient $|a_{m+1}|$ as asserted in (7).

Next, in order to find the bound on the coefficient $|a_{2m+1}|$, we subtract (16) from (14). Observing (17) we get

$$a_{2m+1} = \frac{m+1}{2} a_{m+1}^2 + \frac{\alpha (p_{2m} - q_{2m})}{2(\mu + 2m\lambda)}. \tag{20}$$

It follows from (13) and (20) that

$$a_{2m+1} = \frac{(m+1)\alpha^2 p_m^2}{2(\mu + m\lambda)^2} + \frac{\alpha (p_{2m} - q_{2m})}{2(\mu + 2m\lambda)}. \tag{21}$$

Applying Lemma 1 for (21), we get the desired estimate on the coefficient $|a_{2m+1}|$ as asserted in (8). This completes the proof of Theorem 4. □

By setting $\mu = 1$ in Theorem 4, we obtain the following consequence.

Corollary 5 *Let the function $f(z)$ given by (3) be in the class $\mathcal{A}_{\Sigma,m}^{\alpha,\lambda}$. Then*

$$|a_{m+1}| \leq \begin{cases} \frac{2\alpha}{\sqrt{(1+m\lambda)^2 + m\alpha(1+2m\lambda - m\lambda^2)}} & , \quad 1 \leq \lambda < 1 + \sqrt{\frac{m+1}{m}} \\ \frac{2\alpha}{1+m\lambda} & , \quad \lambda \geq 1 + \sqrt{\frac{m+1}{m}} \end{cases} \tag{22}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)\alpha^2}{(1+m\lambda)^2} + \frac{2\alpha}{1+2m\lambda}.$$

Remark 6 *The above estimates for $|a_{m+1}|$ show that the inequality (22) is an improvement of the estimates obtained by Sümer Eker [23].*

Corollary 7 *(see [23]) Let the function $f(z)$ given by (3) be in the class $\mathcal{A}_{\Sigma,m}^{\alpha,\lambda}$. Then*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(1+m\lambda)^2 + m\alpha(1+2m\lambda - m\lambda^2)}}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)\alpha^2}{(1+m\lambda)^2} + \frac{2\alpha}{1+2m\lambda}.$$

By setting $\mu = 1$ and $\lambda = 1$ in Theorem 4, we obtain the following consequence.

Corollary 8 *(see [22]) Let the function* $f(z)$ *given by* (3) *be in the class* $\mathcal{H}_{\Sigma,m}^{\alpha}$*. Then*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(1+m)(1+m+m\alpha)}}$$

and

$$|a_{2m+1}| \leq \frac{2\alpha^2}{1+m} + \frac{2\alpha}{1+2m}.$$

By setting $\mu = 0$ and $\lambda = 1$ in Theorem 4, we obtain the following consequence.

Corollary 9 *Let the function* $f(z)$ *given by* (3) *be in the class* $\mathcal{S}_{\Sigma,m}^{\alpha}$*. Then*

$$|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{1+\alpha}}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)\alpha^2}{m^2} + \frac{\alpha}{m}.$$

By setting $\lambda = 1$ in Theorem 4, we obtain the following consequence.

Corollary 10 *Let the function* $f(z)$ *given by* (3) *be in the class* $\mathcal{P}_{\Sigma,m}(\alpha,\mu)$*. Then*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(\mu+m)^2 + m\alpha(\mu+m)}}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)\alpha^2}{(\mu+m)^2} + \frac{2\alpha}{\mu+2m}.$$

By setting $m = 1$ in Theorem 4, we obtain the following consequence.

Corollary 11 *Let the function* $f(z)$ *given by* (3) *be in the class* $\mathcal{N}_{\Sigma}^{\mu}(\alpha,\lambda)$*. Then*

$$|a_2| \leq \begin{cases} \frac{2\alpha}{\sqrt{(\mu+\lambda)^2+\alpha(\mu+2\lambda-\lambda^2)}} & , \quad 1 \leq \lambda < 1+\sqrt{\mu+1} \\ \frac{2\alpha}{\mu+\lambda} & , \quad \lambda \geq 1+\sqrt{\mu+1} \end{cases} \tag{23}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\mu+\lambda)^2} + \frac{2\alpha}{\mu+2\lambda}.$$

Remark 12 *The above estimates for* $|a_2|$ *show that the inequality* (23) *is an improvement of the estimates obtained by Çağlar et al. [8].*

3. Coefficient estimates for the function class $\mathcal{N}_{\Sigma,m}^{\mu}(\beta,\lambda)$

Definition 13 *For $\lambda \geq 1$ and $\mu \geq 0$, a function $f \in \Sigma_m$ given by (3) is said to be in the class $\mathcal{N}_{\Sigma,m}^{\mu}(\beta,\lambda)$ if the following conditions are satisfied:*

$$\Re\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) > \beta \tag{24}$$

and

$$\Re\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right) > \beta \tag{25}$$

where $0 \leq \beta < 1$; $m \in \mathbb{N}$; $z, w \in \mathbb{U}$; and $g = f^{-1}$ is defined by (4).

Remark 14 *In the following special cases of Definition 13, we show how the class of analytic bi-univalent functions $\mathcal{N}_{\Sigma,m}^{\mu}(\beta,\lambda)$ for suitable choices of λ, μ, and m lead to certain new as well as known classes of analytic bi-univalent functions studied earlier in the literature.*

(i) *For $\mu = 1$, we obtain the m-fold symmetric bi-univalent function class*

$$\mathcal{N}_{\Sigma,m}^{1}(\beta,\lambda) = \mathcal{A}_{\Sigma,m}^{\lambda}(\beta)$$

introduced by Sümer Eker [23]. In addition, for $m = 1$ we have the bi-univalent function class

$$\mathcal{N}_{\Sigma,1}^{1}(\beta,\lambda) = \mathcal{B}_{\Sigma}(\beta,\lambda)$$

introduced by Frasin and Aouf [12].

(ii) *For $\mu = 1$ and $\lambda = 1$, we have the m-fold symmetric bi-univalent function class*

$$\mathcal{N}_{\Sigma,m}^{1}(\beta,1) = \mathcal{H}_{\Sigma,m}(\beta)$$

introduced by Srivastava et al. [22]. In addition, for $m = 1$ we have the bi-univalent function class

$$\mathcal{N}_{\Sigma,1}^{1}(\beta,1) = \mathcal{H}_{\Sigma}(\beta)$$

introduced by Srivastava et al. [21].

(iii) *For $\mu = 0$ and $\lambda = 1$, we get the class*

$$\mathcal{N}_{\Sigma,m}^{0}(\beta,1)$$

of m-fold symmetric bi-starlike functions of order β (see [14]). In addition, for $m = 1$ we have the bi-starlike function class

$$\mathcal{N}_{\Sigma,1}^{0}(\beta,1) = \mathcal{S}_{\Sigma}^{*}(\beta)$$

introduced by Brannan and Taha [3].

(iv) *For $\lambda = 1$, we have a new class*

$$\mathcal{N}_{\Sigma,m}^{\mu}(\beta,1) = \mathcal{P}_{\Sigma,m}(\beta,\mu),$$

which consists of m-fold symmetric bi-Bazilevič functions.
(v) *For $m=1$, we have the bi-univalent function class*

$$\mathcal{N}_{\Sigma,1}^{\mu}(\beta,\lambda)=\mathcal{N}_{\Sigma}^{\mu}(\beta,\lambda)$$

introduced by Çağlar et al. [8].

Theorem 15 *Let the function $f(z)$ given by (3) be in the class $\mathcal{N}_{\Sigma,m}^{\mu}(\beta,\lambda)$. Then*

$$|a_{m+1}|\leq\begin{cases}\sqrt{\frac{4(1-\beta)}{(\mu+2m\lambda)(\mu+m)}} & , \quad 0\leq\beta<\frac{m\left(\mu+2m\lambda-m\lambda^{2}\right)}{(\mu+2m\lambda)(\mu+m)}\\ \frac{2(1-\beta)}{\mu+m\lambda} & , \quad \frac{m\left(\mu+2m\lambda-m\lambda^{2}\right)}{(\mu+2m\lambda)(\mu+m)}\leq\beta<1\end{cases}\tag{26}$$

and

$$|a_{2m+1}|\leq\begin{cases}\min\left\{\frac{2(m+1)(1-\beta)}{(\mu+2m\lambda)(\mu+m)}, \frac{2(m+1)(1-\beta)^{2}}{(\mu+m\lambda)^{2}}+\frac{2(1-\beta)}{\mu+2m\lambda}\right\} & , \quad 0\leq\mu<1\\ \frac{2(1-\beta)}{\mu+2m\lambda} & , \quad \mu\geq1\end{cases}.\tag{27}$$

Proof It follows from (24) and (25) that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}=\beta+(1-\beta)\,p(z)\tag{28}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}=\beta+(1-\beta)\,q(w),\tag{29}$$

where $p(z)$ and $q(w)$ have the forms (11) and (12), respectively. Now, equating the coefficients in (28) and (29), we have

$$(\mu+m\lambda)\,a_{m+1} = (1-\beta)\,p_{m}\tag{30}$$

$$(\mu+2m\lambda)\left[\frac{\mu-1}{2}a_{m+1}^{2}+a_{2m+1}\right] = (1-\beta)\,p_{2m}\tag{31}$$

$$-(\mu+m\lambda)\,a_{m+1} = (1-\beta)\,q_{m}\tag{32}$$

$$(\mu+2m\lambda)\left[\left(m+\frac{\mu+1}{2}\right)a_{m+1}^{2}-a_{2m+1}\right] = (1-\beta)\,q_{2m}.\tag{33}$$

From (30) and (32), we obtain

$$p_{m}=-q_{m}\tag{34}$$

and

$$2(\mu+m\lambda)^{2}a_{m+1}^{2}=(1-\beta)^{2}\left(p_{m}^{2}+q_{m}^{2}\right).\tag{35}$$

Also, from (31) and (33), we have

$$(\mu+2m\lambda)(\mu+m)\,a_{m+1}^{2}=(1-\beta)\left(p_{2m}+q_{2m}\right).\tag{36}$$

Therefore, from equalities (35) and (36) we find that

$$2\left|a_{m+1}\right|^{2} \leq \frac{(1-\beta)^{2}\left(\left|p_{m}\right|^{2}+\left|q_{m}\right|^{2}\right)}{2\left(\mu+m\lambda\right)^{2}}$$

and

$$\left|a_{m+1}\right|^{2} \leq \frac{(1-\beta)\left(\left|p_{2m}\right|+\left|q_{2m}\right|\right)}{\left(\mu+2m\lambda\right)\left(\mu+m\right)},$$

respectively, and applying Lemma 1, we get the desired estimate on the coefficient $\left|a_{m+1}\right|$ as asserted in (26).

Next, in order to find the bound on the coefficient $\left|a_{2m+1}\right|$, by subtracting (33) from (31), we get

$$a_{2m+1}=\frac{m+1}{2}a_{m+1}^{2}+\frac{(1-\beta)\left(p_{2m}-q_{2m}\right)}{2\left(\mu+2m\lambda\right)}. \tag{37}$$

Upon substituting the value of a_{m+1}^{2} from (35) into (37), it follows that

$$a_{2m+1}=\frac{(m+1)\left(1-\beta\right)^{2}\left(p_{m}^{2}+q_{m}^{2}\right)}{4\left(\mu+m\lambda\right)^{2}}+\frac{(1-\beta)\left(p_{2m}-q_{2m}\right)}{2\left(\mu+2m\lambda\right)}. \tag{38}$$

Applying Lemma 1 for (38), we find that

$$\left|a_{2m+1}\right| \leq \frac{2\left(m+1\right)\left(1-\beta\right)^{2}}{\left(\mu+m\lambda\right)^{2}}+\frac{2\left(1-\beta\right)}{\left(\mu+2m\lambda\right)}. \tag{39}$$

On the other hand, upon substituting the value of a_{m+1}^{2} from (36) into (37), it follows that

$$a_{2m+1}=\frac{1-\beta}{2\left(\mu+2m\lambda\right)\left(\mu+m\right)}\left[\left(\mu+2m+1\right)p_{2m}+\left(1-\mu\right)q_{2m}\right]. \tag{40}$$

Applying Lemma 1 for (40), we have

$$\left|a_{2m+1}\right| \leq \frac{1-\beta}{\left(\mu+2m\lambda\right)\left(\mu+m\right)}\left[\mu+2m+1+\left|1-\mu\right|\right]. \tag{41}$$

By investigating the bound on $\left|a_{2m+1}\right|$ according to μ in (41) and comparing with (39), we get the desired estimate on the coefficient $\left|a_{2m+1}\right|$ as asserted in (27). This completes the proof of the Theorem 15. □

By setting $\mu=1$ in Theorem 15, we obtain the following consequence.

Corollary 16 *Let the function $f(z)$ given by (3) be in the class $\mathcal{A}_{\Sigma,m}^{\lambda}(\beta)$. Then*

$$\left|a_{m+1}\right| \leq \begin{cases} \sqrt{\frac{4(1-\beta)}{(1+2m\lambda)(1+m)}} & , \quad 0 \leq \beta < \frac{m\left(1+2m\lambda-m\lambda^{2}\right)}{(1+2m\lambda)(1+m)} \\ \frac{2(1-\beta)}{1+m\lambda} & , \quad \frac{m\left(1+2m\lambda-m\lambda^{2}\right)}{(1+2m\lambda)(1+m)} \leq \beta < 1 \end{cases}$$

and

$$\left|a_{2m+1}\right| \leq \frac{2\left(1-\beta\right)}{1+2m\lambda}.$$

Remark 17 *Corollary 16 is an improvement of the following estimates obtained by Sümer Eker [23].*

Corollary 18 *(see [23]) Let the function $f(z)$ given by (3) be in the class $\mathcal{A}_{\Sigma,m}^{\lambda}(\beta)$. Then*

$$|a_{m+1}| \leq \sqrt{\frac{4(1-\beta)}{(1+2m\lambda)(1+m)}}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)(1-\beta)^2}{(1+m\lambda)^2} + \frac{2(1-\beta)}{1+2m\lambda}.$$

By setting $\mu = 1$ and $\lambda = 1$ in Theorem 15, we obtain the following consequence.

Corollary 19 *Let the function $f(z)$ given by (3) be in the class $\mathcal{H}_{\Sigma,m}(\beta)$. Then*

$$|a_{m+1}| \leq \begin{cases} \sqrt{\frac{4(1-\beta)}{(1+2m)(1+m)}} & , \quad 0 \leq \beta < \frac{m}{1+2m} \\ \frac{2(1-\beta)}{1+m} & , \quad \frac{m}{1+2m} \leq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{1+2m}.$$

Remark 20 *Corollary 19 is an improvement of the following estimates obtained by Srivastava et al. [22].*

Corollary 21 *(see [22]) Let the function $f(z)$ given by (3) be in the class $\mathcal{H}_{\Sigma,m}(\beta)$. Then*

$$|a_{m+1}| \leq \sqrt{\frac{4(1-\beta)}{(1+2m)(1+m)}}$$

and

$$|a_{2m+1}| \leq \frac{2(1-\beta)^2}{1+m} + \frac{2(1-\beta)}{1+2m}.$$

By setting $\mu = 0$ and $\lambda = 1$ in Theorem 15, we obtain the following consequence.

Corollary 22 *Let the function $f(z)$ given by (3) be in the class $\mathcal{N}_{\Sigma,m}^{0}(\beta,1)$. Then*

$$|a_{m+1}| \leq \begin{cases} \frac{1}{m}\sqrt{2(1-\beta)} & , \quad 0 \leq \beta < \frac{1}{2} \\ \frac{2(1-\beta)}{m} & , \quad \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq \begin{cases} \frac{(m+1)(1-\beta)}{m^2} & , \quad 0 \leq \beta < \frac{2m+1}{2(m+1)} \\ \frac{2(m+1)(1-\beta)^2}{m^2} + \frac{1-\beta}{m} & , \quad \frac{2m+1}{2(m+1)} \leq \beta < 1 \end{cases}.$$

By setting $\lambda = 1$ in Theorem 15, we obtain the following consequence.

Corollary 23 *Let the function* $f(z)$ *given by* (3) *be in the class* $\mathcal{P}_{\Sigma,m}(\beta,\mu)$. *Then*

$$|a_{m+1}| \leq \begin{cases} \sqrt{\frac{4(1-\beta)}{(\mu+2m)(\mu+m)}} & , \quad 0 \leq \beta < \frac{m}{\mu+2m} \\ \frac{2(1-\beta)}{\mu+m} & , \quad \frac{m}{\mu+2m} \leq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq \begin{cases} \min\left\{ \frac{2(m+1)(1-\beta)}{(\mu+2m)(\mu+m)}, \frac{2(m+1)(1-\beta)^2}{(\mu+m)^2} + \frac{2(1-\beta)}{\mu+2m} \right\} & , \quad 0 \leq \mu < 1 \\ \frac{2(1-\beta)}{\mu+2m} & , \quad \mu \geq 1 \end{cases}.$$

By setting $m = 1$ in Theorem 15, we obtain the following consequence.

Corollary 24 (*see [8]*) *Let the function* $f(z)$ *given by* (3) *be in the class* $\mathcal{N}_{\Sigma}^{\mu}(\beta,\lambda)$. *Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{4(1-\beta)}{(\mu+2\lambda)(\mu+1)}} & , \quad 0 \leq \beta < \frac{\mu+2\lambda-\lambda^2}{(\mu+2\lambda)(\mu+1)} \\ \frac{2(1-\beta)}{\mu+\lambda} & , \quad \frac{\mu+2\lambda-\lambda^2}{(\mu+2\lambda)(\mu+1)} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} \min\left\{ \frac{4(1-\beta)}{(\mu+2\lambda)(\mu+1)}, \frac{4(1-\beta)^2}{(\mu+\lambda)^2} + \frac{2(1-\beta)}{\mu+2\lambda} \right\} & , \quad 0 \leq \mu < 1 \\ \frac{2(1-\beta)}{\mu+2\lambda} & , \quad \mu \geq 1 \end{cases}.$$

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