Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2015) 39: 604 – 617
© TÜBİTAK
doi:10.3906/mat-1406-29

Generalized weakly central reduced rings

Ying ZHOU, Junchao WEI*
School of Mathematics, Yangzhou University, Yangzhou, P. R. China

Received: 16.06.2014 • **Accepted/Published Online:** 19.09.2014 • **Printed:** 30.09.2015

Abstract: A ring R is called $GWCN$ if $x^2y^2 = xy^2x$ for all $x \in N(R)$ and $y \in R$, which is a proper generalization of reduced rings and CN rings. We study the sufficient conditions for $GWCN$ rings to be reduced and CN. We first discuss many properties of $GWCN$ rings. Next, we give some interesting characterizations of left min-abel rings. Finally, with the help of exchange $GWCN$ rings, we obtain some characterizations of strongly regular rings.

Key words: $GWCN$ rings, CN rings, reduced rings, strongly regular rings, left min-abel rings, exchange rings, clean rings

1. Introduction

Throughout this paper, all rings are associative with identity. Let R be a ring; we use $P(R)$, $N(R)$, $J(R)$, $E(R)$, $Z(R)$, $U(R)$, and $Z_l(R)$ to denote the prime radical, the set of all nilpotent elements, the Jacobson radical, the set of all idempotent elements, the center, the set of all invertible elements, and the left singular ideal of R, respectively. For $a \in R$, $r(a)$ and $l(a)$ denote the right annihilator of a and the left annihilator of a, respectively.

Since 1950, the commutativity works of associative rings have been discussed by many authors. In [13], James introduced these works in detail. After nearly 70 years of development, the subject is gradually innuendo to the local commutativity conditions of ring. In this paper, the main motive is to study certain local commutativity conditions for rings.

In [22] it is proved that a semiprime ring R in which $x^2y^2 - xy^2x \in Z(R)$ for every $x, y \in R$ is commutative. Hence a semiprime ring R satisfying $x^2y^2 = xy^2x$ for every $x, y \in R$ is commutative.

Recall that a ring R is central reduced (CN for short) [8] if $N(R) \subseteq Z(R)$.

Clearly, for a CN ring R, R satisfies the following equation $\star$

$\star$ $\quad x^2y^2 = xy^2x$ for all $x \in N(R)$ and $y \in R$.

However, the converse is not true by the following example.

Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then R satisfies the above equation $\star$, but R is not a CN ring.

Call a ring R a generalized weakly CN ring (short for $GWCN$) if R satisfies the equation $\star$. Hence

*Correspondence: jcweiyz@126.com

2010 *AMS Mathematics Subject Classification:* 16A30, 16A50, 16E50, 16D30.

Project supported by the Foundation of Natural Science of China (11471282, 11171291) and Natural Science Fund for Colleges and Universities in Jiangsu Province (11KJB110019)

the example mentioned above illustrates that $GWCN$ rings are proper generalizations of CN rings.

An ideal I of a ring R is reduced if $N(R) \cap I = 0$; especially, a ring R is called reduced if $N(R) = 0$.

In section 2, we mainly discuss the properties of $GWCN$ rings. We also show that a $GWCN$ ring containing a reduced maximal left ideal is strongly regular. With the help of $GWCN$ rings, we give some characterizations of reduced rings.

An element $k \in R$ is called left minimal if Rk is a minimal left ideal of R. An idempotent element $e \in R$ is called left minimal idempotent if e is a left minimal element. Write $ME_l(R)$ to denote the set of all left minimal idempotents of R. A ring R is called left min-abel [24] if every left minimal idempotent element of R is left semicentral. The study of left min-abel rings appeared in [23, 24, 25, 29].

In section 3, we mainly study the characterization of left min-abel rings.

In section 4, we discuss some properties of $GWCN$ exchange rings and give some characterizations of strongly regular rings.

2. Properties of $GWCN$ rings

Now we begin with the following proposition.

Proposition 2.1 *Let R be a ring and I an ideal of R. Then we have:*

(1) If $I \subseteq N(R)$ and R is a $GWCN$ ring, then R/I is $GWCN$.

(2) If I is reduced and R/I is a $GWCN$ ring, then R is $GWCN$.

Proof (1) Let $\bar{R} = R/I$. For any $\bar{a} = a + I \in N(\bar{R})$, $\bar{b} = b + I \in \bar{R}$, we have $\bar{a}^n = \bar{0}$, where n is a positive integer, that is $a^n \in I$. Since $I \subseteq N(R)$, $a \in N(R)$, then $a^2b^2 = ab^2a$, which implies $\bar{a}^2\bar{b}^2 = \bar{a}\bar{b}^2\bar{a}$. Hence $\bar{R}$ is a $GWCN$ ring.

(2) Let $0 \neq a \in N(R)$, $b \in R$, then $\bar{a} \in N(\bar{R})$, $\bar{a}^2\bar{b}^2 = \bar{a}\bar{b}^2\bar{a}$. That is $a^2b^2 - ab^2a \in I$. Set $n \geq 2$ as the minimal positive integer such that $a^n = 0$. If $n \geq 3$, then $a^{n-3}(a^2b^2 - ab^2a) = a^{n-1}b^2 - a^{n-2}b^2a \in I$, $a^{n-1}b^2a - a^{n-2}b^2a^2 \in I$. Since $(a^{n-1}b^2a - a^{n-2}b^2a^2)^3 = 0$, $a^{n-1}b^2a = a^{n-2}b^2a^2$ because I is reduced. Therefore, $a^{n-1}b^2 = a^{n-2}b^2a$ because $(a^{n-1}b^2 - a^{n-2}b^2a)^2 = 0$, then $a^{n-3}(a^2b^2 - ab^2a) = 0$. If $n = 3$, then $a^2b^2 = ab^2a$. If $n > 3$, then $(a^{n-4}(a^2b^2 - ab^2a)a)^2 = 0$; this gives $a^{n-4}(a^2b^2 - ab^2a)a = 0$. Continuing the above process, eventually one gets $a^2b^2 = ab^2a$. If $n = 2$, then $a^2b^2 = 0$, $ab^2a = -(a^2b^2 - ab^2a) \in I$. Since $(ab^2a)^2 = 0$, $ab^2a = 0$, this implies $a^2b^2 = ab^2a$. Thus R is a $GWCN$ ring. □

Let R be a ring. Write $Max_l(R)$ to denote the set of all maximal left ideals of R. Then we give some basic properties of $GWCN$ rings.

Proposition 2.2 *Let R be a $GWCN$ ring and $M \in Max_l(R)$, $e \in E(R)$, and $a \in R$. Then we have:*

(1) $ea(1-e)Rea(1-e) = 0$.

(2) Either $e \in M$ or $(1-e)R \subseteq M$.

(3) If $ReR = R$, then $e = 1$.

(4) $Ra + R(ae-1) = R$.

(5) $1 - ae \in M$ always implies $1 - ea \in M$.

(6) $1 - ea \in M$ always implies $1 - ae \in M$.

(7) $Me \subseteq M$.

(8) For every $x_i \in R$, $e_i \in E(R)$, $i = 1, 2 \cdots n$, $\sum_{i=1}^{n} x_i e_i = 1$ *always implies* $\sum_{i=1}^{n} Rx_i = R$.

Proof (1) Write $h = ea(1-e)$, then $he = 0$, $eh = h$, $h^2 = 0$. Since R is a $GWCN$ ring, $hx^2h = h^2x^2 = 0$ for any $x \in R$. Substituting $x + e$ for x, one obtains $hxh = 0$, which implies $hRh = 0$. Hence $ea(1-e)Rea(1-e) = 0$.

(2) If $e \notin M$, then $Re + M = R$, $(1-e)R \subseteq (1-e)Re + (1-e)M$. By (1), $ex(1-e)Rex(1-e) = 0$ for any $x \in R$. Therefore, $(Rex(1-e))^2 = 0$, then $ex(1-e) \in Rex(1-e) \subseteq J(R) \subseteq M$. Hence $eR(1-e) \subseteq M$, which implies $(1-e)Re \subseteq M$. Thus $(1-e)R \subseteq M$.

(3) By hypothesis, $R(1-e) = ReR(1-e) \subseteq J(R)$. Therefore, $1 - e \in J(R)$, which implies $e = 1$.

(4) Supposed that $Ra + R(ae-1) \neq R$, then there exists $M \in Max_l(R)$ such that $Ra + R(ae-1) \subseteq M$. Since $ae - 1 \in M$, then $e \notin M$. By (2), $1 - e \in M$, then $a - ae \in M$. This leads to $1 \in M$, which is a contradiction. Hence $Ra + R(ae-1) = R$.

(5) If $1 - ae \in M$, then $e \notin M$. By (2), $(1-e)R \subseteq M$, it follows that $1 - a = (1-ae) - a(1-e) \in M$. Hence $1 - ea = (1-a) + (1-e)a \in M$.

(6) Assume that $1 - ea \in M$. If $e \in M$, then $1 - e \notin M$, it gives $eR \subseteq M$ by (2). Thus $1 = (1-ea) + ea \in M$ which is a contradiction. Thus $e \notin M$, if follows that $(1-e)R \subseteq M$, which leads to $1 - a = (1-ea) - (1-e)a \in M$. Hence $1 - ae = (1-a) + a(1-e) \in M$.

(7) If $Me \nsubseteq M$, then $R = Me + M$. Write $1 = ne + m$, where $n, m \in M$, then $1 - ne = m \in M$. By (5), $1 - en \in M$. This leads to $1 \in M$, which is a contradiction. Hence $Me \subseteq M$.

(8) Suppose that $\sum_{i=1}^{n} Rx_i \neq R$, then there exists $N \in Max_l(R)$ such that $\sum_{i=1}^{n} Rx_i \subseteq N$. Thus $Rx_i \subseteq \sum_{i=1}^{n} Rx_i \subseteq N$, $i = 1, 2 \cdots n$. Therefore $x_i \in N$ and $x_i e_i \in N$ by (7). Then $1 = \sum_{i=1}^{n} x_i e_i \in N$, which is a contradiction. Hence $\sum_{i=1}^{n} Rx_i = R$. □

Call a left ideal I of a ring R *Abelian* if for any $e \in E(R) \cap I$, we have $ex = xe$ for all $x \in I$. Clearly, A ring R is Abelian if R is an Abelian left ideal of R. It is well known that a ring R is Abelian if and only if $eR(1-e) = 0$ for all $e \in E(R)$. From the Proposition 2.2, we have the following corollary.

Corollary 2.3 *Let R be a $GWCN$ ring. Then we have:*

(1) If idempotents can be lifted modulo $J(R)$, then $R/J(R)$ is Abelian.

(2) If R has an Abelian maximal left ideal M, then R is Abelian.

Proof (1) Let $\bar{R} = R/J(R)$ and $x \in R$ satisfy $x - x^2 \in J(R)$. By hypothesis, there exists $e \in E(R)$, $e - x \in J(R)$. Since R is a $GWCN$ ring, $ea(1-e)Rea(1-e) = 0$ for any $a \in R$ by Proposition 2.2(1), this gives $\bar{e}\bar{a}(\bar{1}-\bar{e})\bar{R}\bar{e}\bar{a}(\bar{1}-\bar{e}) = \bar{0}$. Since $\bar{R}$ is a semiprime ring, $\bar{e}\bar{a}(\bar{1}-\bar{e}) = \bar{0}$ for all $a \in R$. Therefore $\bar{e}\bar{R}(\bar{1}-\bar{e}) = \bar{0}$, that is $\bar{x}\bar{R}(\bar{1}-\bar{x}) = \bar{0}$. Hence $\bar{R}$ is Abelian.

(2) Assume that $e \in E(R)$. If $e \notin M$, then by Proposition 2.2(2), $1 - e \in M$. Therefore for any $x \in R$, $x(1-e) \in M$, it follows that $(x(1-e))(1-e) = (1-e)(x(1-e))$ because M is Abelian, that is, $x(1-e) = (1-e)x(1-e)$. Hence $eR(1-e) = 0$. If $e \in M$, then $1 - e \notin M$ it follows that $eR \subseteq M$ by Proposition 2.2(2). Also by Proposition 2.2(7), $M(1-e) \subseteq M$. Thus $eR(1-e) \subseteq eM(1-e) = Me(1-e) = 0$. In any case, we have $eR(1-e) = 0$; this implies R is Abelian. □

Recall that a ring R is nil-semicommutative [4] if $ab=0$ implies $aRb \subseteq N(R)$ for any $a,b \in R$, and R is said to be NCI if $N(R)=0$ or there exists a nonzero ideal of R contained in $N(R)$. Clearly NI rings (that is, $N(R)$ is an ideal of R) are NCI, but the converse is not true by [12]. In preparation for the proof of our next theorem, we first state the following proposition.

Proposition 2.4 *Let R be a $GWCN$ ring and $a \in R$. Then*

(1) $a^2=0$ always implies $aRaRa=0$.

(2) R is reduced if and only if R is semiprime.

(3) R is a NCI ring.

(4) R is a nonzero divisor ring if and only if R is a prime ring.

(5) R is nil-semicommutative.

Proof (1) By hypothesis, $ax^2a=a^2x^2=0$ for any $x \in R$. Substituting $x+ya$ for x, one has $ayaxa=0$ for all $x,y \in R$. Hence $aRaRa=0$.

(2) One direction is clear.

For the other direction. Let $a^2=0$, where $a \in R$. By (1), $aRaRa=0$, that is $(RaR)^3=0$. Since R is semiprime, $RaR=0$, which implies $a=0$. Hence R is reduced.

(3) If $N(R) \neq 0$, then there exists $0 \neq a \in N(R)$ such that $a^2=0$. By (1), $aRaRa=0$, which implies R is a NCI ring.

(4) Assume that R is a prime ring and $ab=0$, where $a,b \in R$. By (2), R is reduced and so $aRb=0$; it follows that $a=0$ or $b=0$ because R is a prime ring. Therefore R is a nonzero divisor ring.

(5) Let $xy=0$, where $x,y \in R$. Since $(yx)^2=0$, $yxr^2yx=0$ for any $r \in R$. Choose $t \in R$ and substituting $r+tx$ for r, one obtains that $yxtxryx=0$, this gives $yxRxRyx=0$. For any $a \in R$, we have $(yxa)^3=0$. Therefore $(xay)^4=xa(yxa)^3y=0$. Thus $xay \subseteq N(R)$, which implies R is nil-semicommutative. □

The following example shows that there exists a nil-semicommutative ring that is not $GWCN$.

Example 2.5 *Let Z_2 be the field of integers modulo 2 and* $R=T_3(Z_2)=\begin{pmatrix} Z_2 & Z_2 & Z_2 \\ 0 & Z_2 & Z_2 \\ 0 & 0 & Z_2 \end{pmatrix}$. *Let* $A=\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in N(R)$ *and* $B=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R$, *then $A^2B^2 \neq AB^2A$. Hence R is not a $GWCN$ ring. However, R is nil-semicommutative.*

It is well known that R is strongly regular if and only if R is a reduced regular ring or an Abelian regular ring. The following corollary gives a new characterization of strongly regular rings in terms of $GWCN$ rings.

Corollary 2.6 *R is a strongly regular ring if and only if R is a regular $GWCN$ ring.*

Proof We only need to show the Sufficiency: Assume that R is a regular GWCN ring. Since regular rings are semiprime, R is reduced by Proposition 2.4(2). Hence R is strongly regular. □

Recall that a ring R is directly finite if $ab=1$ implies $ba=1$ for $a,b \in R$, and R is said to be $n-$regular (see [26, 27]) if every nilpotent element is regular.

Corollary 2.7 *Let R be a $GWCN$ ring and $a \in R$. Then we have*

(1) If $a \in aRa$, then $a \in a^2R \cap Ra^2$.

(2) R is directly finite.

(3) R is reduced if and only if R is $n-$regular.

(4) R is $\pi-$regular if and only if R is strongly $\pi-$regular.

(5)For any $y_i \in R$, $e_i \in E(R)$, $i = 1, 2 \cdots n$, $u \in R$, if $\sum_{i=1}^{n} y_i u e_i \in U(R)$, then $u \in U(R)$.

Proof (1) Let $a = aba$ for some $b \in R$. Set $e = ab$. Then $e \in E(R)$ and $a = ea$. Clearly, $(a(1-e))^2 = 0$, it follows that $a(1-e)Ra(1-e)Ra(1-e) = 0$ by Proposition 2.4(1), so $a(1-e)ba(1-e)ba(1-e) = 0$, by simple calculation, we have $a = ae(1 + 2ba + baebae - baeba - 2bae) \in a^2R$. Similarly, we can show that $a \in Ra^2$.

(2) Let $ab = 1$. Then $b = bab$, by (1), there exists $c \in R$ such that $b = b^2c$. So $1 = ab = ab^2c = bc$, $a = a1 = a(bc) = (ab)c = c$, this gives $ba = bc = 1$.

(3) Since reduced $\Longrightarrow$ $n-$regular $\Longrightarrow$ semiprime, (3) is an immediate result of Proposition 2.4(2).

(4) It follows from (1).

(5) Write $\sum_{i=1}^{n} y_i u e_i = v \in U(R)$. Then $\sum_{i=1}^{n} (v^{-1} y_i u) e_i = 1$. By Proposition 2.2(8), $\sum_{i=1}^{n} Rv^{-1} y_i u = R$, this gives $R = Ru$. By (1), $u \in U(R)$. □

By Corollary 2.7, we have the following corollary.

Corollary 2.8 *Let R be a $GWCN$ ring with $e \in E(R)$ and $a \in R$. Then*

(1) $1 - ae \in U(R)$ if and only if $1 - ea \in U(R)$.

(2) If $x + z \in xzE(R)$ for $x, z \in R$, then $xR = zR$.

Proof (1) Assume that $1 - ae \in U(R)$. If $R(1 - ea) \neq R$, then there exists $M \in Max_l(R)$ such that $1 - ea \in M$. By Proposition 2.2(6), $1 - ae \in M$. Thus $M = R$, which is a contradiction. Thus $R(1 - ea) = R$ and so $1 - ea \in U(R)$ by Corollary 2.7(1). Similar to show in turn.

(2) Since $x + z \in xzE(R)$, $x + z = xze$ for some $e \in E(R)$, that is $z = xze - x = x(ze - 1)$. Since R is a $GWCN$ ring, $R = Rz + R(ze - 1) = Rx(ze - 1) + R(ze - 1) \subseteq R(ze - 1)$ by Proposition 2.2(4); it follows that $R = R(ze - 1)$. Hence $ze - 1 \in U(R)$ by Corollary 2.7(2); this leads to $xR = zR$. □

According to Cohn [6], a ring R is called symmetric if $abc = 0$ implies $acb = 0$ for $a, b, c \in R$, R is said to be ZC if $ab = 0$ implies $ba = 0$ for $a, b \in R$, and R is said to be ZI if $ab = 0$ implies $aRb = 0$. Clearly, reduced $\Longrightarrow$ symmetric $\Longrightarrow$ ZC $\Longrightarrow$ ZI $\Longrightarrow$ Abelian. A left ideal I of R is called regular if $a \in aIa$ for all $a \in I$. Clearly, every left ideal of strongly regular rings is regular. In preparation for our next theorem, we first state the following lemma.

Lemma 2.9 *Let R be a $GWCN$ ring. If there exists $M \in Max_l(R)$ such that M is regular, then R is a reduced ring.*

Proof Assume that $a \in R$ such that $a^2 = 0$. Then by Proposition 2.4(1), one has $aRaRa = 0$; this gives $a \in J(R) \subseteq M$. Since M is a regular left ideal, then there exists $b \in M$ such that $a = aba = (aba)ba \in aRaRa = 0$. Hence R is reduced. □

Theorem 2.10 *Let R be a $GWCN$ ring. If there exists $M \in Max_l(R)$ such that M is regular, then R is a strongly regular ring.*

Proof By Lemma 2.9, R is reduced. Let $a \in R$. If $a \in M$, then $a = aba$ for some $b \in M$. If $a \notin M$, then $Ra + M = R$. Write $ca + m = 1$ with $c \in R, m \in M$, then $a = aca + am$. Since $am \in M$, $am = amdam$ for some $d \in M$. Write $e = amd$, $g = dam$, then $e^2 = e, g^2 = g$. Since R is reduced, R is Abelian, this gives $eg = ge$, $amd^2am = ge = gamd = agmd = adam^2d$, so $a(md^2am - dam^2d) = 0$. Since R is a ZI ring, $aR(md^2am - dam^2d) = 0$, $(md^2am - dam^2d)^2 = 0$. Therefore $md^2am = dam^2d = gmd = mgd = mdamd$, that is $m(d^2am - damd) = 0$. Repeating the process mentioned above, we obtain $d^2am = damd$, and so $dam = amd$. Since $a(m - mdam) = 0$, $(m - mdam)a = 0$, that is $ma = mdama = mamda$; this gives $ma(1 - mda) = 0$. Since R is symmetric, $m(1 - mda)a = 0$; it follows that $ma = m^2da^2$. Hence $a = (ca + m)a = ca^2 + ma = (c + m^2d)a^2 \in Ra^2$. Hence R is strongly regular. □

Call a ring R $QVNR$ if R contains a regular maximal left ideal. Since every left ideal of strongly regular rings is regular, we have the following corollary.

Corollary 2.11 *The following conditions are equivalent for a ring R.*

(1) R is a strongly regular ring;

(2) R is a reduced $QVNR$ ring;

(3) R is a CN $QVNR$ ring;

(4) R is a $GWCN$ $QVNR$ ring.

Recall that a ring R is left SF if every simple left R-module is flat. Goodearl [10] proved that if R is a left SF ring and $M \in Max_l(R)$, then $m \in mM$ for any $m \in M$. Inspired by [10], Li and Wei [15] introduced the definition of left NSF rings, that is, for any $M \in Max_l(R)$ and any $m \in N(R) \cap M$, one has $m \in mM$. Clearly, reduced rings and left SF rings are left NSF. In preparation for our next proposition, we first state the following lemma.

Lemma 2.12 *If R is a CN ring, then $V_2(R) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| a, b \in R \right\}$ is $GWCN$.*

Proof Let $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in N(V_2(R))$. Then $a \in N(R)$. Since R is a CN ring, $a \in Z(R)$. For any $B = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \in V_2(R)$, we have $A^2B^2 = \begin{pmatrix} a^2x^2 & a^2(xy+yx)+(ab+ba)x^2 \\ 0 & a^2x^2 \end{pmatrix}$, $AB^2A = \begin{pmatrix} ax^2a & ax^2b+a(xy+yx)a+bx^2a \\ 0 & ax^2a \end{pmatrix}$. Since $a \in Z(R)$, $a^2x^2 = ax^2a$, $a^2xy = axya$, $a^2yx = ayxa$. Since $a \in N(R)$, there exists $n \geq 1$ such that $a^n = 0$. Therefore $(ab)^n = a^nb^n = 0$; then $ab \in N(R) \subseteq Z(R)$. Hence $abx^2 = x^2ab = ax^2b$. Thus $A^2B^2 = AB^2A$, which implies $V_2(R)$ is a $GWCN$ ring. □

Proposition 2.13 *R is reduced if and only if $V_2(R)$ is a $GWCN$ ring and R is a left NSF ring.*

Proof By Lemma 2.12, the necessity is clear.

Now let $a^2 = 0$, where $a \in R$. If $a \neq 0$, then $l(a) \neq R$, and so there exists $M \in Max_l(R)$ such that $a \in l(a) \subseteq M$. Since R is left NSF and $a \in N(R) \cap M$, $a \in aM$. Therefore there exists $m \in M$ such that

$a = am$. Let $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, then $A^3 = 0$, and so $A \in N(V_2(R))$. Let $B = \begin{pmatrix} m & m \\ 0 & m \end{pmatrix} \in V_2(R)$. Since $V_2(R)$ is a $GWCN$ ring, $A^2B^2 = AB^2A$, which implies $\begin{pmatrix} 0 & 2am^2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} am^2a & am^2 + 2am^2a + m^2a \\ 0 & am^2a \end{pmatrix}$. Hence $2am^2 = am^2 + 2am^2a + m^2a$; noticing that $a = am$, one obtains that $a = m^2a$. Therefore $1 - m^2 \in l(a) \subseteq M$. This leads to $1 \in M$, which is a contradiction. Thus $a = 0$, which implies R is reduced. □

In [10], Goodearl pointed out regular rings are always left SF. According to Rege [17], reduced left SF rings are strongly regular. Hence by Proposition 2.13, we have the following corollary.

Corollary 2.14 *R is a strongly regular ring if and only if $V_2(R)$ is a $GWCN$ ring and R is left SF ring.*

Let R be a ring and $R[x]$ denote the polynomial ring with indeterminate x over R. Then, clearly, $V_2(R) \cong R[x]/(x^2)$. For a ring R, let $T(R;R) = \{(a;x)|a, x \in R\}$ with the addition componentwise and the multiplication defined by $(a_1;x_1)(a_2;x_2) = (a_1a_2; a_1x_2 + x_1a_2)$. Then $T(R;R)$ is a ring that is called the trivial extension of R by R. Clearly $T(R;R) \cong V_2(R)$. By Proposition 2.13, we have the following corollary that characterizes reduced rings.

Corollary 2.15 *The following conditions are equivalent for a ring R:*

(1) R is a reduced ring;

(2) R is a left NSF ring and $R[x]/(x^2)$ is a $GWCN$ ring;

(3) R is a left NSF ring and $T(R;R)$ is a $GWCN$ ring;

Theorem 2.16 *R is a reduced ring if and only if $T_2(R) = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ is a $GWCN$ ring.*

Proof Assume that R is a reduced ring. Then $N(T_2(R)) = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$, and so $A^2B^2 = 0 = AB^2A$ for all $A \in N(T_2(R))$ and $B \in T_2(R)$. Hence $T_2(R)$ is $GWCN$.

Conversely, assume that $T_2(R)$ is a $GWCN$ ring and $a \in R$ with $a^2 = 0$. Choose $A = \begin{pmatrix} a & x \\ 0 & 0 \end{pmatrix} \in N(T_2(R))$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in T_2(R)$. By hypothesis, $A^2B^2 = AB^2A$; this gives $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and it follows that $a = 0$. Thus R is reduced. □

3. Left min-abel rings

An element $k \in R$ is called left minimal if Rk is a minimal left ideal of R. Write $M_l(R)$ to denote the set of all left minimal elements of R. An idempotent element $e \in R$ is called left minimal idempotent if e is a left minimal element. Write $ME_l(R)$ to denote the set of all left minimal idempotent of R. An idempotent $e \in R$ is called left (resp., right) semicentral if $ae = eae$ (resp., $ea = eae$) for all $a \in R$. A ring R is called left min-abel [24] if every left minimal idempotent element of R is left semicentral, R is said to be strongly left min-abel if $ME_l(R) \subseteq Z(R)$, and R is said to be left $MC2$ ring if $aRe = 0$ implies $eRa = 0$ for all $a \in R$ and $e \in ME_l(R)$. A ring R is called strongly left DS if $k^2 \neq 0$ for each $k \in M_l(R)$, and R is said to be left universally mininjective if $kRk \neq 0$ for each $k \in M_l(R)$.

Proposition 3.1 *If R is a $GWCN$ ring, then R is a left min-abel ring.*

Proof Assume that $e \in ME_l(R)$. For any $a \in R$, write $h = (1-e)ae$, where $e \in ME_l(R)$. If $h \neq 0$, then $Rh = Re$. Let $e = ch$ for some $c \in R$. Then we have $h = he = hch$; by Corollary 2.7(1), $h = dh^2$ for some $d \in R$. Since $h^2 = 0$, $h = 0$, which is a contradiction. Therefore $h = 0$ and R is a left min-abel ring. □

The following corollary follows from [24, Theorem 1.2, 1.8, and 1.11] and Proposition 3.1.

Corollary 3.2 *Let R be a $GWCN$ ring; then*

(1) R is a left quasi-duo ring if and only if R is a $MELT$ ring.

(2) R is a strongly left min-abel ring if and only if R is a left $MC2$ ring.

(3) R is a strongly left DS ring if and only if R is a left universally mininjective ring.

Influenced by the definition of $GWCN$ rings, we give some new characterizations of left min-abel rings that generalize [24, Theorem 1.1]

Theorem 3.3 *R is a left min-abel ring if and only if for any $k \in M_l(R) \cap N(R)$, we have $k^2x^2 = kx^2k$ for all $x \in R$.*

Proof Assume that R is a left min-abel ring and $k \in M_l(R) \cap N(R)$. Then we claim that $k^2 = 0$. Otherwise, $(Rk)^2 \neq 0$; then there exists $e \in ME_l(R)$ such that $Rk = Re$. Set $e = ck$ for some $c \in R$. Since R is left min-abel, e is left semicentral. Then $k = ke = eke = ck^2e = c^2k^3e = \cdots = c^nk^{n+1}e = \cdots$, it follows that $k = 0$ because $k \in N(R)$, which is a contradiction. Hence $k^2 = 0$ and so $k^2x^2 = 0$ for any $x \in R$. In fact the proof mentioned above also implies $kRk = 0$; hence $kx^2k = 0 = k^2x^2$ for all $x \in R$.

Conversely, assume that $e \in ME_l(R)$. For any $a \in R$, write $h = (1-e)ae$. If $h \neq 0$, then $Rh = Re$, $he = h$, $eh = 0$, and $h^2 = 0$. Hence $h \in M_l(R) \cap N(R)$. By hypothesis, $hx^2h = h^2x^2 = 0$ for any $x \in R$. Substituting $x + e$ for x, one has $hxh = 0$, which implies $hRh = 0$. Hence $Re = ReRe = RhRh = 0$, a contradiction. Thus $h = 0$, which implies R is a left min-abel ring. □

Similar to the proof of Theorem 3.3, we easily obtain the following corollary.

Corollary 3.4 *R is a left min-abel ring if and only if for any $k \in M_l(R) \cap N(R)$, we have $x^2k^2 = kx^2k$ for all $x \in R$.*

Theorem 3.5 *R is a left min-abel ring if and only if for every $k \in M_l(R)$, $M \in Max_l(R)$, $1 - ak \in M$ always implies $1 - ka \in M$ for all $a \in R$.*

Proof Suppose that R is a left min-abel ring and $1 - ak \in M$. Clearly, $k \notin M$; this gives $Rk \cap M = 0$ and $R = M \oplus Rk$. Let $M = R(1-e)$ for some $e \in ME_l(R)$. Since R is a left min-abel ring, e is left semi-central, that is $1 - e$ is right semicentral. Hence M is an ideal. Assume that $1 - ka \notin M$. Then $R(1 - ka) + M = R$ and so there exist $c \in R$ and $m \in M$ such that $c(1 - ka) + m = 1$. Therefore $k = (c(1 - ka) + m)k = ck(1 - ak) + mk \in M$, which is a contradiction. Hence $1 - ka \in M$.

Conversely, let $e \in ME_l(R)$. For any $a \in R$, write $h = (1-e)ae$. If $h \neq 0$, then $Rh = Re$ and $h \in M_l(R)$. Let $e = ch$ for some $c \in R$. Since $1 - e \in R(1-e) = l(e) \in Max_l(R)$, $1 - (ec)h \in l(e)$. By hypothesis, $1 - h(ec) \in l(e)$, which implies $(1 - hc)e = 0$. Hence $h = he = h^2ce = 0$, a contradiction. Thus $h = 0$ and so R is left min-abel. □

Theorem 3.6 *R is a left min-abel ring if and only if for every $k \in M_l(R)$, $M \in Max_l(R)$, $1-ka \in M$ always implies $1-ak \in M$ for all $a \in R$.*

Proof Assume that R is a left min-abel ring and $1-ka \in M$. Clearly $ka \notin M$ and so $Rka \oplus M = R$. This implies M is an ideal of R because R is left min-abel. If $k \in M$, then $ka \in M$, which is a contradiction. Hence $k \notin M$; it follows that $Rk \oplus M = R$. Let $Rk = Re$ and $M = R(1-e)$ for some $e \in ME_l(R)$. If $1-ak \notin M$, then $R(1-ak)+M = R$. Write $1 = c(1-ak)+m$ for some $c \in R$ and $m \in M$. Since $e = (c(1-ak)+m)e = c(1-ak)e$ because $me = 0$. Since e is left semicentral, $e = ce(1-ak)e$. Set $e = dk$ for some $d \in R$. Then $e = cd(1-ka)ke = cd(1-ka)k \in M$ because $1-ka \in M$, a contradiction. Hence $1-ak \in M$.

Conversely, let $e \in ME_l(R)$ and $a \in R$, write $h = (1-e)ae$. If $h \neq 0$, then $Rh = Re$ and $h \in M_l(R)$. Let $e = ch$ for some $c \in R$. Write $g = hc$. Then $g \in ME_l(R)$ and $1-h(cg) = 1-g \in l(g) \in Max_l(R)$. By hypothesis, $1-cgh \in l(g)$, that is, $g = cghg = 0$; this leads to $h = gh = 0$, a contradiction. Thus $h = 0$ and so R is left min-abel. □

Recall that a left $R-$module M is $YJ-(nil-)$injective if for any $0 \neq a \in R$ ($a \in N(R)$) there exists a positive integer n such that $a^n \neq 0$ and any left $R-$homomorphism of Ra^n into M extends to one of R into M. Evidently, $YJ-$injective modules are $nil-$injective, but the converse is not true, in general, by Wei and Chen [26]. The following proposition is significant because it is a generalization of [24, Proposition 2.6].

Proposition 3.7 *Let R be a $GWCN$ ring. Then we have:*

(1) If R is a left $MC2$ ring and every simple singular left $R-$ module is $nil-$ injective, then R is reduced.

(2) If every simple left $R-$ module is $nil-$ injective, then R is reduced.

Proof (1) Let $a \in R$ satisfy $aRa = 0$. We claim that $a = 0$. If not, then $l(a) \neq R$, and so there exists a maximal left ideal M of R containing $l(a)$. If M is not an essential left ideal of R, then $M = l(e)$ for some $e \in ME_l(R)$. Since $aR \subseteq l(a) \subseteq M = l(e)$, $aRe = 0$. Since R is a left $MC2$ ring, $eRa = 0$, it follows that $e \in l(a) \subseteq l(e)$, a contradiction. Hence M is an essential left ideal of R; this implies R/M is a simple singular left $R-$module. By hypothesis, R/M is a $nil-$injective left $R-$module. Let $f : Ra \longrightarrow R/M$ be defined by $f(ra) = r+M$. Then f is a well defined left $R-$homomorphism and so there exists a left $R-$ homomorphism $g : R \longrightarrow R/M$ such that $g(a) = f(a)$. Hence there exists $c \in R$ such that $1+M = f(a) = g(a) = ag(1) = ac+M$, that is $1-ac \in M$. Since R is a $GWCN$ ring and $a^2 = 0$, $aRaRa = 0$ by Proposition 2.4(1); this gives $(ac)^3 = 0$. Hence $1 = 1-(ac)^3 = (1+ac+(ac)^2)(1-ac) \in M$, which is a contradiction. Therefore $a = 0$ and so R is a reduced ring by 2.4(2).

(2) Similar to (1), we can show that R is reduced. □

Now we consider whether the result holds if we omit the condition R is a left $MC2$ ring.

Example 3.8 *Let F be a field and $R = T_2(F) = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. By Theorem 2.16, R is a $GWCN$ ring. Now let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in ME_l(R)$ and $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$. Then $ARe = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but $eRA = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, and so R is not a left $MC2$ ring. By [20, Example 2.11], every simple singular left $R-$module is injective. Clearly, $N(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, which implies R is not reduced. Therefore the result of Proposition 3.7(1)(1) is not true* *if we omit the condition R is a left $MC2$ ring.*

Recall that a ring R is said to be left (resp., right) weakly regular if for every $a \in R$, $a \in RaRa$ (resp, $a \in aRaR$). R is said to be weakly regular if all left weakly regular rings are right weakly regular. Clearly, if R is a ZC ring, then R is left weakly regular if and only if R is right weakly regular. Kim et al. [14, Theorem 4] proved that if R is a ZI ring whose every simple singular left $R-$module is $YJ-$injective, then R is a reduced weakly regular ring. The following corollary is a generalization of [14, Theorem 4].

Corollary 3.9 *Let R be a left $MC2$ $GWCN$ ring. If every simple singular left $R-$module is $YJ-$injective, then R is a reduced weakly regular ring.*

Proof It is an immediate result of Proposition 3.7(1) and [14, Theorem 4]. □

It is well known that left quasi-duo left weakly regular rings are strongly regular. Hence we have the following corollary.

Corollary 3.10 *Let R be a left $MC2$ $GWCN$ ring. If R is also a $MELT$ ring whose every simple singular left module is $YJ-$injective, then R is strongly regular.*

Proof By corollary 3.9, R is a reduced weakly regular ring. Since R is $MELT$, R is left quasi-duo by corollary 3.2(1). Hence R is strongly regular. □

4. $GWCN$ exchange rings

An element $x \in R$ is said to be exchange if there exists $e \in E(R)$ such that $e \in xR$ and $1-e \in (1-x)R$. The ring R is said to be exchange if all of its elements are exchange. An element $x \in R$ is said to be clean if $x = u + f$ for some $u \in U(R)$ and $f \in E(R)$. The ring R is said to be clean if all of its elements are clean. Nicholson [16, Proposition 1.8] showed that clean rings are exchange, but the converse is not true by Handelman [11, Example 1]. [16] showed that Abelian exchange rings are clean; [30] showed that left quasi-duo exchange rings are clean; [28] showed that quasi-normal exchange rings are clean.

Theorem 4.1 *Let R be a $GWCN$ exchange ring. Then*

(1) R/P is a local ring for every prime ideal of R.

(2) R/P is a division ring for every left primitive ideal of R.

(3) $R/J(R)$ is reduced.

(4) R is left quasi-duo.

(5) R is clean.

Proof (1) According to Warfield [21, Theorem 1], an exchange ring with only two idempotents is a local ring. Since R is an exchange ring, idempotents can be lifted modulo P. Let $a \in R$ such that $a - a^2 \in P$; then there exists $e \in E(R)$ such that $e - a \in P$. Since R is a $GWCN$ ring, $ex(1-e)Rex(1-e)Rex(1-e) = 0 \subseteq P$ for any $x \in R$ by Proposition 2.4(1). Since P is a prime ideal of R, $ex(1-e) \in P$ for all $x \in R$, and so either $e \in P$ or $1 - e \in P$; this gives either $a \in P$ or $1 - a \in P$. Thus R/P contains only two idempotents. Since R/P is an exchange ring, R/P is a local ring.

(2) Since P is a left primitive ideal, P is a prime ideal. By (1), R/P is a local ring. Since R/P is a left primitive ring, R/P is a division ring.

(3) Let $\bar{R} = R/J(R)$. If $\bar{a}^2 = \bar{0}$, then $a^2 \in J(R)$. Assume that $\bar{a} \neq \bar{0}$. Therefore $a \notin J(R)$ and there exists a left primitive ideal P of R such that $a \notin P$. By (2), $\hat{R} = R/P$ is a division ring. Since $a \notin P$, $\hat{a} \neq \hat{0}$, that is $\hat{a} \in U(\hat{R})$. Hence $\hat{a}\hat{b} = \hat{1} = \hat{b}\hat{a}$ for some $b \in R$, that is $1 - ab \in P$. Then $a - a^2 b = a(1-ab) \in P$, which implies $a \in P$, a contradiction. Thus $\bar{a} = \bar{0}$; it follows that $\bar{R}$ is reduced.

(4) Let $M \in Max_l(R)$. Suppose that M is not an ideal of R; then there exists $m \in M$ and $a \in R$ such that $ma \notin M$. Since R/M is a simple left $R-$module, $P = \{r \in R \mid r \cdot R/M = 0\}$ is a left primitive ideal. By (2), $\bar{R} = R/P$ is a division ring. Since $P \subseteq M$, $ma \notin P$. Therefore there exists $b \in R$ such that $(\bar{m}\bar{a})\bar{b} = \bar{1}$; this gives $\bar{m} \in U(\bar{R})$ and one has $(\bar{a}\bar{b})\bar{m} = \bar{1}$. Hence $1 - abm \in P \subseteq M$. This leads to $1 \in M$, which is a contradiction. Thus M is an ideal of R, and so R is quasi-duo.

(5) By Corollary 2.3(1), $R/J(R)$ is an Abelian exchange ring; by [16], $R/J(R)$ is a clean ring. Hence, by [3], R is clean. □

Recall that a ring R is called a left $tb-$ring [7] if for every pair of distinct maximal left ideals of R there is an idempotent in exactly one of them. Recall that a ring R is said to have stable range 1 (cf. [19]) if for any $a, b \in R$ satisfying $aR + bR = R$ there exists $y \in R$ such that $a + by$ is right invertible. It is well known that an exchange ring has stable range 1 if and only if every (Von Neumann) regular element is unit-regular.

Corollary 4.2 *Let R be a $GWCN$ exchange ring. Then*

(1) R is a left $tb-$ring.

(2) R has stable range 1.

Proof (1) Suppose that M and N are distinct maximal left ideals of R. Let $a \in M \backslash N$. Then $Ra + N = R$ and $1 - xa \in N$ for some $x \in R$. Clearly, $xa \in M \backslash N$. Since R is a $GWCN$ exchange ring, R is clean by Theorem 4.1(5), and so there exist an idempotent $e \in E(R)$ and a unit u in R such that $xa = e + u$. If $e \in M$, then $u = xa - e \in M$ from which it follows that $R = M$, a contradiction. Thus $e \notin M$. If $e \notin N$, then $1 - e \in N$ by Proposition 2.2(2) and hence $u = (1-e) + (xa - 1) \in N$. It follows that $N = R$, which is also not possible. We thus have that e belongs to N only.

(2) It follows from Corollary 2.7(1). □

For several years, whether the set $N(R)$ of nilpotent elements of a $\pi-$regular ring R is an ideal has been studied by many authors. For example, Badawi [1] proved that if R is an Abelian ring, then R is a $\pi-$regular ring if and only if R is a NI ring and $R/N(R)$ is a strongly regular ring, and Chen [5] proved that if R is a semiabilian ring, then R is a $\pi-$regular if and only if R is a NI ring and $R/N(R)$ is a strongly regular ring. We generalize these results as follows.

Theorem 4.3 *Let R be a $GWCN$ ring. Then R is a $\pi-$regular ring if and only if R is a NI ring and $R/N(R)$ is regular.*

Proof Assume that R is $\pi-$regular. Then $J(R) \subseteq N(R)$. Since $\pi-$regular rings are exchange, by Theorem 4.1(4), R is left quasi-duo and so we have $N(R) \subseteq J(R)$ by [30, Lemma 2.3]. Hence $N(R) = J(R)$, which implies R is a NI ring. Clearly, $\bar{R} = R/N(R)$ is a reduced $\pi-$regular ring and so $R/N(R)$ is regular.

Conversely, assume that R is a NI ring and $\bar{R} = R/N(R)$ is regular. Then $\bar{R}$ is strongly regular because $R/N(R)$ is reduced. Let $\bar{x} \in \bar{R}$, where $x \in R$; then there exists $\bar{e} \in E(\bar{R})$ and $\bar{u} \in U(\bar{R})$ such that $\bar{x} = \bar{e}\bar{u} = \bar{u}\bar{e}$. Since R is a NI ring, idempotents and invertible elements can be lifted modulo $N(R)$. Therefore

$\bar{e} = e + N(R)$, $\bar{u} = u + N(R)$ for some $e \in E(R)$, $u \in U(R)$. Set $x - eu = a \in N(R)$, $x - ue = b \in N(R)$, then $(1-e)x = x - ex = a + eu - (ea + eu) = a - ea \in N(R)$. Hence there exists $n \geq 1$ such that $(x - ex)^n = 0$. Let $\hat{R} = R/P$, where P is a prime ideal of R. If $e \in P$, then $x^n \in P$, which implies $\hat{x}^n \in \hat{x}^{n+1}\hat{R}$. If $e \notin P$, then similar to the proof of Theorem 4.1(1), one has $1 - e \in P$, which implies $\hat{e} = \hat{1}$. Hence $\hat{x} = \hat{e}\hat{x} = \widehat{ex} = e\widehat{(a+eu)} = e\widehat{(a+u)} = \hat{e}\widehat{(a+u)} = \widehat{a+u} = \widehat{(au^{-1}+1)}\hat{u}$, that is $\hat{x} \in U(\hat{R})$; it follows that R/P is strongly $\pi-$regular. By [18, Theorem 23.2], R is strongly $\pi-$regular. □

Theorem 4.4 *Let R be a $GWCN$ exchange ring. Then the following conditions are equivalent:*

(1) $P(R) = 0$ and every prime ideal of R is maximal.

(2) $P(R) = 0$ and every prime ideal of R is left primitive.

(3) R is strongly regular.

Proof (1)$\Longrightarrow$(2) Let P be a prime ideal of R. By (1), P is a maximal ideal and $P \neq R$. Therefore, there exists $M \in Max_l(R)$ such that $P \subseteq M$. Since R is a $GWCN$ exchange ring, R is left quasi-duo by Theorem 4.1(4); it follows that M is an ideal. Hence $P = M$. Since R/M is a simple left $R-$module, $I = \{r \in R \mid r \cdot R/M = 0\}$ is a left primitive ideal. Clearly, $P = I$ is a left primitive ideal.

(2)$\Longrightarrow$(3) Let P be a prime ideal of R. By (2), P is a left primitive ideal. By Theorem 4.1(2), R/P is a division ring, which implies R/P is a strongly regular ring. By [18, Theorem 23.2], R is strongly $\pi-$ regular. Since $P(R) = 0$, R is a semiprime ring. By Proposition 2.4(2), R is reduced. Hence R is strongly regular.

(3)$\Longrightarrow$(1) Since R is strongly regular, $P(R) = 0$. Let Q be a prime ideal of R. If $a \in R$ but $a \notin Q$ then there exists $b \in R$ such that $a = aba$. Set $e = ba$; then $a(1-e) = 0, e^2 = e$. Since R is Abelian, $aR(1-e) = 0 \subseteq Q$, which implies $1 - e \in Q$. Let $\bar{R} = R/Q$; then $\bar{1} = \bar{e} = \bar{b}\bar{a}$. Similarly, $\bar{a}\bar{b} = \bar{1}$; it follows that $\overline{R}$ is a division ring. Hence Q is a maximal ideal. □

A ring R is called a left $V-$ring if every simple left $R-$module is injective. One of the three problems considered in [9] asked: For what rings R is it true that R is regular if and only if R is a left $V-$ring?

Corollary 4.5 *For a $GWCN$ exchange ring R, the following conditions are equivalent:*

(1) R is a strongly regular ring;

(2) R is a left weakly regular ring;

(3) R is a left $V-$ring.

Proof (1) $\Longrightarrow$ (3) It follows from [2, Theorem 4.8].

(3) $\Longrightarrow$ (2) follows from [9, Corollary 7].

(2) $\Longrightarrow$ (1) By Theorem 4.1(4), R is a left quasi-duo ring. Since left quasi-duo left weakly regular rings are strongly regular, we are done. □

It is well known that an exchange ring R has stable range 1 if and only if for any $a, x \in R$ and $e \in E(R)$, $ax + e = 1$ implies $a + ey \in U(R)$ for some $y \in R$.

Proposition 4.6 *An exchange ring R has stable range 1 if and only if for every von Neumann regular element a of R, there exists $u \in U(R)$ such that $a - aua \in Z_l(R)$.*

Proof The necessity is clear.

Now assume $ax+e=1$, where $a,x\in R$ and $e\in E(R)$. Then $a=axa+ea$. If $ea=0$, then $a=axa$. By hypothesis, there exists $u\in U(R)$ such that $a-aua\in Z_l(R)$. Let $a=aua+z$ for some $z\in Z_l(R)$. Then $1-e=ax=auax+zx=au(1-e)+zx$ and $(au-e)^2=auau-aue-eau+e=au-zu-aue+e=au(1-e)+e-zu=1-e-zx-zu+e=1-(zx+zu)$. Clearly, $zx+zu\in Z_l(R)$. Since R is an exchange ring, there exists $g\in E(R)$ such that $g\in(zx+zu)R\subseteq Z_l(R)$ and $1-g\in(1-zx-zu)R$; it follows that $g\in Z_l(R)$ and so $g=0$; this gives $1\in(1-zx-zu)R$. Write $1=(1-zx-zu)t$ for some $t\in R$. Then $1-zx-zu=(1-zx-zu)t(1-zx-zu)$ and $1-(1-zx-zu)t\in l(1-zx-zu)$. Since $zx+zu\in Z_l(R)$ and $l(zx+zu)\cap l(1-zx-zu)=0$, $l(1-zx-zu)=0$. Hence $(1-zx-zu)t=1$; it follows that $1-zx-zu\in U(R)$, that is, $au-e\in U(R)$. Let $au-e=v$ for some $v\in U(R)$. Then $a-eu^{-1}=vu^{-1}\in U(R)$. If $ea\neq 0$, then $a\neq axa$. Let $f=ax=1-e$ and $r=fa-a$. Then $rx=(fa-a)x=(axa-a)x=(ax-1)ax=-e(1-e)=0$ and $fr=f^2a-fa=0$. Let $a'=a+r$. Then $a'x=ax+rx=ax=f$, $a'xa'=fa'=fa+fr=fa=r+a=a'$, and $a'x+e=f+e=ax+e=1$. Since $ea'=ea+er=efa=eaxa=e(1-e)a=0$, by a similar proof of above, there exists $w\in U(R)$ such that $a'-ew=s\in U(R)$. Since $fr=0$, $r=(1-f)r=er$; this leads to $s=a'-ew=a+r-ew=a+e(r-w)$. Therefore R has stable range 1. □

References

[1] Badawi A. On Abelian $\pi-$regular rings. Comm Algebra 1997; 25: 1009–1021.

[2] Brown SH. Rings over which every simple module is rationally complete. Canada Math Bull 1973; 25: 693–701.

[3] Camillo VP, Yu HP. Exchange rings, units and idempotents. Comm Algebra 1994; 22: 4737–4749.

[4] Chen HY. On $nil-$semicommutative rings. Thai J Math 2011; 9: 37–47.

[5] Chen WX. On semiabelian $\pi-$regular rings. Intern J Math Sci 2007; 23: 1–10.

[6] Cohn PM. Reversible rings. Bull Korean Math Sci 1999; 31: 641–648.

[7] Contessa M. On $pm-$rings. Comm Algebra 1982; 10: 93–108.

[8] Drazin MP. Rings with central idempotent or nilpotent elements. Trans Amer Math Soc 1956; 187: 157–165.

[9] Fisher JW. Von Neumann regular rings versus $V-$rings. Proc Amer Math Soc 1974; 205: 101–119.

[10] Goodearl KR. Ring Theory: Non-singular Rings and Modules. New York, NY, USA: Marcel-Dekker, 1974.

[11] Handelman D. Perspectivity and cancellation in regular rings. J Algebra 1977; 48: 1–16.

[12] Hwang SU, Joen YC, Park KS. On NCI rings. Bull Korean Math Soc 2007; 44: 215–223.

[13] James PL. Commutativity conditions for rings: 1950-2005. Expositiones Math. 2007; 25: 165–174.

[14] Kim NK, Nam SB, Kim JY. On simple singular $GP-$injective modules. Comm Algebra 1999; 27: 2087–2096.

[15] Li NJ, Wei JC. NSF rings. J Yangzhou Univ 2011; 14: 1–4.

[16] Nicholson WK. Lifting idempotents and exchange rings. Trans Amer Math Soc 1997; 229: 269–178.

[17] Rege MB. On von Neumann regular rings and $SF-$rings. Math Japonica 1986; 31: 927–936.

[18] Tuganbaev A. Rings close to regular. Math Appl 2002; 99: 455–462.

[19] Vaserstein LN. Bass' first stable range condition. J Pure Appl Algebra 1984; 34: 319–330.

[20] Wang L, Wei JC, Li LB. NZI rings. Turk J Math 2013; 37: 781–792.

[21] Warfield RB. A Krull-Schmidt theorem for infinite sums of modules. Proc Amer Math Soc 1969; 22: 460–465.

[22] Wei ZX. A note on the commutativity of semiprime rings. J Math Resea Appl (China) 1985; 5: 109–110.

[23] Wei JC. On simple singular $YJ-$injective modules. Southea Asian Bull Math 2007; 31: 1–10.

[24] Wei JC. Certain rings whose simple singular modules are nil-injective. Turk J Math 2008; 32: 393–408.

[25] Wei JC. $MC2$ rings. Kyungp Math J 2008; 48: 651–663.

[26] Wei JC, Chen JH. $Nil-$injective rings. Intern Electr J Algebra 2007; 2: 1–21.

[27] Wei JC, Chen JH. NPP rings, reduced rings and SNF rings. Intern Electr J Algebra 2008; 4: 9–26.

[28] Wei JC, Li LB. Quasi-normal rings. Comm Algebra 2010; 38: 1855–1868.

[29] Wei JC, Li LB. Strong DS rings. Southea Asian Bull Math 2009; 33: 375–390.

[30] Yu HP. On quasi-duo rings. Glasgow Math J 1995; 37: 21–31.