Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2015) 39: 369 – 394

doi:10.3906/mat-1404-72

Regular poles for the p-adic group GSp_4-II

Yusuf DANIŞMAN*
Elementary Mathematics Education, Faculty of Education, Mevlana University, Selçuklu, Konya, Turkey

Received: 24.04.2014 • **Accepted/Published Online:** 28.01.2015 • **Printed:** 29.05.2015

Abstract: We compute the regular poles of the L-factors of the admissible and irreducible representations of the group GSp_4, which admit a nonsplit Bessel functional and have a Jacquet module length of 3 with respect to the unipotent radical of the Siegel parabolic, over a non-Archimedean local field of odd characteristic. We also compute the L-factors of the generic representations of GSp_4.

Key words: L-function, L-factor, GSp(4), regular pole

1. Introduction

Let k be a non-Archimedean local field of odd characteristics. By Table A.3 of [7], the length of a Jacquet module with respect to the unipotent radical of the Siegel parabolic of an irreducible admissible representation of $GSp_4(k)$ is at most four. In [2], we considered the representations that have Jacquet module length of at most 2 and admit a nonsplit Bessel functional. For such representations we computed the regular poles of the L-factors of the spinor (degree 4) L-functions defined by Piatetski-Shapiro [5]. As a consequence, we also obtained the L-factors of the generic ones among the representations that were considered.

In this paper, we compute the regular poles of the same type of the representations of $GSp_4(k)$ mentioned above, but with Jacquet module length 3. As a result, we also determine the L-factors of the generic ones. Our results agree with the results of [11], [3], and the local Langlands conjecture.

There are three available constructions to define L-factors of representations of $GSp_4(k)$. The first one was defined by Novodvorsky integrals in [4] only for generic representations. The second one was defined by Shahidi in [9] and extended by Gan and Taked in [3] for all representations except nongeneric supercuspidal ones. In this paper, we consider the construction of Piatetski-Shapiro given in [5] for all infinite dimensional representations by using the Bessel model.

Let us first give the definition of the Bessel model. Let S be the unipotent radical of the Siegel parabolic subgroup of $GSp_4(k)$ and let ψ be any nondegenerate character of S. We can realize the group $GL_2(k)$ in the Levi subgroup of the Siegel parabolic subgroup. Let T be the connected component of the stabilizer of ψ in $GL_2(k)$. Then T is isomorphic to units of quadratic extension K of k. Therefore, $R = TS$ is a subgroup of $GSp_4(k)$. Let Λ be any character of T. Then define $\alpha_{\Lambda,\psi}(r) =: \Lambda(t)\psi(s)$ for $r \in R, s \in S, t \in T$ and $r = st$.

Let (Π, V_Π) be an infinite-dimensional, irreducible, and admissible representation of $GSp_4(k)$. The

*Correspondence: ydanisman@mevlana.edu.tr
2010 *AMS Mathematics Subject Classification:* Primary 11F70; 11F85.

dimension of the space

$$\mathrm{Hom}_R(\Pi, \alpha_{\Lambda,\psi}) \cong \mathrm{Hom}_{GSp_4(k)}(\Pi, \mathrm{Ind}_R^{GSP_4(k)} \alpha_{\Lambda,\psi})$$

is at most one. If it is nonzero, the image of Π in the induced space above is called the Bessel model of Π. If Π is infinite-dimensional, then for some choice of R, ψ, and Λ the Bessel model exists.

In the construction of Piatetski-Shapiro the local integrals are

$$L(s; W_u, \Phi, \mu) = \int_{N\backslash G} W_u(g)\Phi[(0,1)g]\mu(\mathrm{detg})|detg|_k^{s+\frac{1}{2}} dg,$$

where $\Phi \in C_c^\infty(K^2)$, μ is a character of k^*, $u \in V_\Pi$, W_u is an element of the Bessel model, and N, G are subgroups of $Gsp_4(k)$, which can be realized in $GL_2(K)$ and will be defined in the next section.

The integral family $\{L(s; W_u, \Phi, \mu) : v \in V_\Pi,\ \Phi \in C_c^\infty(K^2)\}$ admits a greatest common denominator for all its elements. Hence, there exists a function $L(s, \Pi, \mu)$ called an L-factor such that $L(s; W_u, \Phi, \mu)/L(s, \Pi, \mu)$ is entire for all $u \in V_\Pi$ and $\Phi \in C_c^\infty(K^2)$. The poles of the L-factor, coming from an integral with a Schwartz function vanishing at zero, are called regular poles.

Let us summarize the results of [2].

1) Define $\varphi_u(x) := W_u \begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix}$. By using Iwasawa decomposition in Proposition 2.5 of [2] it was proved that the regular poles of the L-factors are the poles of the meromorphic continuation of the integrals

$$\int_{k*} \varphi_u(x)\mu(x)|x|^{s-3/2} d^*x.$$

Hence, the regular poles depend only on the asymptotic behavior of $\varphi_u(x)$.

2) If the length of the Jacquet module is zero, then by Proposition 3.1 of [2], φ_u has a compact support in k^* and there is no regular pole.

3) If the length of the Jacquet module is one, then by Proposition 3.2 of [2], for $|x|$ sufficiently small we have

$$\varphi_u(x) = C\chi(x)$$

for a constant C and a character χ of k^*, where the Jacquet module is $\oplus\chi$ as k^* module. By Lemma 3.4 of [2], a regular pole is the pole of $CL(s, \chi)$.

4) If the length of the Jacquet module is two then the constituents of the Jacquet module are $\oplus\chi_1$ and $\oplus\chi_2$ as k^* module. If $\chi_1 \neq \chi_2$, then by Proposition 3.2 of [2],

$$\varphi_u(x) = C_1\chi_1(x) + C_2\chi_2(x)$$

if $\chi_1 = \chi_2 = \chi$ then by Proposition 3.2 of [2]

$$\varphi_u(x) = C_1\chi(x) + C_2\chi(x)v_k(x),$$

where C_1 and C_2 are constants in k and v_k is the valuation of k. Hence, by Lemma 3.4 and 3.7 of [2], regular poles are the poles of $C_1C_2L(s, \chi_1)L(s, \chi_2)$ or the least common multiple of $C_1L(s, \chi)$ and $C_2L(s, \chi)^2$.

5) After determining the asymptotic behavior of φ_u we showed that for some choice of $u \in V_\Pi$ we have that C, C_1, and C_2 above are nonzero. In Proposition 5.8 and Proposition 5.11 of [2] it was proved that this is a consequence of the existence of the homomorphisms from the constituents of the Jacquet module to the character Λ, which depends on the Bessel existence conditions. By using these results in Theorem 5.9 of [2] and Theorem 5.16 of [2] we computed the regular poles of each representations that were considered.

In this paper, we follow similar steps but extend the results. Unlike the case of the paper [2], we need the Jacquet module structures of the representations more explicitly, namely not only the constituents but also their orders.

This paper is organized as follows. In Section 2, we give the subgroups of $GSp_4(k)$, definitions of Bessel model, local L-factors, and regular poles. In Section 3, we determine the Jacquet module structure of the representations that we consider. In Section 4, we characterize the exact sequences of k^*. In Section 5, we determine the asymptotic behavior of φ_u and possible regular poles. In Section 6, we show that there is a relation between the asymptotic behavior of the φ_u and the homomorphisms from the constituents of the Jacquet module to the character Λ. Then we compute the regular poles of each representation separately. The results of Section 6 with exceptional (nonregular) poles as expected by the local Langlands conjecture and semisimplifications of the Jacquet modules are given in the Appendix.

2. Definitions and preliminaries

We fix some notations.
k is a non-Archimedean local field of odd characteristic.
v_k is the valuation of k.
ν is the absolute value on k.
$\mathcal{O}$ is the ring of integers of k.
$\mathcal{P}$ is the unique maximal prime ideal of $\mathcal{O}$.
ϖ is a fixed generator of $\mathcal{P}$.
q is the cardinality of the residue field of k.
ψ is a nontrivial additive character of k with conductor $\mathcal{O}$.
$dx = d_\psi x$ is the self-dual Haar measure on k.
If ξ is a representation of a group, then its space and central character are denoted by V_ξ and ω_ξ, respectively.

2.1. $\mathbf{GSp_4(k)}$ and its subgroups

Let us define the group $GSp_4(k)$ by

$$GSp_4(k) = \left\{ g \in GL_4(k) : g^t J g = \lambda(g) J \text{ for some } \lambda(g) \in k^* \right\},$$

where $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ and $J = \begin{pmatrix} & w \\ -w & \end{pmatrix}$. Let

$$P = \left\{ g \in GSp_4(k) : g = \begin{pmatrix} A & B \\ & D \end{pmatrix}, A, B, D \in M_2(k) \right\}$$

be the Siegel parabolic subgroup of $GSp_4(k)$.

$$M = \left\{ \begin{pmatrix} A & \\ & \lambda (A')^{-1} \end{pmatrix} : A \in GL_2(k), \lambda \in k^* \right\}$$

is the Levi factor of P and

$$S=\left\{\begin{pmatrix} I_2 & Y \\ & I_2 \end{pmatrix} : Y=Y'\right\},$$

where $X'=w(X^t)w$ for $X\in M_2(k)$ is the unipotent radical of P. Any character of S is of the form

$$\psi_\beta\begin{pmatrix} I_2 & Y \\ & I_2 \end{pmatrix}=\psi[\mathrm{tr}(\beta Y)]$$

for some $\beta=\beta'$. ψ_β is called nondegenerate if $\beta\in GL_2(k)$.

Let ψ_β be a nondegenerate character of S and let T be the connected component of the stabilizer of ψ_β in M; then there is a unique semisimple algebra K over k of index $(K:k)=2$ and $T\cong K^*$. K is either a quadratic field extension of k and $K=k(\sqrt{\rho})$ for some $\rho\notin(k^*)^2$ or $K=k\oplus k$. If K is a field then T is called nonsplit. Otherwise it is called split.

In this paper, we consider the nonsplit case and let $K=k(\rho)$. The group

$$G=\{g\in GL_2(K):\det g\in k^*\}$$

can be realized in $GSp_4(k)$ as

$$\begin{pmatrix} a+b\sqrt{\rho} & c+d\sqrt{\rho} \\ e+f\sqrt{\rho} & m+n\sqrt{\rho} \end{pmatrix}\hookrightarrow\left(\begin{array}{cc|cc} a & b & c & d \\ b\rho & a & d\rho & c \\ \hline e & f & m & n \\ f\rho & e & n\rho & m \end{array}\right),$$

where $a+b\sqrt{\rho}, c+d\sqrt{\rho}, e+f\sqrt{\rho}, m+n\sqrt{\rho}\in K^*, an+bm=cf+de$. Let

$$N=\left\{\begin{pmatrix} I_2 & * \\ & I_2 \end{pmatrix}\in G\right\}\subset G,$$

which can be realized as a subgroup of S. In the nonsplit case,

$$T=\left\{\left(\begin{array}{cc|cc} a & b & & \\ b\rho & a & & \\ \hline & & a & -b \\ & & -b\rho & a \end{array}\right) : a+b\sqrt{\rho}\in K^*\right\}.$$

The center of $GSp_4(k)$ is

$$Z=\left\{\begin{pmatrix} aI_2 & \\ & a^2((aI_2)')^{-1} \end{pmatrix} : a\in k^*\right\}$$

and we define the group

$$H:=\left\{\begin{pmatrix} xI_2 & \\ & x^2((xI_2)')^{-1} \end{pmatrix} : x\in k^*\right\}.$$

2.2. Bessel model, L-factor, and regular poles

If Λ is a character of T and ψ is a nondegenerate character of S then for $r = ts \in R$

$$\alpha_{\Lambda,\psi}(r) := \Lambda(t)\psi(s)$$

is a character of R. The following theorem guarantees the existence and uniqueness of the Bessel model.

Theorem 2.1 *[5] Let (Π, V_Π) be an irreducible smooth, admissible, and preunitary representation of $Gsp_4(k)$; then*

$$dim[Hom_R(\Pi, \alpha_{\Lambda,\psi})] \leq 1.$$

If Π is infinite-dimensional, it is nonzero for some choice of Λ, ψ, and R.

Let l be a nonzero element of $\mathrm{Hom}_R(\Pi, \alpha_{\Lambda,\psi})$. For $u \in V_\Pi$ define Bessel function W_u on $GSp_4(k)$ by $W_u(g) := l(\Pi(g)u)$. Note that for $r \in R$, $g \in GSp_4(k)$ and $v \in V_\Pi$ we have $W_v(rg) = \alpha_{\Lambda,\psi}(r)W_v(g)$. The space

$$\mathbf{W}^{\Lambda,\psi} = \{W_u : u \in V_\Pi\}$$

is called the Bessel model of Π. A representation of $GSp_4(k)$ can be defined on $\mathbf{W}^{\Lambda,\psi}$ by right translation and $\Pi \cong \mathbf{W}^{\Lambda,\psi}$. For $h_x := \begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} \in H$ define $\varphi_u(x) := W_u(h_x)$. The next theorem provides the definitions of L-functions and L-factors.

Theorem 2.2 *[5] Let $\Phi \in C_c^\infty(K^2)$ and μ be a character of k^*. Then for $s \in \mathbb{C}$, the integral*

$$L(s; W_u, \Phi, \mu) = \int_{N\backslash G} W_u(g)\Phi[(0,1)g]\mu(detg)|detg|_k^{s+\frac{1}{2}}dg$$

converges absolutely for $Re(s)$ large enough and has a meromorphic continuation to the whole plane. These integrals form a fractional ideal of the ring $\mathbb{C}[q^s, q^{-s}]$ of the form $L(s;\Pi,\mu)\mathbb{C}[q^s,q^{-s}]$. The factor $L(s;\Pi,\mu)$ is of the form $P(q^{-s})^{-1}$, where $P(X) \in \mathbb{C}[X]$, $P(0) = 1$ and is called the L-factor of Π twisted by μ.

A pole of $L(s;\Pi,\mu)$ is called a regular pole if it is a pole of some $L(s; W_u, \Phi, \mu)$ with $\Phi(0,0) = 0$. Any other pole is called an exceptional pole. Regular poles will be expressed as poles of the Tate L-functions:

$$L(s,\chi) = \begin{cases} 1 & \text{if } \chi \text{ is ramified} \\ (1-\chi(\varpi)q^{-s})^{-1} & \text{if } \chi \text{ is unramified} \end{cases}$$

where χ is a character of k^*.

By using Iwasawa decomposition regular poles can be characterized as follows.

Proposition 2.3 *Regular poles of $L(s;\Pi,\mu)$ are the poles of the integrals*

$$\int_{k*} \varphi_u(x)\mu(x)|x|^{s-3/2}d^*x, \qquad v \in V_\Pi.$$

Proof Proposition 2.5 of [2] □

By the following theorem, finding regular poles of generic representations is equivalent to finding L-factors.

Theorem 2.4 *[5] If Π is generic, then its L-factor has only regular poles.*

2.3. Parabolic induction and Jacquet module

Let P_o be a maximal standard parabolic subgroup of $GSp_4(k)$ with Levi decomposition $P_o = M_oU_o$. Let (τ, V_τ) be a representation of M_o and let δ_{P_o} be the modular character of P_o . The normalized parabolic induction from P_o to $GSp_4(k)$ is defined as $\mathrm{ind}_{P_o}^{GSp_4}\tau = \{f : GSp_4(k) \to V_\tau : f(mug) = \delta_{P_o}(m)^{1/2}\tau(m)f(g), \quad \text{for } m \in M_o \text{ and } u \in U_o\}$. The action of $GSp_4(k)$ on $\mathrm{ind}_{P_o}^{GSp_4}\tau$ is by right translation. The unnormalized one is denoted by Ind.

Let (Π, V_Π) be an admissible and irreducible representation of $GSp_4(k)$. Define

$$V_S(\Pi) := \mathrm{span}\{v - \Pi(s)v : s \in S, v \in V_\Pi\}.$$

The normalized Jacquet module with respect to S is the smooth representation of M defined by

$$R_S(\Pi) = \Pi_S \otimes \delta_P^{-1/2},$$

where $(\Pi_S, V_\Pi/V_S(\Pi))$ is the regular Jacquet module.

If $p = \begin{pmatrix} A & * \\ & \lambda(A')^{-1} \end{pmatrix} \in P$ for $A \in GL_2(k)$, then $\delta_P(p) = |\det(A)^3\lambda^{-3}|$.

Let B denote the Borel subgroup of $GL_2(k)$. For the characters χ_1, χ_2 of k^*, define $ind_B^{GL_2(k)}(\chi_1,\chi_2) = \{f : GL_2(k) \to \mathbb{C} : f(\begin{pmatrix} a & b \\ & d \end{pmatrix} g) = \delta_B(\begin{pmatrix} a & b \\ & d \end{pmatrix})^{1/2}\chi_1(a)\chi_2(d)f(g), \quad g \in GL_2(k)\}$ to be the normalized induction from B to $GL_2(k)$, where $\delta_B(\begin{pmatrix} a & b \\ & d \end{pmatrix}) = |\frac{a}{d}|$. Let τ be a representation of $GL_2(k)$, Jacquet module of τ with respect to the unipotent subgroup $N_{GL_2(k)} = \left\{\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \in GL_2(k)\right\}$ is

$$J(\tau) = V_\tau/\{\tau(n)u - u : n \in N_{GL_2(k)},\, u \in V_\tau\}$$

and it is a representation of the diagonal subgroup of $GL_2(k)$. By Theorem 4.5.1 of [1], $ind_B^{GL_2(k)}(\nu^{1/2}, \nu^{-1/2})$ has an irreducible subrepresentation of codimension one denoted by $St_{GL_2(k)}$ and one-dimensional quotient denoted by $1_{GL_2(k)}$. Also, by Theorem 4.5.4 of [1],

$$J(St_{GL_2(k)})\begin{pmatrix} a & \\ & d \end{pmatrix} = \left|\frac{a}{d}\right| \text{ and } J(1_{GL_2(k)}) = 1_{k^*}.$$

We also denote the contragredient representation of τ by $\tilde{\tau}$.

3. Jacquet module structure

In this section we determine the Jacquet module structures of the representations of $Gsp_4(k)$, which are induced from the Siegel parabolic subgroup. First we provide the root system and Weyl group of $GSp_4(k)$. For details, one can look at Section 2.3 of [7].

3.1. Weyl group

A base for a root system of $GSp_4(k)$ is α, β and the positive roots are

$$\alpha, \beta, \alpha+\beta,\ 2\alpha+\beta.$$

A Weyl group is

$$\{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2\}$$

where

$$s_1 = \begin{pmatrix} w & \\ & w \end{pmatrix}, s_2 = \left(\begin{array}{cc|cc} 1 & & & \\ & & 1 & \\ \hline & -1 & & \\ & & & 1 \end{array}\right), s_2s_1s_2 = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}$$

and

$s_1(\alpha) = -\alpha,\ s_1(\beta) = 2\alpha+\beta,\ \ s_2(\alpha) = \alpha+\beta,\ s_2(\beta) = -\beta.$
$s_1s_2(\alpha) = \alpha+\beta,\ s_1s_2(\beta) = \alpha,\ \ s_2s_1(\alpha) = -\alpha-\beta,\ s_2s_1(\beta) = 2\alpha+\beta.$
$s_2s_1s_2(\alpha) = \alpha,\ s_2s_1s_2(\beta) = \alpha+\beta,\ \ s_1s_2s_1(\alpha) = -\alpha-\beta,\ s_2s_1s_2(\beta) = \beta.$
$s_1s_2s_1s_2(\alpha) = -\alpha,\ s_2s_1s_2(\beta) = \alpha+\beta.$

We need the following sets:

$[W/W_\alpha] := \{w : w\alpha > 0\} = \{1, s_2, s_1s_2, s_2s_1s_2\}$

$[W_\alpha\backslash W] := \{w : w^{-1}\alpha > 0\} = \{1, s_2, s_2s_1, s_2s_1s_2\}$

$[W_\alpha\backslash W/W_\alpha] := [W/W_\alpha] \cap [W_\alpha\backslash W] = \{1,\ w_1 = s_2s_1s_2,\ w_2 = s_2\}.$

Siegel parabolic subgroup of $GSp_4(k)$ corresponds to the positive root α and $P = P_\alpha$.

Proposition 3.1 *i)* $w_1^{-1}Pw_1 \cap P = M$.
ii) $w_1^{-1}Pw_1 \cap M = M$.
iii) $w_1^{-1}Pw_1 \cap S = I_4$.
iv) $M \cap w_1Sw_1^{-1} = I_4$.

Proof For $p = \begin{pmatrix} A & B \\ & D \end{pmatrix} \in P$ we have

$$\begin{aligned} w_1^{-1}pw_1 &= \begin{pmatrix} & -I_2 \\ I_2 & \end{pmatrix}\begin{pmatrix} A & B \\ & D \end{pmatrix}\begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix} \\ &= \begin{pmatrix} & -D \\ A & B \end{pmatrix}\begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix} \\ &= \begin{pmatrix} D & \\ -B & A \end{pmatrix}. \end{aligned}$$

Hence,

$$w_1^{-1}Pw_1 \cap P = M, \;\; w_1^{-1}Pw_1 \cap M = M, \;\; w_1^{-1}Pw_1 \cap S = I_4.$$

Since

$$w_1^{-1}Pw_1 \cap S = I_4 \Rightarrow P \cap w_1 S w_1^{-1} = I_4,$$

we have $M \cap w_1 S w_1^{-1} = I_4$. □

Proposition 3.2 *Let*

$$M_2 = \left(\begin{array}{cc|cc} * & * & & \\ & * & & \\ \hline & & * & * \\ & & & * \end{array}\right), \; M_2' = \left(\begin{array}{cc|cc} 1 & * & & \\ & 1 & & \\ \hline & & 1 & * \\ & & & 1 \end{array}\right),$$

$$P_2 = \left(\begin{array}{cc|cc} * & * & * & * \\ & * & & * \\ \hline & & * & * \\ & & & * \end{array}\right), \; S_2 = \left(\begin{array}{cc|cc} 1 & & * & * \\ & 1 & & * \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$$

be subgroups of $GSp_4(k)$*.*

i) $w_2^{-1}Pw_2 \cap P = P_2$*.*

ii) $w_2^{-1}Pw_2 \cap M = M_2$*.*

iii) $w_2^{-1}Pw_2 \cap S = S_2$*.*

iv) $M \cap w_2 S w_2^{-1} = M_2'$*.*

Proof For $p = \left(\begin{array}{cc|cc} a & b & c & d \\ e & f & g & h \\ \hline & & m & n \\ & & r & s \end{array}\right) \in P$ we have

$$\begin{aligned} w_2^{-1}pw_2 &= \left(\begin{array}{cc|cc} 1 & & & \\ & & -1 & \\ \hline & 1 & & \\ & & & 1 \end{array}\right) \left(\begin{array}{cc|cc} a & b & c & d \\ e & f & g & h \\ \hline & & m & n \\ & & r & s \end{array}\right) \left(\begin{array}{cc|cc} 1 & & & \\ & & 1 & \\ \hline & -1 & & \\ & & & 1 \end{array}\right) \\ &= \left(\begin{array}{cc|cc} a & b & c & d \\ & & -m & -n \\ \hline e & f & g & h \\ & & r & s \end{array}\right) \left(\begin{array}{cc|cc} 1 & & & \\ & & 1 & \\ \hline & -1 & & \\ & & & 1 \end{array}\right) \\ &= \left(\begin{array}{cc|cc} a & -c & b & d \\ & m & & -n \\ \hline e & -g & f & h \\ & -r & & s \end{array}\right). \end{aligned}$$

Hence,

$$M_2 = w_2^{-1}Pw_2 \cap M, \;\; w_2^{-1}Pw_2 \cap P = P_2, \;\; w_2^{-1}Pw_2 \cap S = S_2.$$

For $s=\left(\begin{array}{cc|cc} 1 & & c & d \\ & 1 & g & h \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\in S$ we have

$$\begin{aligned} w_2 s w_2^{-1} &= \left(\begin{array}{cc|cc} 1 & & & \\ & & 1 & \\ \hline & -1 & & \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & & c & d \\ & 1 & g & h \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & & & \\ & & -1 & \\ \hline & 1 & & \\ & & & 1 \end{array}\right) \\ &= \left(\begin{array}{cc|cc} 1 & & c & d \\ & & 1 & \\ \hline & -1 & -g & -h \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & & & \\ & & -1 & \\ \hline & 1 & & \\ & & & 1 \end{array}\right) \\ &= \left(\begin{array}{cc|cc} 1 & c & & d \\ & 1 & & \\ \hline & -g & 1 & -h \\ & & & 1 \end{array}\right). \end{aligned}$$

Hence, $M\cap w_2 S w_2^{-1}=M_2'$. ◻

Proposition 3.3 *i)* $dim(P\backslash Pw_1P)=3$.
ii) $dim(P\backslash Pw_2P)=2$.

Proof i) By Proposition 3.1(i), since $w_1Mw_1^{-1}\subset P$, we have

$$P\backslash Pw_1P=P\backslash P(w_1Mw_1^{-1})w_1S=P\backslash Pw_1S.$$

Hence, the representatives for the quotient are the elements of w_1S and $dim(P\backslash Pw_1P)=3$.

ii) First note that $w_2Mw_2^{-1}\cap P=w_2M_2w_2^{-1}=M_2$, and if $g=\begin{pmatrix} * & * \\ c & * \end{pmatrix}\in GL_2(k)$ and $c\neq 0$ then there exists $b\in B$ and $z=\begin{pmatrix} & 1 \\ 1 & * \end{pmatrix}\in GL_2(k)$ such that $g=bz$. Hence, $M=M_2\cap M_2M'$ where $M'=\left\{\left(\begin{array}{cc|cc} & 1 & & \\ 1 & n & & \\ \hline & & & 1 \\ & & 1 & -n \end{array}\right): n\in k\right\}$. Also let $S'=\left\{\left(\begin{array}{cc|cc} 1 & & & \\ & 1 & * & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\right\}$. Note that $M'S=SM'$ and $S=S_2S'$. Hence,

$$\begin{aligned} P\backslash Pw_2P &= P\backslash Pw_2MS \\ &= P\backslash Pw_2M_2S\cup P\backslash Pw_2M_2M'S \\ &= P\backslash Pw_2S\cup P\backslash Pw_2M'S \\ &= P\backslash Pw_2S\cup P\backslash Pw_2SM' \\ &= P\backslash Pw_2S\cup P\backslash Pw_2SM' \\ &= P\backslash Pw_2S_2S'\cup P\backslash Pw_2S_2S'M' \\ &= P\backslash Pw_2S'\cup P\backslash Pw_2S'M' \end{aligned}$$

and the representatives for $P\backslash Pw_2P$ are the elements of $S'\cup S'M'$. Hence, $dim(P\backslash Pw_2P)=2$. □

3.2. Jacquet module structure of induced representations

In this section we summarize the Jacquet module structures of induced representations due to Section 3.4 of [10] and Section 6.3 of the unpublished book Introduction to the Theory of Admissible Representations of p-adic Reductive Groups by William Casselman. Let θ be a representation of M and

$$I_n=\{f\in ind_P^G\theta : supp(f)\subset G_n=\cup_{dim(PwP)\geq n}PwP\}.$$

The elements of I_3, I_2, I_1, and $I_0=I$ have support in Pw_1P, $Pw_1P\cup Pw_2P$, $Pw_1P\cup Pw_2P$, and $Gsp_4(k)$, respectively. Note that $I_1=I_2$. We also have

$$(I_n/I_{n+1})_S\cong Ind^M_{w^{-1}Pw\cap M}w^{-1}(\theta_{M\cap wSw^{-1}})(w^{-1}\delta_P^{1/2})\gamma, \tag{1}$$

where $dim(PwP)=n$ and γ is the modulus of the unique rational character of $w^{-1}Pw\cap P$ acting on $S/(w^{-1}Pw\cap S)$, and $\theta_{M\cap wSw^{-1}}$ is the Jacquet module of θ with respect to $M\cap wSw^{-1}$.

Now we give the Jacquet module structure of representations induced from the Siegel parabolic by using (1).

Proposition 3.4 *Let $\theta=\tau\otimes\rho$ be a representation of M. If $I=ind_P^{GSp_4(k)}\theta$ then $0\subset I_3\subset I_2\subset I$ and*

$$0\subset (I_3)_S\subset (I_2)_S\subset I_S,$$

where
i) $(I_3)_S=\widetilde{\tau}\otimes\omega_\tau\rho.$
ii) $(I_2/I_3)_S=ind_B^{GL_2(k)}J'(\tau)\otimes\rho\nu^{1/2}$ *where $J(\tau)$ is the Jacquet module of τ and*

$$J'(\tau)\otimes\rho\nu^{1/2}\left(\begin{array}{cc|cc} a & & & \\ & d & & \\ \hline & & \frac{\lambda}{d} & \\ & & & \frac{\lambda}{a}\end{array}\right)=J(\tau)\begin{pmatrix} a & \\ & \lambda/d\end{pmatrix}|ad|^{-1/2}\otimes\rho(\lambda)|\lambda|^{1/2}.$$

iii) $(I/I_2)_S=\theta.$

Proof First constituent: $w=w_1$
By (1),

$$(I_3)_S=Ind_M^Mw^{-1}(\theta_{I_4})(w^{-1}\delta^{1/2})\gamma=w^{-1}(\theta\delta^{1/2})\gamma$$

and γ is the modulus of the unique rational character of M acting on S; hence, it is δ.

$$\begin{aligned} w^{-1}(\theta\delta_P^{1/2})\delta\begin{pmatrix} A & \\ & \lambda(A')^{-1}\end{pmatrix} &= \theta\delta_P^{1/2}\begin{pmatrix}\lambda(A')^{-1} & \\ & \lambda(\lambda^{-1})\end{pmatrix}\left|\frac{det(A)}{\lambda}\right|^3 \\ &= \tau(\lambda(A')^{-1})\otimes\rho(\lambda)\left|\frac{det(\lambda(A')^{-1})}{\lambda}\right|^{3/2}\left|\frac{det(A)}{\lambda}\right|^3 \\ &= \tau((A')^{-1})\otimes\omega_\tau\rho(\lambda)\left|\frac{det(A)}{\lambda}\right|^{3/2} \\ &= \delta_P^{1/2}\begin{pmatrix} A & \\ & \lambda(A')^{-1}\end{pmatrix}\widetilde{\tau}(A)\otimes\omega_\tau\rho(\lambda).\end{aligned}$$

In Table A.2, the Jacquet module is normalized by $\delta_P^{-1/2}$; hence, the first constituent is $\widetilde{\tau} \otimes \omega_\tau \rho$.

Second constituent: $w = w_2$

By (1),

$$(I_2/I_3)_S = Ind_{M_2}^{M} w^{-1}(\theta_{M_2'})(w^{-1}\delta_P^{1/2})\gamma$$

and γ is the modulus of the unique rational character of P_2 acting on $S/S_2 = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & * & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$. Since

$$\left(\begin{array}{cc|cc} a & & & \\ & d & & \\ \hline & & \frac{\lambda}{d} & \\ & & & \frac{\lambda}{a} \end{array}\right)\left(\begin{array}{cc|cc} 1 & & & \\ & 1 & s & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} a & & & \\ & d & & \\ \hline & & \frac{\lambda}{d} & \\ & & & \frac{\lambda}{a} \end{array}\right)^{-1} = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & \frac{d^2 s}{\lambda} & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$$

$\gamma\left(\begin{array}{cc|cc} a & & & \\ & d & & \\ \hline & & \frac{\lambda}{d} & \\ & & & \frac{\lambda}{a} \end{array}\right) = \frac{d^2}{\lambda}$. Also,

$$\begin{aligned} w_2\left(\begin{array}{cc|cc} a & & & \\ & d & & \\ \hline & & \frac{\lambda}{d} & \\ & & & \frac{\lambda}{a} \end{array}\right)w_2^{-1} &= \left(\begin{array}{cc|cc} 1 & & & \\ & & 1 & \\ \hline & -1 & & \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} a & & & \\ & d & & \\ \hline & & \frac{\lambda}{d} & \\ & & & \frac{\lambda}{a} \end{array}\right)\left(\begin{array}{cc|cc} 1 & & & \\ & & -1 & \\ \hline & 1 & & \\ & & & 1 \end{array}\right) \\ &= \left(\begin{array}{cc|cc} a & & & \\ & & \frac{\lambda}{d} & \\ \hline & & & \\ & & & \frac{\lambda}{a} \end{array}\right)\left(\begin{array}{cc|cc} 1 & & & \\ & & -1 & \\ \hline & 1 & & \\ & & & 1 \end{array}\right) \\ &= \left(\begin{array}{cc|cc} a & & & \\ & \frac{\lambda}{d} & & \\ \hline & & \lambda\left(\frac{d}{\lambda}\right) & \\ & & & \lambda\left(\frac{1}{a}\right). \end{array}\right) \end{aligned}$$

Hence,

$$(I_2/I_3)_S = Ind_{B\times k^*}^{M} w^{-1}(\theta_N)(w^{-1}\delta_P^{1/2})\gamma$$

and $\theta_N = J(\tau)\otimes\rho$ where $J(\tau)$ is Jacquet module of τ. Thus we have

$$\begin{aligned}
& w^{-1}(\theta_N)(w^{-1}\delta^{1/2})\gamma\left(\begin{array}{cc|cc} a & & & \\ & d & & \\ \hline & & \frac{\lambda}{d} & \\ & & & \frac{\lambda}{a}\end{array}\right) \\
= \;& [J(\tau)\otimes\rho]\delta^{1/2}\left[w\left(\begin{array}{cc|cc} a & & & \\ & d & & \\ \hline & & \frac{\lambda}{d} & \\ & & & \frac{\lambda}{a}\end{array}\right)w^{-1}\right]\gamma\left(\begin{array}{cc|cc} a & & & \\ & d & & \\ \hline & & \frac{\lambda}{d} & \\ & & & \frac{\lambda}{a}\end{array}\right) \\
= \;& [J(\tau)\otimes\rho]\delta^{1/2}\left(\begin{array}{cc|cc} a & & & \\ & \frac{\lambda}{d} & & \\ \hline & & \lambda\left(\frac{d}{\lambda}\right) & \\ & & & \lambda\left(\frac{1}{a}\right)\end{array}\right)\gamma\left(\begin{array}{cc|cc} a & & & \\ & d & & \\ \hline & & \frac{\lambda}{d} & \\ & & & \frac{\lambda}{a}\end{array}\right) \\
= \;& J(\tau)\begin{pmatrix} a & \\ & \frac{\lambda}{d}\end{pmatrix}\rho(\lambda)\left|\frac{\frac{a\lambda}{d}}{\lambda}\right|^{3/2}\left|\frac{d^2}{\lambda}\right| \\
= \;& J(\tau)\begin{pmatrix} a & \\ & \frac{\lambda}{d}\end{pmatrix}\rho(\lambda)|a|^{3/2}|d|^{1/2}|\lambda|^{-1}. \qquad (2)
\end{aligned}$$

In Table A.2 the Jacquet module is normalized by $\delta_P^{-1/2}$. Hence, (2) becomes

$$\begin{aligned}
& \left|\frac{ad}{\lambda}\right|^{-3/2} J(\tau)\begin{pmatrix} a & \\ & \frac{\lambda}{d}\end{pmatrix}|a|^{3/2}|d|^{1/2}|\otimes\rho(\lambda)|\lambda|^{-1} \\
= \;& J(\tau)\begin{pmatrix} a & \\ & \frac{\lambda}{d}\end{pmatrix}|d|^{-1}\otimes\rho(\lambda)|\lambda|^{1/2} \\
= \;& \delta_B^{1/2}\delta_B^{-1/2}\begin{pmatrix} a & \\ & d\end{pmatrix}J(\tau)\begin{pmatrix} a & \\ & \frac{\lambda}{d}\end{pmatrix}|d|^{-1}\otimes\rho(\lambda)|\lambda|^{1/2} \\
= \;& \delta_B^{1/2}\begin{pmatrix} a & \\ & d\end{pmatrix}J(\tau)\begin{pmatrix} a & \\ & \frac{\lambda}{d}\end{pmatrix}|ad|^{-1/2}\otimes\rho(\lambda)|\lambda|^{1/2}
\end{aligned}$$

and the result follows.

Third constituent: $w = 1$
By (1),

$$(I/I_2)_S = Ind_M^M\theta_{I_4}\delta_P^{1/2}\gamma = \theta\delta_P^{1/2}\gamma$$

and γ is the modulus of the unique rational character of P acting on I_4; hence, it is 1. In Table A.2 the Jacquet module is normalized by $\delta_P^{-1/2}$ and hence the third constituent is θ. □

3.3. Representations and their Jacquet modules

In this section we determine Jacquet module structures of the representations that we consider in this paper. Irreducible and admissible representations of $GSp_4(k)$, which has Jacquet module length of 3, are given in Table

A.1 due to the Sally–Tadic classification in [8]. In this table, the representations that we consider are named as **IIa, IIb, Via, and VId**. Additionally, semisimplifications of these representations are given in Table A.2.

3.3.1. II-a: $\chi\mathrm{St}_{\mathrm{GL}_2(k)} \rtimes \sigma$

Proposition 3.5 *If* $I = \chi\mathrm{St}_{\mathrm{GL}_2(k)} \rtimes \sigma$ *then we have*

$$0 \underbrace{\subset}_{\chi^{-1}\mathrm{St}_{\mathrm{GL}_2(k)}\otimes\chi^2\sigma} (I_3)_S \underbrace{\subset}_{ind_B^{GL_2(k)}(\chi\nu^{1/2},\chi^{-1}\nu^{1/2})\otimes\chi\nu^{-1/2}\sigma} (I_2)_S \underbrace{\subset}_{\chi\mathrm{St}_{\mathrm{GL}_2(k)}\otimes\sigma} (I)_S.$$

Proof **First constituent:** By Proposition 3.4(i), the first constituent is

$$\widetilde{\chi\mathrm{St}_{\mathrm{GL}_2(k)}} \otimes \omega_{\chi\mathrm{St}_{\mathrm{GL}_2(k)}}\sigma = \chi^{-1}\mathrm{St}_{\mathrm{GL}_2(k)} \otimes \chi^2\sigma.$$

Second constituent: By Proposition 3.4(ii),

$$J(\chi\mathrm{St}_{\mathrm{GL}_2(k)})\begin{pmatrix} a & \\ & \frac{\lambda}{d}\end{pmatrix}|ad|^{-1/2} \otimes \sigma(\lambda)|\lambda|^{1/2}$$

$$= \chi\left(\frac{a\lambda}{d}\right)\left|\frac{a}{\frac{\lambda}{d}}\right||ad|^{1/2} \otimes \sigma(\lambda)|\lambda|^{1/2}$$

$$= \chi(a)|a|^{1/2}\chi^{-1}(d)|d|^{1/2} \otimes \chi\sigma(\lambda)|\lambda|^{-1/2}.$$

Hence, the second constituent is

$$ind_B^{GL_2(k)}(\chi\nu^{1/2},\chi^{-1}\nu^{1/2}) \otimes \chi\sigma\nu^{-1/2}.$$

Third constituent: By Proposition 3.4(iii), the third constituent is $\chi\mathrm{St}_{GL_2(k)} \otimes \sigma$. □

3.3.2. II-b: $\chi 1_{GL_2(k)} \rtimes \sigma$

Proposition 3.6 *If* $I = \chi 1_{\mathrm{GL}_2(k)} \rtimes \sigma$ *then we have*

$$0 \underbrace{\subset}_{\chi^{-1}1_{\mathrm{GL}_2(k)}\otimes\chi^2\sigma} (I_3)_S \underbrace{\subset}_{ind_B^{GL_2(k)}(\chi\nu^{-1/2},\chi^{-1}\nu^{-1/2})\otimes\chi\nu^{1/2}\sigma} (I_2)_S \underbrace{\subset}_{\chi 1_{GL_2(k)}\otimes\sigma} (I)_S.$$

Proof **First constituent:** By Proposition 3.4(i), the first constituent is

$$\widetilde{\chi 1_{GL_2(k)}} \otimes \omega_{\chi 1_{GL_2(k)}}\sigma = \chi^{-1}1_{GL_2(k)} \otimes \chi^2\sigma.$$

Second constituent: By Proposition 3.4(ii),

$$J(\chi 1_{GL_2(k)})\begin{pmatrix} a & \\ & \frac{\lambda}{d}\end{pmatrix}|ad|^{-1/2} \otimes \sigma(\lambda)|\lambda|^{1/2}$$

$$= \chi\left(\frac{a\lambda}{d}\right)|ad|^{-1/2} \otimes \sigma(\lambda)|\lambda|^{1/2}$$

$$= \chi(a)|a|^{-1/2}\chi^{-1}(d)|d|^{-1/2} \otimes \chi\sigma(\lambda)|\lambda|^{1/2}.$$

Hence, the second constituent is

$$ind_B^{GL_2(k)}(\chi\nu^{-1/2}, \chi^{-1}\nu^{-1/2}) \otimes \chi\nu^{1/2}\sigma.$$

Third constituent: By Proposition 3.4(iii), the third constituent is $\chi 1_{GL_2(k)} \otimes \sigma$. □

3.3.3. VI-a: $\tau(S, \nu^{-1/2}\sigma)$

By Section 2.2 of [7], we have

$$0 \longrightarrow VI - a \longrightarrow \nu^{1/2}\mathrm{St}_{GL_2(k)} \rtimes \nu^{-1/2}\sigma \longrightarrow VI - c \longrightarrow 0$$

and the Jacquet module of VI-c is $\nu^{-1/2}\mathrm{St}_{GL_2(k)} \otimes \nu^{1/2}\sigma$.

Proposition 3.7 *If* $I = \nu^{1/2}\mathrm{St}_{GL_2(k)} \rtimes \nu^{-1/2}\sigma$ *then we have*

$$0 \underbrace{\subset}_{\nu^{-1/2}\mathrm{St}_{GL_2(k)}\otimes\nu^{1/2}\sigma} (I_3)_S \underbrace{\subset}_{\nu^{1/2}ind_B^{GL_2(k)}(\nu^{1/2},\nu^{-1/2})\otimes\nu^{-1/2}\sigma} (I_2)_S \underbrace{\subset}_{\nu^{1/2}\mathrm{St}_{GL_2(k)}\otimes\nu^{-1/2}\sigma} (I)_S.$$

Proof **First constituent:** By Proposition 3.4(i), the first constituent is

$$\widetilde{\nu^{1/2}\mathrm{St}_{GL_2(k)}} \otimes \omega_{\nu^{1/2}\mathrm{St}_{GL_2(k)}}\nu^{-1/2}\sigma = \nu^{-1/2}\mathrm{St}_{\mathrm{GL}(2)} \otimes \nu^{1/2}\sigma.$$

Second constituent: By Proposition 3.4(ii),

$$\begin{aligned} &J(\nu^{1/2}\mathrm{St}_{GL_2(k)})\begin{pmatrix} a & \\ & \frac{\lambda}{d} \end{pmatrix}|ad|^{-1/2} \otimes |\lambda|^{-1/2}\sigma(\lambda)|\lambda|^{1/2} \\ &= \left|\frac{a\lambda}{d}\right|^{1/2} \left|\frac{a}{\frac{\lambda}{d}}\right| |ad|^{-1/2} \otimes \sigma(\lambda) \\ &= |a| \otimes \sigma(\lambda)|\lambda|^{-1/2}. \end{aligned}$$

Hence, the second constituent is

$$ind_B^{GL_2(k)}(\nu, 1) \otimes \nu^{-1/2}\sigma = \nu^{1/2}ind_B^{GL_2(k)}(\nu^{1/2}, \nu^{-1/2}) \otimes \nu^{-1/2}\sigma.$$

Third constituent: By Proposition 3.4(iii), the third constituent is $\nu^{1/2}\mathrm{St}_{GL_2(k)} \otimes \nu^{-1/2}\sigma$. □

Corollary 3.8

$$0 \underbrace{\subset}_{\nu^{1/2}ind_B^{GL_2(k)}(\nu^{1/2},\nu^{-1/2})\otimes\nu^{-1/2}\sigma} ((VI-a)_2)_S \underbrace{\subset}_{\nu^{1/2}\mathrm{St}_{GL_2(k)}\otimes\nu^{-1/2}\sigma} (VI-a)_S$$

3.3.4. VI-d: $L(\nu, 1_{K^*} \rtimes \nu^{-1/2}\sigma)$

By Section 2.2 of [7], we have

$$0 \longrightarrow VI-b \longrightarrow \nu^{1/2}1_{GL_2(k)} \rtimes \nu^{-1/2}\sigma \longrightarrow VI-d \longrightarrow 0$$

and the Jacquet module of VI-b is $\nu^{1/2}1_{GL_2(k)} \otimes \nu^{-1/2}\sigma$.

Proposition 3.9 *If* $I = \nu^{1/2}1_{GL_2(k)} \rtimes \nu^{-1/2}\sigma$ *then we have*

$$0 \underbrace{\subset}_{\nu^{-1/2}1_{GL_2(k)}\otimes\nu^{1/2}\sigma} (I_3)_S \underbrace{\subset}_{\nu^{-1/2}ind_B^{GL_2(k)}(\nu^{1/2},\nu^{-1/2})\otimes\nu^{1/2}\sigma} (I_2)_S \underbrace{\subset}_{\nu^{1/2}1_{GL_2(k)}\otimes\nu^{-1/2}\sigma} (I)_S.$$

Proof **First constituent:** By Proposition 3.4(i), the first constituent is

$$\widetilde{\nu^{1/2}1_{GL_2(k)}} \otimes \omega_{\nu^{1/2}1_{GL_2(k)}}\nu^{-1/2}\sigma = \nu^{-1/2}1_{\mathrm{GL}(2)} \otimes \nu^{1/2}\sigma.$$

Second constituent: By Proposition 3.4(ii),

$$\begin{aligned} &J(\nu^{1/2}1_{GL_2(k)})\begin{pmatrix} a & \\ & \frac{\lambda}{d} \end{pmatrix}|ad|^{-1/2} \otimes |\lambda|^{-1/2}\sigma(\lambda)|\lambda|^{1/2} \\ &= \left|\frac{a\lambda}{d}\right|^{1/2}|ad|^{-1/2} \otimes \sigma(\lambda) \\ &= |d|^{-1} \otimes \sigma(\lambda)|\nu|^{1/2}. \end{aligned}$$

Hence, the second constituent is

$$ind_B^{GL_2(k)}(1,\nu^{-1}) \otimes \nu^{1/2}\sigma = \nu^{-1/2}ind_B^{GL_2(k)}(\nu^{1/2},\nu^{-1/2}) \otimes \nu^{1/2}\sigma.$$

Third constituent: By Proposition 3.4(iii), the third constituent is $\nu^{-1/2}1_{GL_2(k)} \otimes \nu^{1/2}\sigma$. □

Corollary 3.10

$$0 \underbrace{\subset}_{\nu^{-1/2}ind_B^{GL_2(k)}(\nu^{1/2},\nu^{-1/2})\otimes\nu^{1/2}\sigma} ((VI-d)_2)_S \underbrace{\subset}_{\nu^{-1/2}1_{GL_2(k)}\otimes\nu^{1/2}\sigma} (VI-d)_S.$$

4. Representations of k^*

In this section, we describe some exact sequences of representations of k^* that we will need in the following section.

Proposition 4.1 *Let* χ_1 *and* χ_2 *be characters of* k^*, (ρ, U) *be a representation of* k^*, *and*

$$0 \longrightarrow \oplus\chi_1 \longrightarrow U \longrightarrow \oplus\chi_2 \longrightarrow 0.$$

i)If $\chi_1 \neq \chi_2$ *then* $U = \oplus\chi_1 \bigoplus \oplus\chi_2$. *Hence, if* $u \in U$ *then there exists* $u_1 \in \oplus\chi_1$ *and* $u_2 \in \oplus\chi_2$ *such that* $u = u_1 + u_2$ *and*

$$\rho(x)u = \chi_1(x)u_1 + \chi_2(x)u_2.$$

ii)If $\chi_1 = \chi_2 = \chi$ and $u \in U$ then for some $w \in \oplus\chi$ we have

$$\rho(x)u = \chi(x)u + \chi(x)v_k(x)w.$$

Proof Lemma 5.10 of [2]. □

Proposition 4.2 *Let χ_1, χ_2, and χ_3 be characters of k^* and let (ρ, U) and (ρ, U_1) be representations of k^* such that*

$$0 \longrightarrow U_1 \longrightarrow U \longrightarrow \oplus\chi_3 \longrightarrow 0 \tag{3}$$

and

$$0 \longrightarrow \oplus\chi_1 \longrightarrow U_1 \longrightarrow \oplus\chi_2 \longrightarrow 0.$$

i) If χ_1, χ_2, and χ_3 are all different then $U = \oplus\chi_1 \bigoplus \oplus\chi_2 \bigoplus \oplus\chi_3$. Hence, for all $u \in U$ there exists $u_i \in \oplus\chi_i$ for $i = 1, 2, 3$ such that $u = u_1 + u_2 + u_3$ and

$$\rho(x)u = \chi_1(x)u_1 + \chi_2(x)u_2 + \chi_3(x)u_3.$$

ii) If $\chi_1 = \chi_3$ and $\chi_2 \neq \chi_3$ then $U = U_1' \bigoplus \oplus\chi_2$ where $0 \longrightarrow \oplus\chi_1 \longrightarrow U_1' \longrightarrow \oplus\chi_1 \longrightarrow 0$. Hence, for all $u \in U$ there exists $u_1 \in U_1'$ and $u_2 \in \oplus\chi_2$ such that $u = u_1 + u_2$ and for some $u_1' \in \oplus\chi_1$ we have

$$\rho(x)u = \chi_1(x)u_1 + \chi_1(x)v_k(x)u_1' + \chi_2(x)u_2.$$

Proof i) By Proposition 4.1 (i), $U_1 = \oplus\chi_1 \bigoplus \oplus\chi_2$. If $u \in U$ then we have

$$\rho(x)u = \chi_3(x)u + w_1(x) + w_2(x) \tag{4}$$

for some $w_1(x) \in \oplus\chi_1$ and $w_2(x) \in \oplus\chi_2$. Hence,

$$\rho(y)w_1(x) = \chi_1(y)w_1(x), \;\; \rho(y)w_2(x) = \chi_2(y)w_2(x). \tag{5}$$

From Eq. (4),

$$\rho(xy)u = \chi_3(xy)u + w_1(xy) + w_2(xy). \tag{6}$$

By Eqs. (4) and (5),

$$\begin{aligned}
\rho(xy)u &= \rho(y)[\rho(x)u] \\
&= \rho(y)[\chi_3(x)u + w_1(x) + w_2(x)] \\
&= \chi_3(x)\rho(y)u + \rho(y)w_1(x) + \rho(y)w_2(x)] \\
&= \chi_3(x)[\chi_3(y)u + w_1(y) + w_2(y)] + \chi_1(y)w_1(x) + \chi_2(y)w_2(x) \\
&= \chi_3(xy)u + \chi_3(x)w_1(y) + \chi_3(x)w_2(y) + \chi_1(y)w_1(x) + \chi_2(y)w_2(x).
\end{aligned} \tag{7}$$

Hence, by Eqs. (6) and (7) we have

$$[w_1(xy) - \chi_3(x)w_1(y) - \chi_1(y)w_1(x)] + [w_2(xy) - \chi_3(x)w_2(y) - \chi_2(y)w_2(x)] = 0,$$

and since we have a direct sum,

$$w_1(xy) = \chi_3(x)w_1(y) + \chi_1(y)w_1(x) \tag{8}$$

$$w_2(xy) = \chi_3(x)w_2(y) + \chi_2(y)w_2(x). \tag{9}$$

As in the proof of Proposition 4.1(i) , for some $w_1 \in \oplus\chi_1$ and $w_2 \in \oplus\chi_2$ we have

$$w_1(x) = w_1[\chi_1(x) - \chi_3(x)], \ \ w_2(x) = w_2[\chi_2(x) - \chi_3(x)].$$

Hence, by Eq. (4),

$$\rho(x)u = \chi_3(x)u + w_1[\chi_1(x) - \chi_3(x)] + w_2[\chi_2(x) - \chi_3(x)]$$

and

$$\rho(x)[u - w_1 - w_2] = \chi_3(x)[u - w_1 - w_2].$$

Thus, the exact sequence (3) splits.

ii) By Eqs. (8) and (9) and as in the proof of Proposition 4.1(i) and (ii) for some $w_2 \in \oplus\chi_2$ and $w_1 \in \oplus\chi_1$ we have

$$w_2(x) = w_2[\chi_2(x) - \chi_3(x)],$$

$$w_1(x) = w_1\chi_1(x)v_k(x).$$

By Eq. (4),

$$\rho(x)u = \chi_1(x)u + w_1\chi_1(x)v_k(x) + w_2[\chi_2(x) - \chi_1(x)]$$

and

$$\rho(x)[u - w_2] = \chi_1(x)[u - w_2] + \chi_1(x)v_k(x)w_1.$$

Hence, we have a direct sum of $\oplus\chi_2$ and U_1'. □

5. Asymptotic behavior of φ_u

In this section, we determine the behavior of $\varphi_u(x)$ for small enough $|x|$, which depends on the Jacquet module structure. We also compute the possible poles of the integrals in Proposition 2.3 in the line of [2] but by extending the results.

Proposition 5.1 *Let (Π, V_Π) be a smooth representation of $GSp_4(k)$.*
1) If $u \in V_\Pi$, then there exists a constant C, depending on u, such that $\varphi_u(x) = 0$ for $|x| > C$.
2) If $u \in V_S(\Pi)$, then there exists a constant $\epsilon > 0$, depending on u, such that $\varphi_u(x) = 0$ for $|x| < \epsilon$. Therefore, φ_u has compact support in k^*.

Proof Proposition 3.1 of [2]. □

Proposition 5.2 *Let $u \in V_\Pi$. If $\Pi(h_x)u - \chi(x)u \in V_S(\Pi)$ for every $x \in k^*$, then there exists a constant C and positive integer j_o such that*

$$\varphi_u(x) = C\chi(x)$$

for $|x| \leq q^{-j_o}$.

Proof Proposition 3.2 of [2]. □

Lemma 5.3 *If* $\varphi_u(x) = C|x|^{3/2}\chi(x)$ *for some character* χ *of* k^* *and* $|x| \leq q^{-j_o}$*, then the pole of* $\int_{k^*} \varphi_u(x)|x|^{s-3/2}d^*x$ *is the pole of* $CL(s,\chi)$*.*

Proof Lemma 3.4 of [2]. □

Proposition 5.4 *Let* $u, w \in V_\Pi$*. If* $\Pi(h_x)u - \chi(x)u - \chi(x)v_k(x)w \in V_S(\Pi)$ *and* $\Pi(h_x)w - \chi(x)w \in V_S(\Pi)$ *for sufficiently small* $|x|$ *then for some constants* C_1 *and* C_2 *we have*

$$\varphi_u(x) = C_1\chi(x) + C_2\chi(x)v_k(x).$$

Proof Proposition 3.5 of [2]. □

Lemma 5.5 *If* $\varphi_u(x) = C_1|x|^{3/2}\chi(x) + C_2|x|^{3/2}\chi(x)v_k(x)$ *for some character* χ *of* k^* *and* $|x| \leq q^{-j_o}$*, then the poles of* $\int_{k^*} \varphi_u(x)|x|^{s-3/2}d^*x$ *are the poles of the least common multiple of* $C_1L(s,\chi)$ *and* $C_2L(s,\chi)^2$*.*

Proof Lemma 3.7 of [2]. □

Theorem 5.6 *Let* $u \in V_\Pi$ *and* $|x|$ *be small enough; then the asymptotic behavior of the Bessel model of*
i)II-a is if $\chi^2 \neq 1$

$$\varphi_u(x) = C_1|x|^{3/2}\sigma(x) + C_2|x|^{3/2}|x|^{1/2}\chi\sigma(x) + C_3|x|^{3/2}\chi^2\sigma(x);$$

otherwise

$$\varphi_u(x) = C_1|x|^{3/2}\sigma(x) + C_2|x|^{3/2}|x|^{1/2}\chi\sigma(x) + C_3|x|^{3/2}v_k(x)\sigma(x).$$

ii)II-b is if $\chi^2 \neq 1$

$$\varphi(x) = C_1|x|^{3/2}\sigma(x) + C_2|x|^{3/2}|x|^{-1/2}\chi\sigma(x) + C_3|x|^{3/2}\chi^2\sigma(x);$$

otherwise

$$\varphi_u(x) = C_1|x|^{3/2}\sigma(x) + C_2|x|^{3/2}|x|^{-1/2}\chi\sigma(x) + C_3|x|^{3/2}v_k(x)\sigma(x).$$

iii) VI-a is

$$\varphi_u(x) = C_1|x|^{3/2}|x|^{1/2}\sigma(x) + C_2|x|^{3/2}|x|^{1/2}v_k(x)\sigma(x).$$

iv) VI-d is

$$\varphi_u(x) = C_1|x|^{3/2}|x|^{-1/2}\sigma(x) + C_2|x|^{3/2}|x|^{-1/2}v_k(x)\sigma(x).$$

Proof i) By Proposition 3.5, the constituents of the Jacquet module of II-a are $\chi^{-1}St_{GL_2(k)} \otimes \chi^2\sigma$, $ind_B^{GL_2(k)}(\chi\nu^{1/2}, \chi^{-1}\nu^{1/2}) \otimes \chi\nu^{-1/2}\sigma$, and $\chi St_{GL_2(k)} \otimes \sigma$. As a representation of H, the constituents are $\oplus\sigma$, $\oplus\nu^{1/2}\chi\sigma$, and $\oplus\chi^2\sigma$ and in this case $\chi \neq \nu^{\mp 3/2}$ and $\chi^2 \neq \nu^{\mp 1}$ by Section 2.2 of [7]. Hence, if $\chi^2 \neq 1$, then these three constituents are different, so we are in the case of Proposition 4.2(i) and the result follows from Proposition 5.2. If $\chi^2 = 1$ then the constituents are $\oplus\sigma$, $\oplus\nu^{1/2}\chi\sigma$ and $\oplus\sigma$ then we are in the case of Proposition 4.2(ii), and the result follows from Proposition 5.2 and Proposition 5.4.

ii) Similar to the previous proof.

iii) By Corollary 3.8 the constituents of the Jacquet module of VI-a are $\nu^{1/2} ind_B^{GL_2(k)}(\nu^{1/2}, \nu^{-1/2}) \otimes \nu^{-1/2}\sigma$ and $\nu^{1/2} St_{GL_2(k)} \otimes \nu^{-1/2}\sigma$. As a representation of H, the constituents are two $\oplus\nu^{1/2}\sigma$. Thus we are in the case of Proposition 4.1(ii) and the result follows from Proposition 5.4.

iv)Similar to the previous proof. □

6. Computation of regular poles

The following theorem and lemmas are required to determine whether constants in Theorem 5.6 are nonzero or not.
Let

$$V_{T,S}(\Lambda, \Pi) := \{\Pi(ts)v - \Lambda(t)v : v \in V_\Pi\}$$

and

$$\overline{V}_T(\Lambda, \Pi) := \{\Pi(t)\overline{v} - \Lambda(t)\overline{v} : \overline{v} \in V_\Pi / V_S(\Pi)\}.$$

Theorem 6.1

$$\varphi_v \in C_c^\infty(k^*) \Longleftrightarrow v \in V_{T,S}(\Lambda, \Pi).$$

Proof Theorem 4.9 of [2] □

Lemma 6.2 *Let K be a quadratic extension of k and $T \cong K^*$; then $Hom_T(\sigma St_{GL_2(k)}, \Lambda)$ is nonzero for a character of K^*, which satisfies $\sigma^2 = \Lambda|_{k^*}$ if and only if $\Lambda \neq \sigma \circ N_{K/k}$. If $Hom_T(\sigma St_{GL_2(k)}, \Lambda)$ is nonzero then it is one-dimensional.*

Proof Proposition 1.7 in [12]. □

Lemma 6.3 *Let K be a quadratic extension of k and $T \cong K^*$. If π is an irreducible representation of $GL_2(k)$, which is induced from a character of the torus of $GL_2(k)$, then $Hom_T(\pi, \Lambda)$ is nonzero for every character Λ of K^* such that $\omega_\pi = \Lambda|_{k^*}$ and $Hom_T(\pi, \Lambda)$ is one-dimensional.*

Proof Proposition 1.6 in [12]. □

From now on, we assume that (Π, V_Π) has a Bessel model with respect to ψ and Λ. Also, for simplicity, we take $\mu = 1$.

6.1. Representations with Jacquet module length 3

In this section we compute the regular poles for each representation separately.

6.1.1. II-a: $\chi \mathrm{St}_{GL_2(k)} \rtimes \sigma$

By Proposition 3.5, if $I = \chi \mathrm{St}_{GL_2(k)} \rtimes \sigma$, then we have

$$0 \underbrace{\subset}_{\chi^{-1}\mathrm{St}_{GL_2(k)} \otimes \chi^2\sigma} (I_3)_S \underbrace{\subset}_{ind_B^{GL_2(k)}(\chi\nu^{1/2}, \chi^{-1}\nu^{1/2}) \otimes \chi\nu^{-1/2}\sigma} (I_2)_S \underbrace{\subset}_{\chi \mathrm{St}_{GL_2(k)} \otimes \sigma} (I)_S$$

and as a representation of H we have

$$0 \underbrace{\subset}_{\oplus\sigma} (I_3)_S \underbrace{\subset}_{\oplus\chi\nu^{1/2}\sigma} (I_2)_S \underbrace{\subset}_{\oplus\chi^2\sigma} (I)_S.$$

In this case, we have $\chi \neq \nu^{\mp 3/2}, \chi^2 \neq \nu^{\mp 1}$.

Proposition 6.4 *If **II-a** has a Bessel model with respect to the characters* Λ *and* Ψ *then* $\Lambda \neq \chi\sigma \circ N_{K/k}$.

Proof By Proposition 2.1 of [6],

$$(\chi \mathrm{St}_{GL_2(k)} \rtimes \sigma)_\theta = (\chi \mathrm{St}_{GL_2(k)} \otimes \sigma)|_T = (\chi\sigma \mathrm{St}_{GL_2(k)})|_T.$$

Hence, **II-a** has a Bessel model if and only if $Hom_T[\chi\sigma \mathrm{St}_{GL_2(k)}, \Lambda] \neq 0$ and the result follows from Lemma 6.2. □

Case 1: If $\chi^2 \neq 1$ then by Theorem 5.6(i),

$$\varphi_u(x) = C_1|x|^{3/2}\sigma(x) + C_2|x|^{3/2}|x|^{1/2}\chi\sigma(x) + C_3|x|^{3/2}\chi^2\sigma(x)$$

and by Proposition 4.2(i)

$$I_S = \oplus\sigma \bigoplus \oplus\chi\nu^{1/2}\sigma \bigoplus \oplus\chi^2\sigma.$$

Proposition 6.5 *For some choice of* u, *the constants* C_1, C_2, *and* C_3 *are all nonzero.*

Proof If $C_1 = 0$, then by Theorem 6.1, $(I_3)_S \subset \overline{V}_T(\Lambda, \Pi)$. Therefore, if $\overline{u} \in \oplus\sigma = (I_3)_S$ then

$$\overline{u} = \sum_{i=1}^{N_1} a_i[\Pi_S(t_i)u_1^i - \Lambda(t_i)u_1^i] + \sum_{j=1}^{N_2} b_j[\Pi_S(t_j)u_2^j - \Lambda(t_j)u_2^j] + \sum_{l=1}^{N_3} c_l[\Pi_S(t_l)u_3^l - \Lambda(t_l)u_3^l]$$

where $a_i, b_j, c_l \in k$, $t_i, t_j, t_l \in T$ and $u_1^i \in \oplus\sigma, u_2^l \in \oplus\chi\nu^{1/2}\sigma, u_3^l \in \oplus\chi^2\sigma$. Note that
$\sum_{i=1}^{N_1} a_i[\Pi_S(t_i)u_1^i - \Lambda(t_i)u_1^i] \in \oplus\sigma$,
$\sum_{j=1}^{N_2} b_j[\Pi_S(t_j)u_2^j - \Lambda(t_j)u_2^j] \in \oplus\chi\nu^{1/2}\sigma$,
$\sum_{l=1}^{N_3} c_l[\Pi_S(t_l)u_2^l - \Lambda(t_l)u_3^l] \in \oplus\chi^2\sigma$.

Since we have a direct sum $\sum_{j=1}^{N_2} b_j[\Pi_S(t_j)u_2^j - \Lambda(t_j)u_2^j] = 0$ and $\sum_{l=1}^{N_3} c_l[\Pi_S(t_l)u_3^l - \Lambda(t_l)u_3^l]$=0, hence

$$u = \sum_{i=1}^{N_1} a_i[\Pi_S(t_i)u_1^i - \Lambda(t_i)u_1^i]$$

and

$$\begin{aligned} 0 &= Hom_T[(I_3)_S, \Lambda] \\ &= Hom_T[\chi^{-1}\mathrm{St}_{GL_2(k)} \otimes \chi^2\sigma, \Lambda] \\ &= Hom_T[\chi\sigma\mathrm{St}_{GL_2(k)}, \Lambda]. \end{aligned}$$

Hence by Lemma 6.2, $\Lambda = \chi\sigma \circ N_{K/k}$, which contradicts the Bessel existence condition.

If $C_2 = 0$ then by Theorem 6.1, $\oplus\chi\nu^{1/2}\sigma \subset \overline{V}_T(\Lambda, \Pi)$. Therefore, if $\overline{u} \in \oplus\chi\nu^{1/2}\sigma$ then

$$\overline{u} = \sum_{i=1}^{N_1} a_i[\Pi_S(t_i)u_1^i - \Lambda(t_i)u_1^i] + \sum_{j=1}^{N_2} b_j[\Pi_S(t_j)u_2^j - \Lambda(t_j)u_2^j] + \sum_{l=1}^{N_3} c_l[\Pi_S(t_l)u_3^l - \Lambda(t_l)u_3^l]$$

where $a_i, b_j, c_l \in k$, $t_i, t_j, t_l \in T$ and $u_1^i \in \oplus\sigma, u_2^l \in \oplus\chi\nu^{1/2}\sigma, u_3^l \in \oplus\chi^2\sigma$. Note that
$\sum_{i=1}^{N_1} a_i[\Pi_S(t_i)u_1^i - \Lambda(t_i)u_1^i] \in \oplus\sigma$,
$\sum_{j=1}^{N_2} b_j[\Pi_S(t_j)u_2^j - \Lambda(t_j)u_2^j] \in \oplus\chi\nu^{1/2}\sigma$,
$\sum_{l=1}^{N_3} c_l[\Pi_S(t_l)u_3^l - \Lambda(t_l)u_3^l] \in \oplus\chi^2\sigma$.

Since we have a direct sum $\sum_{i=1}^{N_1} a_i[\Pi_S(t_i)u_1^i - \Lambda(t_i)u_1^i] = 0$ and $\sum_{l=1}^{N_3} c_l[\Pi_S(t_l)u_3^l - \Lambda(t_l)u_3^l] = 0$, hence

$$u = \sum_{j=1}^{N_2} b_j[\Pi_S(t_j)u_2^j - \Lambda(t_j)u_2^j] \in \oplus\chi\nu^{1/2}\sigma$$

and

$$\begin{aligned} 0 &= Hom_T[(I_2)_S/(I_3)_S, \Lambda] \\ &= Hom_T[ind_B^{GL_2(k)}(\chi\nu^{1/2}, \chi^{-1}\nu^{1/2}) \otimes \chi\sigma\nu^{-1/2}, \Lambda] \\ &= Hom_T[\chi\sigma\nu^{-1/2} ind_B^{GL_2(k)}(\chi\nu^{1/2}, \chi^{-1}\nu^{1/2}), \Lambda], \end{aligned}$$

which is a contradiction by Lemma 6.3.

If $C_3 = 0$ then by Theorem 6.1, $\oplus\chi^2\sigma \subset \overline{V}_T(\Lambda, \Pi)$. Therefore, if $\overline{u} \in \oplus\chi^2\sigma$ then

$$\overline{u} = \sum_{i=1}^{N_1} a_i[\Pi_S(t_i)u_1^i - \Lambda(t_i)u_1^i] + \sum_{j=1}^{N_2} b_j[\Pi_S(t_j)u_2^j - \Lambda(t_j)u_2^j] + \sum_{l=1}^{N_3} c_l[\Pi_S(t_l)u_3^l - \Lambda(t_l)u_3^l]$$

where $a_i, b_j, c_l \in k$, $t_i, t_j, t_l \in T$ and $u_1^i \in \oplus\sigma, u_2^l \in \oplus\chi\nu^{1/2}\sigma, u_3^l \in \oplus\chi^2\sigma$. Note that
$\sum_{i=1}^{N_1} a_i[\Pi_S(t_i)u_1^i - \Lambda(t_i)u_1^i] \in \oplus\sigma$,
$\sum_{j=1}^{N_2} b_j[\Pi_S(t_j)u_2^j - \Lambda(t_j)u_2^j] \in \oplus\chi\nu^{1/2}\sigma$,
$\sum_{l=1}^{N_3} c_l[\Pi_S(t_l)u_2^l - \Lambda(t_l)u_3^l] \in \oplus\chi^2\sigma$.

Since we have a direct sum $\sum_{i=1}^{N_1} a_i[\Pi_S(t_i)u_1^i - \Lambda(t_i)u_1^i] = 0$ and $\sum_{j=1}^{N_2} b_j[\Pi_S(t_j)u_2^j - \Lambda(t_j)u_2^j] = 0$, hence

$$u = \sum_{l=1}^{N_3} c_l[\Pi_S(t_l)u_3^l - \Lambda(t_l)u_3^l]$$

and

$$\begin{aligned} 0 &= Hom_T[(I)_S/(I_2)_S, \Lambda] \\ &= Hom_T[\chi \mathrm{St}_{GL_2(k)} \otimes \sigma, \Lambda] \\ &= Hom_T[\chi\sigma \mathrm{St}_{GL_2(k)}, \Lambda]. \end{aligned}$$

Hence, by Lemma 6.2, $\Lambda = \chi\sigma \circ N_{K/k}$, which contradicts the Bessel existence condition. □

Case 2: If $\chi^2 = 1$ then, by Theorem 5.6(ii), for every $u \in V_\Pi$ we have

$$\varphi_u(x) = C_1|x|^{3/2}\sigma(x) + C_2|x|^{3/2}|x|^{1/2}\chi\sigma(x) + C_3|x|^{3/2}v_k(x)\sigma(x),$$

and by Proposition 4.2(ii)

$$I_S = U_1' \bigoplus \oplus\chi\nu^{1/2}\sigma$$

where U_1' is an extension of two $\oplus\sigma$.

Proposition 6.6 *For some choice of* u, *the constants* C_2 *and* C_3 *are nonzero.*

Proof If $C_2 = 0$ then by Theorem 6.1 $\oplus\chi\nu^{1/2}\sigma \subset \overline{V}_T(\Lambda, \Pi)$ $((I_2)_S = \oplus\sigma \bigoplus \oplus\chi\nu^{1/2}\sigma)$. Therefore, if $\overline{u} \in \oplus\chi\nu^{1/2}\sigma$, then

$$\overline{u} = \sum_{i=1}^{N_1} a_i[\Pi_S(t_i)u_1^i - \Lambda(t_i)u_1^i] + \sum_{j=1}^{N_2} b_j[\Pi_S(t_j)u_2^j - \Lambda(t_j)u_2^j]$$

where $a_i, b_j \in k$, $t_i, t_j \in T$ and $u_1^i \in U_1', u_2^l \in \oplus\chi\nu^{1/2}\sigma$. Note that
$\sum_{i=1}^{N_1} a_i[\Pi_S(t_i)u_1^i - \Lambda(t_i)u_1^i] \in \oplus U_1'$,
$\sum_{j=1}^{N_2} b_j[\Pi_S(t_j)u_2^j - \Lambda(t_j)u_2^j] \in \oplus\chi\nu^{1/2}\sigma$.
Since we have a direct sum $\sum_{i=1}^{N_1} a_i[\Pi_S(t_i)u_1^i - \Lambda(t_i)u_1^i] = 0$, hence

$$u = \sum_{j=1}^{N_2} b_j[\Pi_S(t_j)u_2^j - \Lambda(t_j)u_2^j]$$

and

$$\begin{aligned} 0 &= Hom_T[(I_2)_S/(I_3)_S, \Lambda] \\ &= Hom_T[ind_B^{GL_2(k)}(\chi\nu^{1/2}, \chi^{-1}\nu^{1/2}) \otimes \chi\sigma\nu^{-1/2}, \Lambda] \\ &= Hom_T[\chi\sigma\nu^{-1/2} ind_B^{GL_2(k)}(\chi\nu^{1/2}, \chi^{-1}\nu^{1/2}), \Lambda], \end{aligned}$$

which is a contradiction by Lemma 6.3.

If $C_3 = 0$, then for every $\overline{u} \in (I)_S$ there exists $\overline{u}_2 \in (I_2)_S$ such that $\overline{u} - \overline{u}_2 \in \overline{V}_T(\Lambda, \Pi)$. Hence,

$$\begin{aligned} 0 &= Hom_T[(I)_S/(I_2)_S, \Lambda] \\ &= Hom_T[\chi \mathrm{St}_{GL_2(k)} \otimes \sigma, \Lambda] \\ &= Hom_T[\chi\sigma \mathrm{St}_{GL_2(k)}, \Lambda]. \end{aligned}$$

Hence by Lemma 6.2, $\Lambda = \chi\sigma \circ N_{K/k}$, which contradicts the Bessel existence condition. □

6.1.2. II-b: $\chi 1_{GL_2(k)} \rtimes \sigma$

By Proposition 3.6, if $I = \chi 1_{GL_2(k)} \rtimes \sigma$, then we have

$$0 \underbrace{\subset}_{\chi^{-1}1_{GL_2(k)} \otimes \chi^2\sigma} (I_3)_S \underbrace{\subset}_{ind_B^{GL_2(k)}(\chi\nu^{-1/2}, \chi^{-1}\nu^{-1/2}) \otimes \chi\nu^{1/2}\sigma} (I_2)_S \underbrace{\subset}_{\chi 1_{\mathrm{GL}(2)} \otimes \sigma} (I)_S$$

and as a representation of H we have

$$0 \underbrace{\subset}_{\oplus\sigma} (I_3)_S \underbrace{\subset}_{\oplus\chi\nu^{-1/2}\sigma} (I_2)_S \underbrace{\subset}_{\oplus\chi^2\sigma} (I)_S.$$

In this case, we have $\chi \neq \nu^{\mp 3/2}, \chi^2 \neq \nu^{\mp 1}$. By a similar proof to that of **II-a** we have:

Proposition 6.7 *If* ***II-b*** *has a Bessel model with respect to the characters* Λ *and* Ψ *then* $\Lambda = \chi\sigma \circ N_{K/k}$.

Proposition 6.8 *If* $\chi^2 \neq 1$ *then for some choice of* u *we have*

$$\varphi_u(x) = C_1|x|^{3/2}\sigma(x) + C_2|x|^{3/2}|x|^{1/2}\chi\sigma(x) + C_3|x|^{3/2}\chi^2\sigma(x)$$

and the constants C_1, C_2*, and* C_3 *are nonzero.*

Proposition 6.9 *If* $\chi^2 = 1$ *then for some choice of* u *we have*

$$\varphi_u(x) = C_1|x|^{3/2}\sigma(x) + C_2|x|^{3/2}|x|^{1/2}\chi\sigma(x) + C_3|x|^{3/2}v_k(x)\sigma(x)$$

and the constants C_2 *and* C_3 *are nonzero.*

6.1.3. VI-a: $\tau(S, \nu^{1/2}\sigma)$

By Corollary 3.8, we have

$$0 \underbrace{\subset}_{\nu^{1/2} ind_B^{GL_2(k)}(\nu^{1/2}, \nu^{-1/2}) \otimes \nu^{-1/2}\sigma} ((VI-a)_2)_S \underbrace{\subset}_{\nu^{1/2}\mathrm{St}_{GL_2(k)} \otimes \nu^{-1/2}\sigma} (VI-a)_S$$

and as a representation of H we have

$$0 \underbrace{\subset}_{\oplus\nu^{1/2}\sigma} ((VI-a)_2)_S \underbrace{\subset}_{\oplus\nu^{1/2}\sigma} (VI-a)_S.$$

Proposition 6.10 *If* ***VI-a*** *has a Bessel model with respect to the characters* Λ *and* Ψ *then* $\Lambda \neq \sigma \circ N_{K/k}$.

Proof By Sally–Tadic classification and exactness of the twisted Jacquet module, we have

$$0 \longrightarrow (VI-a)_\psi \longrightarrow (\nu^{1/2}St_{GL_2(k)} \rtimes \nu^{-1/2}\sigma)_\psi \longrightarrow (VI-c)_\psi \longrightarrow 0.$$

Since T is nonsplit this sequence splits and by Proposition 2.1 of [6], $(\nu^{1/2}St_{GL_2(k)} \rtimes \nu^{-1/2}\sigma)_\psi = \sigma St_{GL_2(k)}$. Thus, if VI-a or VI-c has a Bessel model with respect to ψ and Λ, then by Lemma 6.2 $\Lambda \neq \sigma \circ N_{K/k}$.

□

By Theorem 5.6(iii)

$$\varphi_u(x) = C_1|x|^{3/2}|x|^{1/2}\sigma(x) + C_2|x|^{3/2}|x|^{1/2}v_k(x)\sigma(x).$$

Proposition 6.11 *For some choice of u we have a nonzero C_2.*

Proof If $C_2 = 0$ then for all $\overline{u} \in (VI-a)_S$ there exists $\overline{u}_2 \in ((VI-a)_2)_S$ such that $\overline{u} - \overline{u}_2 \in \overline{V}_T(\Lambda, \Pi)$. Hence,

$$\begin{aligned} 0 &= Hom_T[(VI-a)_S/((VI-a)_2)_S, \Lambda] \\ &= Hom_T[\nu^{1/2}\mathrm{St}_{GL_2(k)} \otimes \nu^{-1/2}\sigma, \Lambda] \\ &= Hom_T[\sigma\mathrm{St}_{GL_2(k)}, \Lambda]. \end{aligned}$$

Hence, by Lemma 6.2 $\Lambda = \sigma \circ N_{K/k}$, which contradicts the Bessel existence condition. □

6.1.4. VI-d: $L(\nu, 1_{K^*} \rtimes \nu^{-1/2}\sigma)$

By Corollary 3.10, we have

$$0 \underbrace{\subset}_{\nu^{-1/2}ind_B^{GL_2(k)}(\nu^{1/2},\nu^{-1/2})\otimes\nu^{1/2}\sigma} ((VI-d)_2)_S \underbrace{\subset}_{\nu^{-1/2}1_{GL_2(k)}\otimes\nu^{1/2}\sigma} (VI-d)_S$$

and as a representation of H we have

$$0 \underbrace{\subset}_{\oplus\nu^{-1/2}\sigma} ((VI-d)_2)_S \underbrace{\subset}_{\oplus\nu^{-1/2}\sigma} (VI-d)_S.$$

Proposition 6.12 *If **VI-d** has a Bessel model with respect to the characters Λ and Ψ, then $\Lambda = \sigma \circ N_{K/k}$.*

Proof By Sally–Tadic classification and exactness of the twisted Jacquet module, we have

$$0 \longrightarrow (VI-b)_\psi \longrightarrow (\nu^{1/2}1_{GL_2(k)} \rtimes \nu^{-1/2}\sigma)_\psi \longrightarrow (VI-d)_\psi \longrightarrow 0.$$

Since T is nonsplit this sequence splits and by Proposition 2.1 of [6], $(\nu^{1/2}1_{GL_2(k)} \rtimes \nu^{-1/2}\sigma)_\psi = \sigma 1_{GL_2(k)}$. Thus, if VI-b or VI-d has a Bessel model with respect to ψ and Λ, then $\Lambda = \sigma \circ N_{K/k}$. □

By Theorem 5.6(iii)

$$\varphi_u(x) = C_1|x|^{3/2}|x|^{-1/2}\sigma(x) + C_2|x|^{3/2}|x|^{-1/2}v_k(x)\sigma(x).$$

Proposition 6.13 *For some choice of u we have a nonzero C_2.*

Proof If $C_2 = 0$ then for all $\overline{u} \in (VI-d)_S$ there exists $\overline{u}_2 \in ((VI-d)_2)_S$ such that $\overline{u} - \overline{u}_2 \in \overline{V}_T(\Lambda, \Pi)$. Hence,

$$\begin{aligned} 0 &= Hom_T[(VI-d)_S/((VI-d)_2)_S, \Lambda] \\ &= Hom_T[\nu^{-1/2}1_{GL_2(k)} \otimes \nu^{1/2}\sigma, \Lambda] \\ &= Hom_T[\sigma 1_{GL_2(k)}, \Lambda]. \end{aligned}$$

Hence, $\Lambda \neq \sigma \circ N_{K/k}$, which contradicts the Bessel existence condition. □

Theorem 6.14 *i) The L-factor of **II-a** is $L(s,\sigma)L(s,\nu^{1/2}\chi\sigma)L(s,\chi^2\sigma)$.*

*ii) Regular poles of **II-b** are poles of $L(s,\sigma)L(s,\nu^{-1/2}\chi\sigma)L(s,\chi^2\sigma)$.*

*iii) The L-factor of **VI-a** is $L(s,\nu^{1/2}\sigma)^2$.*

*iv) Regular poles of **VI-d** are poles of $L(s,\nu^{-1/2}\sigma)^2$.*

Proof i) If $\chi^2 \neq 1$ then the result follows from Proposition 6.5, Lemma 5.3, and Theorem 2.4. If $\chi^2 = 1$ then the result follows from Proposition 6.6, Lemma 5.3, Lemma 5.5, and Theorem 2.4.

ii) Similar to (i).

iii) The result follows from Proposition 6.11, Lemma 5.5, and Theorem 2.4.

iv) The result follows from Proposition 6.13 and Lemma 5.5. □

Appendix

A. Tables

Table A.1 displays the regular poles of the nonsupercuspidal representations due to [8], which have Jacquet module length of 3, in terms of the poles of Tate L-functions. The last column shows the expected exceptional poles from the local Langlands conjecture. Table A.2 shows the semisimplifications of the Jacquet modules with respect to the Siegel parabolic, given in the appendix of [7]. $'\#'$ and $'g'$ columns indicate the number of constituents of the Jacquet module and generic representations, respectively.

A.1. Regular and exceptional poles.

		Representation	Regular poles	Exceptional
II	a	$\chi St_{GL(2)} \rtimes \sigma$	$L(s,\sigma)L(s,\nu^{1/2}\chi\sigma)L(s,\chi^2\sigma)$	-
II	b	$\chi 1_{GL(2)} \rtimes \sigma$	$L(s,\sigma)L(s,\nu^{-1/2}\chi\sigma)L(s,\chi^2\sigma)$	$L(s,\nu^{1/2}\chi\sigma)$
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)^2$	-
VI	d	$L(\nu, 1_{K^*} \rtimes \nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\sigma)^2$	$L(s,\nu^{1/2}\sigma)^2$

A.2. Jacquet modules: the Siegel parabolic.

		Representation	Semisimplification	#	g
II	a	$\chi St_{GL_2(k)} \rtimes \sigma$	$\chi^{-1}St_{GL_2(k)} \otimes \chi^2\sigma$ $ind_B^{GL_2(k)}(\chi\nu^{1/2}, \chi^{-1}\nu^{1/2}) \otimes \chi\nu^{-1/2}\sigma$ $\chi St_{GL_2(k)} \otimes \sigma$	3	•
II	b	$\chi 1_{GL_2(k)} \rtimes \sigma$	$\chi^{-1}1_{GL_2(k)} \otimes \chi^2\sigma$ $ind_B^{GL_2(k)}(\chi\nu^{-1/2}, \chi^{-1}\nu^{-1/2}) \otimes \chi\nu^{1/2}\sigma$ $\chi 1_{GL_2(k)} \otimes \sigma$	3	
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$2(\nu^{1/2}St_{GL_2(k)} \otimes \nu^{-1/2}\sigma)$ $\nu^{1/2}1_{GL_2(k)} \otimes \nu^{-1/2}\sigma$	3	•
VI	d	$L(\nu, 1_{k^*} \rtimes \nu^{-1/2}\sigma)$	$2(\nu^{-1/2}1_{GL_2(k)} \otimes \nu^{1/2}\sigma)$ $\nu^{-1/2}St_{GL_2(k)} \otimes \nu^{1/2}\sigma$	3	

References

[1] Bump D. Automorphic Forms and Representations. New York, NY, USA: Cambridge Studies in Advanced Mathematics, 1998.

[2] Danışman Y. Regular poles for the p-adic group GSp_4. Turk J Math 2014; 38: 587–613.

[3] Gan W, Takeda S. The local Langlands conjecture for GSp(4). Ann Math 2011; 173: 1841–1882.

[4] Novodvorsky M. Automorphic L-functions for the symplectic group GSp(4). P Sympos Pure Math 1977; 33: 87–95.

[5] Piatetski-Shapiro I. L-functions for GSp_4. Pacific J Math Olga Taussky Todd Memorial Issue 1997; 259–275.

[6] Prasad D, Takloo-Bighash R. Bessel models for GSp(4). J Reine Angew Math 2011; 655: 189–243.

[7] Roberts B, Schmidt R. New Forms for GSp(4). Springer Lecture Notes in Mathematics 1918. Heidelberg, Germany: Springer-Verlag, 2007.

[8] Sally PJ Jr, Tadic M. Induced representations and classifications for GSp(2, F) and Sp(2, F). Memoires Societe Mathematique de France 1993; 52: 75–133.

[9] Shahidi F. A proof of Langlands conjecture on plancherel measures; complementary series for p-adic groups. Ann Math 1990; 132: 273–330.

[10] Shahidi F. Eisenstein Series and Automorphic L-Functions. American Mathematical Society Colloquium Publications, Vol. 58. Providence, RI, USA: American Mathematical Society, 2010.

[11] Takloo-Bighash R. L-functions for the p-adic group GSp(4). Am J Math 2000; 122: 1085–1120.

[12] Tunnell J. Local L-factors and characters of GL(2). Am J Math 1983; 105: 1277–1307.