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Oscillation of second order differential equations with mixed nonlinearities

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Abstract: By refining the standard integral averaging technique, in this paper, new oscillation criteria as well as interval oscillation criteria are established for the second order delay differential equation with mixed nonlinearities

$$(r(t)|x'(t)|^{\alpha-1}x'(t))' + q_0(t)|x(\tau_0(t))|^{\alpha-1}x(\tau_0(t)) + \sum_{i=1}^{n} q_i(t)|x(\tau_i(t))|^{\alpha_i-1}x(\tau_i(t)) = 0,$$

where $\alpha > 0$, $\alpha_i > 0, i = 1, 2, \cdots, n$. Our results generalize and improve the known results in the literature. Examples are also given to illustrate the importance of our results.

Key words: Oscillation, interval oscillation, mixed nonlinearities, delay differential equations

1. Introduction

In this paper, we are concerned with the oscillation properties of the following second order differential equation with mixed nonlinearities:

$$(r(t)|x'(t)|^{\alpha-1}x'(t))' + q_0(t)|x(t)|^{\alpha-1}x(t) + \sum_{i=1}^{n} q_i(t)|x(t)|^{\alpha_i-1}x(t) = 0, \tag{1.1}$$

and

$$(r(t)|x'(t)|^{\alpha-1}x'(t))' + q_0(t)|x(\tau_0(t))|^{\alpha-1}x(\tau_0(t)) + \sum_{i=1}^{n} q_i(t)|x(\tau_i(t))|^{\alpha_i-1}x(\tau_i(t)) = 0, \tag{1.2}$$

where $t \geq t_0 \geq 0$. Throughout this paper, we assume that the following conditions hold:

(A1) $\alpha > 0$, $\alpha_i > 0$ $(i = 1, 2, \cdots, n)$ are mutually different constants, and let $\eta_i > 0$ $(i = 1, 2, \cdots, n)$ be given constants satisfying $\sum_{i=1}^{n} \eta_i = 1$ and $\sum_{i=1}^{n} \alpha_i \eta_i = \alpha$.

(A2) $r \in C^1([t_0, \infty), \mathbb{R}^+)$, $q_0 \in C([t_0, \infty), \mathbb{R})$ and $q_i \in C([t_0, \infty), [0, +\infty))$, $i = 1, \cdots, n$;

(A3) $\tau_i \in C([t_0, \infty), \mathbb{R})$ with $\tau_i(t) \leq t$, $\lim_{t\to\infty} \tau_i(t) = \infty$, $i = 0, 1, \cdots, n$.

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Here we note that the constants η_i from (A1) exist if and only if $\alpha_i > \alpha$ for some i and $\alpha_j < \alpha$ for some j. When $n = 2$, and $\alpha_2 > \alpha > \alpha_1 > 0$, we can choose

$$\eta_1 = \frac{\alpha_2 - \alpha}{\alpha_2 - \alpha_1}, \quad \eta_2 = \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1},$$

then $\eta_1 + \eta_2 = 1$ and $\eta_1\alpha_1 + \eta_2\alpha_2 = \alpha$.

By a solution of Eq. (1.1) (or Eq. (1.2)), we mean a function $x \in C^1([T_x, \infty), \mathbb{R})$, $T_x \geq t_0$, which has the property $r|x'|^{\alpha-1}x' \in C^1([T_x, \infty), \mathbb{R})$ and satisfies Eq. (1.1) (or Eq. (1.2)) for all $t \geq T_x$. We restrict our attention to those solutions $x(t)$ of Eq. (1.1) (or Eq. (1.2)) that exist on some half-line $[T_x, \infty)$ with $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq T_x$; see [8]. As usual, a nontrivial solution is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Eq. (1.1) (or Eq. (1.2)) itself is said to be oscillatory if all its solutions are oscillatory.

The theory of oscillation is an important branch of the qualitative theory of differential equations. In the last 60 years, there has been increasing interest in obtaining the sufficient conditions for the oscillation/nonoscillation of solutions of different classes of differential equations such as linear and nonlinear ordinary, and functional differential equations; we refer the reader to the monographs (see, e.g., [2, 5, 14]).

It is well known that an important tool in the study of oscillation of the considerable number of equations is the averaging technique, which goes back as far as the classical results of Hartman [10] and Wintner [23], giving a sufficient condition for oscillation of second order linear differential equations. The result of Wintner was improved by Kamenev [12], and further extensions of Kamenev's criteria have been obtained by Yan [26] and Philos [16]. Following Philos's idea, Xu et al. [25] and Rogovchenko and Tuncay [17] recently developed a modified integral averaging technique and derived a variety of simpler oscillation theorems for second order neutral differential equations and ordinary differential equations with damping, respectively. On the other hand, as pointed out previously [13], oscillation is an interval property, that is, it is more reasonable to investigate solutions on an infinite set of bounded intervals. Therefore, the problem is to find oscillation criteria that use only the information about the involved functions on these intervals; outside of these intervals the behavior of the functions is irrelevant. Such type of oscillation criteria are referred to as interval oscillation criteria, see, e.g., [4, 20, 22].

As stated in [19], differential equations with mixed nonlinearities arise in the growth of bacterial populations with competitive species and therefore require more attention. To the best of our knowledge, the first study concerning the oscillation of equations with mixed nonlinearities was performed by Sun and Wong [19]. More precisely, they considered the following second order forced differential equation:

$$(p(t)x')' + q(t)x + \sum_{i=1}^{n} q_i(t)|^{\alpha-1}x = e(t), \ \ t \geq t_0, \tag{1.3}$$

where $\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0$, and obtained interval criteria for Eq. (1.3); see also [18]. For more related works, see, e.g., [1, 3, 11, 6, 7, 21, 27].

Motivated by the ideas in [11, 13, 16, 17, 25], in this paper, by employing the Riccati technique and refining the standard integral averaging method, we will establish oscillation criteria as well as interval oscillation criteria for Eqs. (1.1)–(1.2). Our results generalize and improve some known results in [4, 13, 15, 16, 20, 22]. Finally, several examples are also given to illustrate the importance of our results.

2. Oscillation criteria for Eq. (1.1)

In the sequel, we say that a function $H = H(t,s)$ belongs to a function class $\mathcal{H}$, denoted by $H \in \mathcal{H}$, if $H \in C(D,[0,\infty))$, where $D = \{(t,s) : t \geq s \geq t_0\}$, $D_0 = \{(t,s) : t > s \geq t_0\}$, which satisfies

$$H(t,t) = 0, \quad t \geq t_0; \;\; H(t,s) > 0, \quad (t,s) \in D_0,$$

and has partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H}{\partial t}(t,s) = h_1(t,s)\sqrt{H(t,s)} \quad \text{and} \quad \frac{\partial H}{\partial s}(t,s) = -h_2(t,s)\sqrt{H(t,s)},$$

where $h_1, h_2 \in L_{loc}(D, \mathbb{R}^+)$.

In what follows, throughout this paper, we use the notation. For $H \in \mathcal{H}$, $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$, and $\phi \in C([t_0,\infty),\mathbb{R})$, define

$$\psi_1(s,t) = \frac{h_1(s,t)}{\sqrt{H(s,t)}} + \frac{\rho'(s)}{\rho(s)}, \quad \psi_2(t,s) = -\frac{h_2(t,s)}{\sqrt{H(t,s)}} + \frac{\rho'(s)}{\rho(s)}, \quad (t,s) \in D_0,$$

and

$$Q(t) = q_0(t) + \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t), \quad \phi_+(t) = \max\{0, \phi(t)\}, \quad k_0 = \frac{1}{(\alpha+1)^{\alpha+1}}.$$

We begin with a preparatory lemma.

Lemma 2.1 *Assume that $x(t)$ is a solution of Eq. (1.1) such that $x(t) > 0$ on $[c,b)$. For any $\rho \in C^1([c,b),\mathbb{R}^+)$, let*

$$w(t) = \rho(t)\frac{r(t)|x'(t)|^{\alpha-1}x'(t)}{|x(t)|^{\alpha-1}x(t)} \tag{2.1}$$

on $[c,b)$. Then, for any $H \in \mathcal{H}$ and any $\beta \geq 1$, we have

$$\int_c^b H(b,s)\rho(s)Q(s)ds \leq H(b,c)w(c) + k_0\beta^\alpha \int_c^b H(b,s)\rho(s)r(s)|\psi_2(b,s)|^{\alpha+1}ds. \tag{2.2}$$

Proof Differentiating (2.1) and making use of (1.1), we have

$$w'(t) = \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)\Big[q_0(t) + \sum_{i=1}^{n} q_i(t)x^{\alpha_i-\alpha}(t)\Big] - \alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(\rho(t)r(t))^{1/\alpha}}, \;\; t \in [c,b). \tag{2.3}$$

Recall the arithmetic-geometric mean inequality [9, page 17]: $\sum_{i=1}^{n} \eta_i u_i \geq \prod_{i=1}^{n} u_i^{\eta_i}$, $u_i \geq 0$, where η_1, η_2, $\cdots$, η_n are chosen to satisfy (A1). Applying the inequality with $u_i = \eta_i^{-1}q_i(t)x^{\alpha_i}(t)$, $i = 1, 2, \cdots, n$, we have

$$\sum_{i=1}^{n} q_i(t)x^{\alpha_i-\alpha}(t) \geq \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(s). \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$\rho(t)Q(t) \leq -w'(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(\rho(t)r(t))^{1/\alpha}}. \tag{2.5}$$

Multiplying (2.5) (with t replaced by s) by $H(t,s)$, integrating with respect to s from c to t for $t \in [c,b)$, and using integration by parts, we find

$$\begin{aligned}
&\int_c^t H(t,s)\rho(s)Q(s)ds \\
&\leq H(t,c)w(c) + \int_c^t H(t,s)\psi_2(t,s)w(s)ds - \alpha\int_c^t H(t,s)\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}ds \\
&\leq H(t,c)w(c) + \int_c^t H(t,s)\Big[|\psi_2(t,s)||w(s)| - \alpha\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}\Big]ds.
\end{aligned} \tag{2.6}$$

By Young's inequality [9, Theorem 37], for $\beta \geq 1$,

$$|\psi_2(t,s)||w(s)| \leq \frac{\alpha}{\beta}\frac{|w(t)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}} + k_0\beta^\alpha\rho(s)r(s)|\psi_2(t,s)|^{\alpha+1}.$$

From (2.6) and the above inequality,

$$\begin{aligned}
\int_c^t H(t,s)\rho(s)Q(s)ds \leq &H(t,c)w(c) + k_0\beta^\alpha\int_c^t H(t,s)\rho(s)r(s)|\psi_2(t,s)|^{\alpha+1}ds \\
&- \frac{\alpha(\beta-1)}{\beta}\int_c^t H(t,s)\frac{|w(s)|^{(\alpha+1)/\alpha}}{\big(r(s)\rho(s)\big)^{1/\alpha}}ds,
\end{aligned} \tag{2.7}$$

which yields

$$\int_c^t H(t,s)\rho(s)Q(s)ds \leq H(t,c)w(c) + k_0\beta^\alpha\int_c^t H(t,s)\rho(s)r(s)|\psi_2(t,s)|^{\alpha+1}ds. \tag{2.8}$$

Letting $t \to b^-$ in the above inequality, we obtain that (2.2) holds. □

Firstly, we establish oscillation criteria for Eq. (1.1).

Theorem 2.1 *Assume that there exist $H \in \mathcal{H}$ and $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ such that for some $\beta \geq 1$ and any $T \geq t_0$,*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)\Big[Q(s) - k_0\beta^\alpha r(s)|\psi_2(t,s)|^{\alpha+1}\Big]ds = \infty. \tag{2.9}$$

Then Eq. (1.1) is oscillatory.

Proof Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Without loss of generality, we assume that $x(t) > 0$ for all $t \geq T_0 \geq t_0$. Define $w(t)$ as in (2.1), and proceeding as in the proof of Lemma 2.1, we find (2.8) holds, consequently, for all $t > T \geq T_0$,

$$\int_T^t H(t,s)\rho(s)\Big[Q(s) - k_0\beta^\alpha r(s)|\psi_2(t,s)|^{\alpha+1}\Big]ds \leq H(t,T)|w(T)|,$$

which yields

$$\frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)\Big[Q(s)-k_0\beta^\alpha r(s)|\psi_2(t,s)|^{\alpha+1}\Big]ds \le |w(T)|,$$

which contradicts (2.9). □

It may well happen that condition (2.9) of Theorem 2.2 is not satisfied, in which case the following result proves to be useful.

Theorem 2.2 *Assume that there exists $H \in \mathcal{H}$ such that*

$$0 < \inf_{s\ge T_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,T_0)}\right] \le \infty,\ \forall\, T_0 \ge t_0. \tag{2.10}$$

Furthermore, if there exist $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ and $\phi \in C([t_0,\infty),\mathbb{R})$ such that for some $\beta > 1$ and any $T \ge t_0$,

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)\Big[Q(s)-k_0\beta^\alpha r(s)|\psi_2(t,s)|^{\alpha+1}\Big]ds \ge \phi(T), \tag{2.11}$$

and

$$\lim_{t\to\infty}\int_{t_0}^t \frac{(\phi_+(s))^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}ds = \infty, \tag{2.12}$$

then Eq. (1.1) is oscillatory.

Proof Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Without loss of generality, we assume that $x(t) > 0$ for all $t \ge T_0 \ge t_0$. Define $w(t)$ as in (2.1), proceeding as in the proof of Lemma 2.1, we arrive at the inequality (2.7), which follows, for all $t \ge T \ge T_0$,

$$\begin{aligned}&\frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)\Big[Q(s)-k_0\beta^\alpha r(s)|\psi_2(t,s)|^{\alpha+1}\Big]ds\\ &\quad\le w(T)-\frac{\alpha(\beta-1)}{\beta}\frac{1}{H(t,T)}\int_T^t H(t,s)\frac{|w(s)|^{(\alpha+1)/\alpha}}{\big(\rho(s)r(s)\big)^{1/\alpha}}ds.\end{aligned}$$

Therefore, for $t > T \ge T$,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)\Big[Q(s)-k_0\beta^\alpha r(s)|\psi_2(t,s)|^{\alpha+1}\Big]ds\\ &\quad\le w(T)-\frac{\alpha(\beta-1)}{\beta}\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\frac{|w(s)|^{(\alpha+1)/\alpha}}{\big(r(s)\rho(s)\big)^{1/\alpha}}ds.\end{aligned}$$

It follows from (2.11) that

$$w(T) \ge \phi(T)+\frac{\alpha(\beta-1)}{\beta}\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\frac{|w(s)|^{(\alpha+1)/\alpha}}{\big(r(s)\rho(s)\big)^{1/\alpha}}ds.$$

Consequently, for all $T \ge T_0$,

$$w(T) \ge \phi(T), \tag{2.13}$$

and

$$\liminf_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^{t}H(t,s)\frac{|w(s)|^{(\alpha+1)/\alpha}}{\big(r(s)\rho(s)\big)^{1/\alpha}}ds\le\frac{\beta}{\alpha(\beta-1)}[w(T_0)-\phi(T_0)]<\infty. \tag{2.14}$$

Now we prove that

$$\int_{T_0}^{\infty}\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}ds<\infty. \tag{2.15}$$

If (2.15) fails, then there exists a $T_1\ge T_0$ such that

$$\int_{T_0}^{t}\frac{|w(s)|^{(\alpha+1)/\alpha}}{\big(\rho(s)r(s)\big)^{1/\alpha}}ds\ge\frac{\varepsilon}{\eta},\quad t\ge T_1, \tag{2.16}$$

where ε is an arbitrary positive number and η is a positive constant satisfying

$$\inf_{s\ge T_0}\Big[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,T_0)}\Big]>\eta.$$

Then there exists a $T_2\ge T_1$ such that $H(t,T_2)/H(t,T_0)\ge\eta$ for all $t\ge T_2$. Using integration by parts, we conclude that, for all $t\ge T_2$,

$$\begin{aligned}&\frac{1}{H(t,T_0)}\int_{T_0}^{t}H(t,s)\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}ds\\&=\frac{1}{H(t,T_0)}\int_{T_0}^{t}\Big(-\frac{\partial H(t,s)}{\partial s}\Big)\Big(\int_{T_0}^{s}\frac{|w(\tau)|^{(\alpha+1)/\alpha}}{(\rho(\tau)r(\tau))^{1/\alpha}}d\tau\Big)ds\\&\ge\frac{\varepsilon}{\eta}\frac{1}{H(t,T_0)}\int_{T_2}^{t}\Big(-\frac{\partial H(t,s)}{\partial s}\Big)ds=\frac{\varepsilon}{\eta}\frac{H(t,T_2)}{H(t,T_0)}\ge\varepsilon.\end{aligned}$$

Since ε is an arbitrary positive constant,

$$\liminf_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^{t}H(t,s)\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}ds=+\infty,$$

which contradicts (2.14). Consequently, (2.15) holds, and by virtue of (2.13),

$$\int_{T_0}^{\infty}\frac{(\phi_+(s))^{1+1/\alpha}}{(r(s)\rho(s))^{1/\alpha}}ds\le\int_{T_0}^{\infty}\frac{|w(s)|^{\alpha+1)/\alpha}}{(r(s)\rho(s))^{1/\alpha}}ds<+\infty,$$

which contradicts (2.12). Hence, Eq. (1.1) is oscillatory. □

Remark 2.1 Let (2.11) be replaced by

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}H(t,s)\rho(s)\big[Q(s)-k_0\beta^{\alpha}r(s)|\psi_2(t,s)|^{\alpha+1}\big]ds\ge\phi(T).$$

Then the conclusion of Theorem 2.2 also holds.

As an immediate consequence of Theorem 2.1, we get the following conclusions.

Corollary 2.1 *Let condition (2.9) in Theorem 2.1 be replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\rho(s)Q(s)ds = \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\rho(s)r(s)|\psi_2(t,s)|^{\alpha+1}ds < \infty.$$

Then Eq. (1.1) is oscillatory.

Corollary 2.2 *Assume that there exists $\rho \in C([t_0,\infty),\mathbb{R}^+)$ such that $\lim_{t\to\infty} G(t) = \infty$ and*

$$\int_{t_0}^{\infty} r(s)\frac{|\rho'(s)|^{\alpha+1}}{\rho^\alpha(s)}ds < \infty. \tag{2.17}$$

Then Eq. (1.1) is oscillatory provided that for some $\lambda > \alpha$, one of the following cases holds:

(i)
$$\limsup_{t\to\infty} \frac{1}{G^\lambda(t)} \int_{t_0}^t [G(t)-G(s)]^\lambda \rho(s)Q(s)ds = \infty; \tag{2.18}$$

(ii) There exists $\phi \in C([t_0,\infty),\mathbb{R})$ such that (2.12) holds, and for all $T \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{G^\lambda(t)} \int_T^t [G(t)-G(s)]^\lambda \rho(s)Q(s)ds \geq \phi(T), \tag{2.19}$$

where $G(t) = \int_{t_0}^t (\rho(s)r(s))^{-1/\alpha}ds$.

Proof Let $H(t,s) = [G(t)-G(s)]^\lambda$. Then

$$h_2(t,s) = \frac{\lambda[G(t)-G(s)]^{(\lambda-2)/2}}{(\rho(s)r(s))^{1/\alpha}}.$$

By the elementary inequality $(X+Y)^{\alpha+1} \leq 2^\alpha(X^{\alpha+1}+Y^{\alpha+1})$, X, $Y \geq 0$, we obtain

$$\begin{aligned} &\int_T^t H(t,s)\rho(s)r(s)|\psi_2(t,s)|^{\alpha+1}ds \\ &\leq 2^\alpha\Big[\int_T^t \rho(s)r(s)\frac{|h_2(t,s)|^{\alpha+1}}{H^{(\alpha-1)/2}(t,s)}ds + \int_T^t H(t,s)r(s)\frac{|\rho'(s)|^{\alpha+1}}{\rho^\alpha(s)}ds\Big]. \end{aligned} \tag{2.20}$$

Note that

$$\int_T^t \rho(s)r(s)\frac{|h_2(t,s)|^{\alpha+1}}{H^{(\alpha-1)/2}(t,s)}ds = \frac{\lambda^{\alpha+1}}{\lambda-\alpha}[G(t)-G(T)]^{\lambda-\alpha},$$

which follows

$$\lim_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \rho(s)r(s)\frac{|h_2(t,s)|^{\alpha+1}}{H^{(\alpha-1)/2}(t,s)}ds = 0, \tag{2.21}$$

and by (2.17) and [24, Lemma (14)],

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)r(s)\frac{|\rho'(s)|^{\alpha+1}}{\rho^\alpha(s)}ds=0. \tag{2.22}$$

Hence, by (2.20)–(2.22), we have

$$\lim_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)r(s)\rho(s)|\psi_2(t,s)|^{\alpha+1}ds=0.$$

Thus,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)\Big[Q(s)-k_0\beta^\alpha r(s)|\psi_2(t,s)|^{\alpha+1}\Big]ds\\&=\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t [G(t)-G(s)]^\lambda\rho(s)Q(s)ds.\end{aligned}$$

Then, by Theorems 2.1 and 2.2, we find the conclusion of Corollary 2.2 holds. □

Next, we establish interval oscillation criteria for Eq. (1.1). For this, we need the following lemma.

Lemma 2.2 *Assume that $x(t)$ is a solution of Eq. (1.1) such that $x(t)>0$ on $(a,c]$. For any $\rho\in C^1((a,c],\mathbb{R}^+)$, let $w(t)$ be defined on $(a,c]$ as (2.1). Then, for any $H\in\mathcal{H}$ and any $\beta\geq 1$, we have*

$$\int_a^c H(s,a)\rho(s)Q(s)ds\leq -H(c,a)w(c)+k_0\beta^\alpha\int_a^c H(s,a)\rho(s)r(s)|\psi_1(s,a)|^{\alpha+1}ds. \tag{2.23}$$

Proof Similar to the proof of Lemma 2.1, multiplying (2.5) (with t replaced by s) by $H(s,t)$, integrating it with respect to s from t to c for $t\in(a,c]$, we have

$$\begin{aligned}&\int_t^c H(s,t)\rho(s)Q(s)ds\\&\leq -H(c,t)w(c)+\int_t^c H(s,t)\psi_1(s,t)w(s)ds-\alpha\int_t^c H(s,t)\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}ds\\&\leq -H(c,t)w(c)+\int_t^c H(s,t)\Big[|\psi_1(s,t)||w(s)|-\alpha\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}\Big]ds\\&\leq -H(c,t)w(c)+k_0\beta^\alpha\int_t^c H(s,t)\rho(s)r(s)|\psi_1(s,t)|^{\alpha+1}ds.\end{aligned}$$

Letting $t\to a^+$ in the above inequality leads to (2.23). □

The following theorem is an immediate result from Lemmas 2.1 and 2.2.

Theorem 2.3 *Assume that, for each $T\geq t_0$, there exist $H\in\mathcal{H}$, $\rho\in C^1([t_0,\infty),\mathbb{R}^+)$, and $a,b,c\in\mathbb{R}$ such that $T\leq a<c<b$, and for some $\beta\geq 1$,*

$$\frac{1}{H(b,c)}\int_c^b H(b,s)\rho(s)Q(s)ds+\frac{1}{H(c,a)}\int_a^c H(s,a)\rho(s)Q(s)ds$$

$$> k_0\beta^\alpha\Big[\frac{1}{H(b,c)}\int_c^b H(b,s)\rho(s)r(s)|\psi_2(b,s)|^{\alpha+1}ds$$
$$+\frac{1}{H(c,a)}\int_a^c H(s,a)\rho(s)r(s)|\psi_1(s,a)|^{\alpha+1}ds\Big]. \tag{2.24}$$

Then Eq. (1.1) is oscillatory.

Proof Firstly, we claim that every solution of (1.1) has at least one zero in (a,b). If not, then, without loss of generality, we assume that there exists a solution $x(t)$ of Eq. (1.1) such that $x(t)>0$ for $t\in(a,b)$. It follows from Lemmas 2.1 and 2.2 that (2.2) and (2.23) hold. By dividing (2.2) and (2.23) by $H(b,c)$ and $H(c,a)$, respectively, and then adding them, we have

$$\frac{1}{H(b,c)}\int_c^b H(b,s)\rho(s)Q(s)ds+\frac{1}{H(c,a)}\int_a^c H(s,a)\rho(s)Q(s)ds$$
$$\le k_0\beta^\alpha\Big[\frac{1}{H(b,c)}\int_c^b H(b,s)\rho(s)r(s)|\psi_2(b,s)|^{\alpha+1}ds$$
$$+\frac{1}{H(c,a)}\int_a^c H(s,a)\rho(s)r(s)|\psi_1(s,a)|^{\alpha+1}ds\Big].$$

This contradicts (2.24) and completes the proof of the claim.

Secondly, we show Eq. (1.1) is oscillatory. Pick up a sequence $\{T_i\}\subset[t_0,\infty)$ such that $T_i\to\infty$ as $i\to\infty$. By the assumption, for each $i\in N$, there exist $a_i,b_i,c_i\in R$ such that $T_i\le a_i<c_i<b_i$, and (2.24) holds, where a,b,c are replaced by a_i,b_i,c_i, respectively. As claimed, every solution $x(t)$ of Eq. (1.1) has at least one zero, $t_i\in(a_i,b_i)$. Noting that $t_i>a_i\ge T_i$, $i\in\{1,2,\cdots\}$, we see that every solution has arbitrary large zeros. Thus, Eq. (1.1) is oscillatory. □

Theorem 2.4 *Assume that there exist $H\in\mathcal{H}$, $\rho\in C^1([t_0,\infty),\mathbb{R}^+)$ such that for some $\beta\ge1$ and each $\tau\ge t_0$,*

$$\limsup_{t\to\infty}\int_\tau^t H(t,s)\rho(s)\Big[Q(s)-k_0\beta^\alpha r(s)|\psi_2(t,s)|^{\alpha+1}\Big]ds>0 \tag{2.25}$$

and

$$\limsup_{t\to\infty}\int_\tau^t H(s,\tau)\rho(s)\Big[Q(s)-k_0\beta^\alpha r(s)|\psi_1(s,\tau)|^{\alpha+1}\Big]ds>0. \tag{2.26}$$

Then Eq. (1.1) is oscillatory.

Proof For any $T\ge t_0$, let $a=T$. In (2.25), we choose $\tau=a$. Then there exists $c>a$ such that

$$\int_a^c H(s,a)\rho(s)\Big[Q(s)-k_0\beta^\alpha r(s)|\psi_1(s,a)|^{\alpha+1}\Big]ds>0. \tag{2.27}$$

In (2.26), we choose $\tau=c$. Then there exists $b>c$ such that

$$\int_c^b H(b,s)\rho(s)\Big[Q(s)-k_0\beta^\alpha r(s)|\psi_2(b,s)|^{\alpha+1}\Big]ds>0. \tag{2.28}$$

Combining (2.27) and (2.28), we find (2.24) holds. Thus, by Theorem 2.3, Eq. (1.1) is oscillatory. □

For the case where $H := H(t-s) \in \mathcal{H}$, we have $h_1(t-s) \equiv h_2(t-s)$ and denote these by $h(t-s)$. The subclass of $\mathcal{H}$ containing such $H(t-s)$ is denoted by $\mathcal{H}_0$. Applying Theorem 2.4 to $\mathcal{H}_0$, we obtain

Theorem 2.5 *Assume that, for each $T \geq t_0$, there exist $H \in \mathcal{H}_0$, $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$, and $a, c \in \mathbb{R}$ such that $T \leq a < c$ and for some $\beta \geq 1$,*

$$\begin{aligned}&\int_a^c H(s-a)[\rho(s)Q(s)+\rho(2c-s)Q(2c-s)]ds\\&> k_0\beta^\alpha \int_a^c H(s-a)\Big[r(s)\rho(s)\Big|\frac{h(s-a)}{\sqrt{H(s-a)}}+\frac{\rho'(s)}{\rho(s)}\Big|^{\alpha+1}\\&+r(2c-s)\rho(2c-s)\Big|\frac{h(s-a)}{\sqrt{H(s-a)}}-\frac{\rho'(2c-s)}{\rho(2c-s)}\Big|^{\alpha+1}\Big]ds.\end{aligned} \tag{2.29}$$

Then Eq. (1.1) is oscillatory.

Proof Let $b = 2c-a$. Then $H(b-c) = H(c-a) = H((b-a)/2)$, and for any $f \in L[a,b]$, we have

$$\int_c^b f(s)ds = \int_a^c f(2c-s)ds.$$

Hence,

$$\int_c^b H(b-s)\rho(s)Q(s)ds = \int_a^c H(s-a)\rho(2c-s)Q(2c-s)ds.$$

Thus, (2.29) implies that (2.24) holds for $H \in \mathcal{H}_0$. Hence, Eq. (1.1) is oscillatory by Theorem 2.3. □

Define

$$R(t) = \int_{t_0}^t \frac{ds}{r^{1/\alpha}(s)}, \ \ t > t_0.$$

Theorem 2.6 *Let $\lim_{t\to\infty} R(t) = \infty$. Assume that, for some $\beta \geq 1$ and some $\lambda > \max\{1,\alpha\}$, the following 2 inequalities hold:*

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_\tau^t [R(t)-R(s)]^\lambda Q(s)ds > k_0\beta^\alpha \frac{\lambda^{\alpha+1}}{\lambda-\alpha} \tag{2.30}$$

and

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_\tau^t [R(s)-R(\tau)]^\lambda Q(s)ds > k_0\beta^\alpha \frac{\lambda^{\alpha+1}}{\lambda-\alpha}. \tag{2.31}$$

Then Eq. (1.1) is oscillatory.

Proof Let $H(t,s) = [R(t)-R(s)]^\lambda$, $(t,s) \in D_0$. Then

$$h_1(t,s) = \lambda[R(t)-R(s)]^{(\lambda-2)/2} r^{-1/\alpha}(t), \quad h_2(t,s) = \lambda[R(t)-R(s)]^{(\lambda-2)/2} r^{-1/\alpha}(s).$$

Note that

$$\int_\tau^t H(s,\tau)\rho(s)r(s)|\psi_1(s,\tau)|^{\alpha+1}ds = \int_\tau^t \lambda^{\alpha+1}[R(s)-R(\tau)]^{\lambda-\alpha-1}r^{-1/\alpha}(s)ds$$
$$=\frac{\lambda^{\alpha+1}}{\lambda-\alpha}[R(t)-R(\tau)]^{\lambda-\alpha}$$

and

$$\int_\tau^t H(t,s)\rho(s)r(s)\psi_2^{\alpha+1}(t,s)ds = \int_\tau^t \lambda^{\alpha+1}[R(t)-R(s)]^{\lambda-\alpha-1}r^{-1/\alpha}(s)ds$$
$$=\frac{\lambda^{\alpha+1}}{\lambda-\alpha}[R(t)-R(\tau)]^{\lambda-\alpha},$$

and recall that $\lim_{t\to\infty} R(t)=\infty$ and $\lambda-\alpha>0$; it follows

$$\lim_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_\tau^t H(s,\tau)\rho(s)r(s)|\psi_1(s,\tau)|^{\alpha+1}ds = \frac{\lambda^{\alpha+1}}{\lambda-\alpha} \tag{2.32}$$

and

$$\lim_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_\tau^t H^\alpha(t,s)\rho(s)r(s)|\psi_2(t,s)|^{\alpha+1}ds = \frac{\lambda^{\alpha+1}}{\lambda-\alpha}. \tag{2.33}$$

From (2.30) and (2.32), we have

$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_\tau^t [R(s)-R(\tau)]^\lambda\Big[Q(s)-k_0\beta^\alpha|\psi_1(s,\tau)|^{\alpha+1}\Big]ds$$
$$=\limsup_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_\tau^t [R(s)-R(\tau)]^\lambda Q(s)ds - k_0\beta^\alpha\frac{\lambda^{\alpha+1}}{\lambda-\alpha}>0,$$

i.e. (2.25) holds. Similarly, (2.31) and (2.33) imply that (2.26) holds. By Theorem 2.4, Eq. (1.1) is oscillatory. □

Remark 2.1 The above theorems are presented in the form of a high degree of generality. They extend, improve, and complement a number of existing results in [4, 13, 15, 16, 22] and handle some cases not covered by known criteria. They also give rather wide possibilities of deriving different explicit oscillation criteria for Eq. (1.1) with appropriate choices of the functions $H(t,s)$ and $\rho(t)$.

Example 2.1 Consider the following second order equation with mixed nonlinearities

$$(t^{-\nu}|x'(t)|^{\alpha-1}x'(t))' + t^\mu\big(\frac{\mu\cos t}{t}-\sin t\big)|x(t)|^{\alpha-1}x(t) + \sum_{i=1}^{2} q_i(t)|x(t)|^{\alpha_i-1}x(t) = 0, \tag{2.34}$$

where $t\ge 1$, and $n=2$, $\alpha>0$, $\alpha\neq 2$, $\alpha_1=\frac{3}{2}\alpha$, $\alpha_2=\frac{1}{2}\alpha$, $r(t)=t^{-\nu}$, $q_0(t)=t^\mu(\mu t^{-1}\cos t-\sin t)$, ν, $\mu>0$, and q_1, $q_2\in C([1,\infty),(0,\infty))$.

Next, we show that if there exists $c>\mu^2$ such that

$$q_1(t)q_2(t)\ge c\,t^{2(\mu-1)}, \tag{2.35}$$

and then Eq. (2.34) is oscillatory.

Indeed, let $\eta_1 = \eta_2 = \frac{1}{2}$, then

$$Q(t) = t^{\mu}\Big(\frac{\mu\cos t}{t} - \sin t\Big) + 2\sqrt{q_1(t)q_2(t)} \geq t^{\mu}\Big(\frac{\mu\cos t}{t} - \sin t\Big) + 2\sqrt{c}t^{\mu-1}.$$

Hence, for any $t \geq 1$,

$$\begin{aligned}\int_1^t Q(s)ds \geq & \int_1^t d[s^{\mu}\cos s] + 2\sqrt{c}\int_{t_0}^t s^{\mu-1}ds \\ = & t^{\mu}\cos t - \cos 1 + \frac{2\sqrt{c}}{\mu}(t^{\mu}-1) \geq \delta t^{\mu}, \quad 0<\delta<1.\end{aligned}$$

Taking $\rho(t)\equiv 1$, $H(t,s) = (t-s)^2$, and $\beta = \alpha+1$, we have, for $t>1$,

$$\begin{aligned}&\frac{1}{t^2}\int_1^t (t-s)^2\Big[Q(s) - k_0\beta^{\alpha} r(s)\Big|\frac{2}{t-s}\Big|^{\alpha+1}\Big]ds \\ &= \frac{1}{t^2}\int_1^t \Big[(t-s)^2 Q(s) - k_2\frac{(t-s)^{1-\alpha}}{s^{\nu}}\Big]ds \quad \Big(k_2 = \frac{2^{\alpha+1}}{\alpha+1}\Big) \\ &= \frac{1}{t^2}\int_1^t \Big[2(t-s)\int_1^s Q(\tau)d\tau\Big]ds - \frac{k_2}{t^2}\int_1^t \frac{(t-s)^{1-\alpha}}{s^{\nu}}ds \\ &\geq \frac{2\delta}{t^2}\int_1^t (t-s)s^{\mu}ds - \frac{k_2}{t^2}\int_1^t (t-s)^{1-\alpha}ds \\ &= \frac{2\delta}{(\mu+1)(\mu+2)}t^2 - \frac{2\delta}{\mu+1}\frac{1}{t} + \frac{2\delta}{\mu+2}\frac{1}{t^2} + \frac{1}{2-\alpha}\frac{1}{t^{\alpha}}\big(1-\frac{1}{t}\big)^{2-\alpha} \to \infty, \ t\to\infty,\end{aligned}$$

i.e. (2.9) holds. Hence, by Theorem 2.1, Eq. (2.34) is oscillatory when (2.35) holds.

Example 2.2 Consider the following equation

$$(r(t)|x'(t)|^{\alpha-1}x'(t))' + q_0(t)|x(t)|^{\alpha-1}x(t) + \sum_{i=1}^{2} q_i(t)|x(t)|^{\alpha_i-1}x(t) = 0, \tag{2.36}$$

where $t\geq 1$, $r \in C(\mathbb{R}^+, (0,1))$, $n=2$, $\alpha>0$, $\alpha_1 = \frac{3}{2}\alpha$, $\alpha_2 = \frac{1}{2}\alpha$, and

$$q_0(t) = \begin{cases} c_0(t-3n), & 3n\leq t\leq 3n+1, \\ c_0(-t+3n+2), & 3n+1<t<3n+2, \\ -n, & 3n+2<t\leq 3n+3, \end{cases}$$

and

$$q_i(t) = \begin{cases} c_i(t-3n), & 3n\leq t\leq 3n+1, \\ c_i(-t+3n+2), & 3n+1<t<3n+2, \\ 0, & 3n+2<t\leq 3n+3, \end{cases}$$

where $i=1,\,2$, $c_0,\,c_1,\,c_2>0$.

Now, we verify that Eq. (2.36) is oscillatory if there exists $\lambda > \alpha_0 = \max\{1,\alpha\}$ such that

$$c_0 + 2\sqrt{c_1c_2} > \frac{\lambda+2}{(\lambda-\alpha)(\alpha+1)^{\alpha+1}}. \tag{2.37}$$

In fact, let $\eta_1 = \eta_2 = \frac{1}{2}$; then $Q(t) = q_0(t) + 2\sqrt{q_1(t)q_2(t)}$. For each $\tau > 0$, there exists $n \in \{0, 1, 2, \cdots\}$ such that $3n > \tau$. Let $\rho(t) = 1$, $H(t-s) = (t-s)^\lambda$, $a = 3n$, and $c = 3n+1$. It follows

$$\begin{aligned}\int_a^c H(s-a)\big[Q(s) + Q(2c-s)\big]ds = &\int_{3n}^{3n+1} (s-3n)^\lambda \big[Q(s) + Q(6n+2-s)\big]ds \\ =&2(c_0 + 2\sqrt{c_1 c_2}) \int_{3n}^{3n+1} (s-3n)^{\lambda+1} ds \\ =&\frac{2(c_0 + 2\sqrt{c_1c_2})}{\lambda+2},\end{aligned}$$

and

$$\int_a^c H(s-a)\big[r(s) + r(2c-s)\big]\Big|\frac{h(s-a)}{\sqrt{H(s-a)}}\Big|^{\alpha+1} ds \le 2\int_{3n}^{3n+1} (s-3n)^{\lambda-(\alpha+1)} ds = \frac{2}{\lambda-\alpha}.$$

This implies that (2.29) holds with $\beta = 1$. Hence, by Theorem 2.5, Eq. (2.36) is oscillatory when (2.37) holds. Note that we have $\int_0^\infty q_0(t)dt = -\infty$ in Eq. (2.36).

3. Oscillation criteria for Eq. (1.2)

In what follows, we obtain counterparts of oscillation criteria derived earlier in section 2 for Eq. (1.2). For simplicity, we define, for any $T_0 \ge t_0$,

$$\phi_i(t, T_0) = \begin{cases} \Big(\displaystyle\int_{\tau_i(T_0)}^t \frac{ds}{r^{1/\alpha}(s)}\Big)^{-1} \displaystyle\int_{\tau_i(T_0)}^{\tau_i(t)} \frac{ds}{r^{1/\alpha}(s)}, & \tau_i(t) < t, \\ 1, & \tau_i(t) = t, \end{cases}$$

where $i = 1, 2, \cdots$, and

$$Q(t, T_0) = q_0(t)\phi_i^\alpha(t, T_0) + \prod_{i=1}^n \eta_i^{-\eta_i}(q_i(t)\phi_i(t, T_0))^{\eta_i}.$$

Before starting our main results, we prove the following theorem.

Theorem 3.1 *Assume that, for any $T_0 \ge t_0$, $q_0(t) \ge 0$ for $t \ge T_0$, and Eq. (1.2) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)$. Then, for $t \ge T_0$,*

$$(r(t)|x'(t)|^{\alpha-1}x'(t))' + q_0(t)\phi_0^\alpha(t, T_0)|x(t)|^{\alpha-1}x(t) + \sum_{i=1}^n q_i(t)\phi_i^{\alpha_i}(t, T_0)|x(t)|^{\alpha_i-1}x(t) \le 0. \tag{3.1}$$

Proof Without loss of generality, we can pick $T_0 \in [t_0, \infty)$ so that $x(t) > 0$ and $x(\tau_i(t)) > 0$ $(i = 0, 1, \cdots, n)$ for all $t \ge T_0$, (when $x(t)$ and $x(\tau_i(t)), i = 0, 1, \cdots, n$, are eventually negative, the proof follows a similar argument). It follows from Eq. (1.2) that $(r(t)|x'(t)|^{\alpha-1}x'(t))' \le 0$ for $t \ge T_0$, i.e. $r(t)|x'(t)|^{\alpha-1}x'(t)$ is nonincreasing, consequently, $x'(t)$ is eventually of constant sign.

If $x'(t) > 0$ for $t \ge T_0$, then, by (A3), noting that $r(t)|x'(t)|^\alpha x(t) = r(t)(x'(t))^\alpha$, we have, for $t \ge T_0$,

$$x(t) - x(\tau_i(t)) = \int_{\tau_i(t)}^t \frac{(r(s)(x'(s))^\alpha)^{1/\alpha}}{r^{1/\alpha}(s)} ds$$

$$\leq (r(\tau_i(t))(x'(\tau_i(t)))^\alpha)^{1/\alpha}\int_{\tau_i(t)}^t \frac{ds}{r^{1/\alpha}(s)},$$

i.e.

$$\frac{x(t)}{x(\tau_i(t))} \leq 1 + \frac{(r(\tau_i(t))(x'(\tau_i(t)))^\alpha)^{1/\alpha}}{x(\tau_i(t))}\int_{\tau_i(t)}^t \frac{ds}{r^{1/\alpha}(s)}. \tag{3.2}$$

Also, we see, for $t \geq T_0$,

$$x(\tau_i(t)) > x(\tau_i(t)) - x(\tau_i(T_0)) = \int_{\tau_i(T_0)}^{\tau_i(t)} \frac{(r(s)(x'(s))^\alpha)^{1/\alpha}}{r^{1/\alpha}(s)}ds$$
$$\geq (r(\tau_i(t))(x'(\tau_i(t)))^\alpha)^{1/\alpha}\int_{\tau_i(T_0)}^{\tau_i(t)} \frac{ds}{r^{1/\alpha}(s)},$$

which follows

$$\frac{(r(\tau_i(t))(x'(\tau_i(t)))^\alpha)^{1/\alpha}}{x(\tau_i(t))} < \frac{1}{\int_{\tau_i(T_0)}^{\tau_i(t)} \frac{ds}{r^{1/\alpha}(s)}}, \quad t \geq T_0, \tag{3.3}$$

Thus, (3.2) and (3.3) imply that

$$\frac{x(t)}{x(\tau_i(t))} < \int_{\tau_i(T_0)}^t \frac{ds}{r^{1/\alpha}(s)}\Big(\int_{\tau_i(T_0)}^{\tau_i(t)} \frac{ds}{r^{1/\alpha}(s)}\Big)^{-1} = \frac{1}{\delta_i(t,T_0)}, \quad t \geq T_0.$$

Hence,

$$x(\tau_i(t)) \geq \phi_i(t,T_0)x(t) \text{ for } i = 0,1,\cdots,n, \quad t \geq T_0. \tag{3.4}$$

If $x'(t) < 0$ for $t \geq T_0$, note that

$$0 \geq (r(t)|x'(t)|^{\alpha-1}x'(t))' = -(r(t)(-x'(t))^\alpha)',$$

then $r(t)(-x'(t))^\alpha$ is nondecreasing for $t \geq T$. Similar to the above proof, we also find (3.4) holds. Using (3.4) in Eq. (1.2), we find (3.1) holds. □

Using Theorem 3.1, and by some slight modifications of the proofs of the corresponding theorems in the previous section, we can obtain the following results stated here for completion.

Theorem 3.2 *Assume that there exist $H \in \mathcal{H}$ and $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ such that for some $\beta \geq 1$ and any $T_0 \geq t_0$, $T \geq t_0$,*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)\Big[Q(s,T_0) - k_0\beta^\alpha r(s)|\psi_2(t,s)|^{\alpha+1}\Big]ds = \infty. \tag{3.5}$$

Then Eq. (1.2) is oscillatory.

Theorem 3.3 *Assume that there exists $H \in \mathcal{H}$ such that (2.10) holds. Furthermore, if there exist $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ and $\phi \in C([t_0,\infty),\mathbb{R})$ such that for some $\beta > 1$ and any $T_0 \geq t_0$, $T \geq t_0$,*

$$\limsup_{t\to\infty} \frac{1}{H(t,T_0)}\int_T^t H(t,s)\rho(s)\Big[Q(s,T_0) - k_0\beta^\alpha r(s)|\psi_2(t,s)|^{\alpha+1}\Big]ds \geq \phi(T), \tag{3.6}$$

and (2.12) holds, then Eq. (1.2) is oscillatory.

Theorem 3.4 *Assume that, for each* $T \geq t_0$, *there exist* $H \in \mathcal{H}$, $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$, *and* $a, b, c \in \mathbb{R}$ *such that* $T \leq a < c < b$ *and for some* $\beta \geq 1$ *and any* $T_0 \geq t_0$

$$\begin{aligned}&\frac{1}{H(b,c)}\int_c^b H(b,s)\rho(s)Q(s,T_0)ds + \frac{1}{H(c,a)}\int_a^c H(s,a)\rho(s)Q(s,T_0)ds\\ &\quad > k_0\beta^\alpha\Big[\frac{1}{H(b,c)}\int_c^b H(b,s)\rho(s)r(s)|\psi_2(b,s)|^{\alpha+1}ds\\ &\qquad + \frac{1}{H(c,a)}\int_a^c H(s,a)\rho(s)r(s)|\psi_1(s,a)|^{\alpha+1}ds\Big]. \end{aligned} \tag{3.7}$$

Then Eq. (1.2) is oscillatory.

Theorem 3.5 *Assume that there exist* $H \in \mathcal{H}$, $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that, for some* $\beta \geq 1$ *and each* $\tau \geq t_0$ *and any* $T_0 \geq t_0$,

$$\limsup_{t\to\infty}\int_\tau^t H(t,s)\rho(s)\Big[Q(s,T_0) - k_0\beta^\alpha r(s)|\psi_2(t,s)|^{\alpha+1}\Big]ds > 0 \tag{3.8}$$

and

$$\limsup_{t\to\infty}\int_\tau^t H(s,\tau)\rho(s)\Big[Q(s,T_0) - k_0\beta^\alpha r(s)|\psi_1(s,\tau)|^{\alpha+1}\Big]ds > 0. \tag{3.9}$$

Then Eq. (1.2) is oscillatory.

Theorem 3.6 *Assume that, for each* $T \geq t_0$, *there exist* $H \in \mathcal{H}_0$, $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$, *and* $a, c \in \mathbb{R}$ *such that* $T \leq a < c$ *and for some* $\beta \geq 1$ *and any* $T_0 \geq t_0$,

$$\begin{aligned}&\int_a^c H(s-a)[\rho(s)Q(s,T_0) + \rho(2c-s)Q(2c-s,T_0)]ds\\ &\quad > k_0\beta^\alpha\int_a^c H(s-a)\Big[r(s)\rho(s)\Big|\frac{h(s-a)}{\sqrt{H(s-a)}} + \frac{\rho'(s)}{\rho(s)}\Big|^{\alpha+1}\\ &\qquad + r(2c-s)\rho(2c-s)\Big|\frac{h(s-a)}{\sqrt{H(s-a)}} - \frac{\rho'(2c-s)}{\rho(2c-s)}\Big|^{\alpha+1}\Big]ds.\end{aligned} \tag{3.10}$$

Then Eq. (1.2) is oscillatory.

Theorem 3.7 *Let* $\lim_{t\to\infty}R(t) = \infty$. *Assume that, for some* $\beta \geq 1$, *some* $\lambda > \max\{1,\alpha\}$, *and any* $T_0 \geq t_0$, *the following 2 inequalities hold:*

$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_\tau^t [R(t)-R(s)]^\lambda Q(s,T_0)ds > k_0\beta^\alpha\frac{\lambda^{\alpha+1}}{\lambda-\alpha} \tag{3.11}$$

and

$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_\tau^t [R(s)-R(\tau)]^\lambda Q(s,T_0)ds > k_0\beta^\alpha\frac{\lambda^{\alpha+1}}{\lambda-\alpha}. \tag{3.12}$$

Then Eq. (1.1) is oscillatory.

Remark 3.1 Let $q_i(t) = 0$ for $i = 1, \cdots, n$ in Eq. (1.2), Theorems 3.4–3.7 reduce to corresponding all theorems given in [20]. However, our conditions do not require $\int^{\infty} r^{-1/\alpha}(s)ds = \infty$ for Eq. (1.2).

Remark 3.2 Note that the parameter β introduced in our results plays an important role in the results. Clearly, it is easy to see that if Theorems 2.1–2.6 and Theorems 3.2–3.7 hold for some β_0, then they shall also hold for any $\beta > \beta_0$. In particular, the results obtained in [4, 13, 15, 16, 24, 22, 26] fixed the value $\beta = 2$.

Example 3.1 Consider the following mixed nonlinear delay differential equation:

$$\begin{aligned}&(|x'(t)|^{\alpha-1}x'(t))' + c_0 t^{-\nu}|x(t-\tau_0)|^{\alpha-1}x(t-\tau_0)\\&\quad + q_1(t)|x(t-\tau_1)|^{3\alpha/2-1}x(t-\tau_1) + q_2(t)|x(t-\tau_2)|^{\alpha/2-1}x(t-\tau_2) = 0,\end{aligned} \tag{3.13}$$

where $t \geq 1$, $\alpha > 0$, and $\alpha \neq 2$, $\tau_0, \tau_1, \tau_2 \geq 0$, $1 < \nu \leq \frac{3}{2}$, $c_0 > 0$, $q_i \in C([1,\infty),[0,\infty))$.

Next, we show that Eq. (3.13) is oscillatory if there exists $c_1 \geq 0$ such that

$$q_1(t)q_2(t) \geq c_1 t^{-2\nu}. \tag{3.14}$$

In fact, for any $T_0 \geq 1$, $t \geq T_0 + \max\{\tau_0, \tau_1, \tau_2\}$,

$$\phi_i(t, T_0) = \frac{t - T_0}{t + \tau_i - T_0} > \frac{1}{2},\ i = 0, 1, 2.$$

Let $\eta_1 = \eta_2 = \frac{1}{2}$. Then, for $t \geq T_0 + \max\{\tau_0, \tau_1, \tau_2\}$,

$$Q(t, T_0) \geq c_0 2^{-\alpha} t^{-\nu} + \sqrt{q_1(t)q_2(t)} \geq (c_0 2^{-\alpha} + c_1)t^{-\nu} =: C_0 t^{-\nu}.$$

For Theorem 3.3, let $H(t,s) = (t-s)^2$, $\beta = \alpha + 1$, $\rho(t) = 1$. A straightforward computation yields, for all $T \geq 1$,

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{t^2}\int_T^t \left\{(t-s)^2\left[Q(t,T_0) - k_0\beta^\alpha r(s)\left|\frac{2}{t-s}\right|^{\alpha+1}\right]\right\}ds\\&\geq \limsup_{t\to\infty} \frac{1}{t^2}\int_T^t \left[C_0(t-s)^2 s^{-\nu} - \frac{2^{\alpha+1}}{\alpha+1}(t-s)^{1-\alpha}\right]ds\\&= \frac{C_0}{\nu-1}\frac{1}{T^{\nu-1}} =: \phi(T).\end{aligned}$$

Consequently,

$$\limsup_{t\to\infty}\int_T^t \frac{\phi_+^2(s)}{r(s)}ds = \frac{C_0^2}{(\nu-1)^2}\int_1^t \frac{1}{s^{2(\nu-1)}}ds = \infty.$$

Hence, by Theorem 3.3, Eq. (3.13) is oscillatory when (3.14) holds.

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