Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 664 – 671
© TÜBİTAK
doi:10.3906/mat-1302-50

On the size of the third homotopy group of the suspension of an Eilenberg–MacLane space

Peyman NIROOMAND[1,*], **Francesco G. RUSSO**[2]
[1]School of Mathematics and Computer Science, Damghan University, Damghan, Iran
[2]Institute of Mathematics, Federal University of Rio de Janeiro, Rio de Janeiro, Brazil

Received: 26.02.2013 • **Accepted:** 08.10.2013 • **Published Online:** 25.04.2014 • **Printed:** 23.05.2014

Abstract: The nonabelian tensor square $G \otimes G$ of a group G of $|G| = p^n$ and $|G'| = p^m$ (p prime and $n, m \geq 1$) satisfies a classic bound of the form $|G \otimes G| \leq p^{n(n-m)}$. This allows us to give an upper bound for the order of the third homotopy group $\pi_3(SK(G,1))$ of the suspension of an Eilenberg–MacLane space $K(G,1)$, because $\pi_3(K(G,1))$ is isomorphic to the kernel of $\kappa : x \otimes y \in G \otimes G \mapsto [x,y] \in G'$. We prove that $|G \otimes G| \leq p^{(n-1)(n-m)+2}$, sharpening not only $|G \otimes G| \leq p^{n(n-m)}$ but also supporting a recent result of Jafari on the topic. Consequently, we discuss restrictions on the size of $\pi_3(SK(G,1))$ based on this new estimation.

Key words: Schur multipliers, p–groups, nonabelian tensor square, homotopy

1. Introduction and statement of results

All groups considered in the present paper are supposed to be finite. Following [2, 3], 2 groups G and H act *compatibly* on each other if

$$(x^y)^z = ((x^{z^{-1}})^y)^z, \qquad (t^z)^y = ((t^{y^{-1}})^z)^y,$$

for $x, z \in G$ and $y, t \in H$, and if they act on themselves by conjugation (notice that $x^y = y^{-1}xy$). The group presented by

$$G \otimes H = \langle x \otimes y \mid xz \otimes y = (z^x \otimes y^x)(x \otimes y),\ x \otimes yt = (x \otimes y)(x^y \otimes t^y)\rangle$$

is called the *nonabelian tensor product* of G and H. If G and H act trivially on each other, then $G^{ab} \otimes H^{ab}$ is the usual abelian tensor product (see [3, Proposition 2.4]) of the abelian groups $G^{ab} = G/G'$ and $H^{ab} = H/H'$, called *abelianizations* of G and H, respectively. On the other hand, if $G = H$ and all actions are by conjugation, then $G \otimes G$ is called *nonabelian tensor square* of G and

$$\nabla(G) = \langle g \otimes g \mid g \in G\rangle$$

turns out to be a central subgroup of $G \otimes G$, inducing a short exact sequence

$$1 \longrightarrow \nabla(G) \longrightarrow J_2(G) \longrightarrow M(G) \longrightarrow 1 \tag{1.1}$$

*Correspondence: p_niroomand@yahoo.com
2010 *AMS Mathematics Subject Classification:* Primary: 20D15, 20J99; Secondary: 20E34, 20F18.

where $M(G)$ is the *Schur multiplier* of G and $J_2(G) = \ker \kappa$ is the kernel of the epimorphism of groups

$$\kappa : x \otimes y \in G \otimes G \longmapsto [x, y] = x^{-1}y^{-1}xy \in G'.$$

From [2, 3], one can see that κ induces the central extension

$$1 \longrightarrow J_2(G) \longrightarrow G \otimes G \overset{\kappa}{\longrightarrow} G' \longrightarrow 1. \tag{1.2}$$

Brown and Loday [3] described widely the role of $J_2(G)$ and of $G \otimes G$ in algebraic topology, showing that the third homotopy group of the suspension of an Eilenberg–MacLane space $K(G, 1)$ satisfies the condition

$$\pi_3(SK(G, 1)) \simeq J_2(G) = \ker \kappa. \tag{1.3}$$

We can find the definitions of homotopy groups and Eilenberg–MacLane spaces in [17] and in [8, pp. 87, 365, 393, 410, 453, 475]. Further information can be found also in [4], which is a recent work on nonabelian algebraic topology.

Briefly, we note that the nonabelian tensor product is used to describe the third relative homotopy group of a triad as a nonabelian tensor product of the second homotopy groups of appropriate subspaces. More specifically, let a CW-complex X be the union $X = A \cup B$ of 2 path-connected CW-subcomplexes A and B, whose intersection $C = A \cap B$ is path-connected. Suppose that the canonical homomorphisms $\pi_1(C) \to \pi_1(A)$ and $\pi_1(C) \to \pi_1(B)$ are surjective. Then we have $\pi_3(X, A, B) \simeq \pi_2(A, C) \otimes \pi_2(B, C)$, where the groups $\pi_2(A, C)$ and $\pi_2(B, C)$ act on each another via $\pi_1(C)$. Therefore, restricting the structure of the nonabelian tensor product $\pi_2(A, C) \otimes \pi_2(B, C)$, we have information on $\pi_3(X, A, B)$ (for more specific properties of (1.3), the reader may refer also to [6, 9]).

Rocco [15] and later Ellis [5] showed that

$$|G \otimes G| \leq p^{n(n-m)} \tag{1.4}$$

for every p-group G of $|G| = p^n$ such that $|G'| = p^m$, where p is a prime and n, m are positive integers. In a successive paper, Ellis and McDermott [7] sharpened (1.4), proving that

$$|G \otimes G| \leq p^{nd} \tag{1.5}$$

where $d(G) = d$ is the minimum number of generators of G. More recently, Jafari [10] worked on (1.4) and (1.5) and proved that

$$|G \otimes G| \leq p^{d(n-e)+n-m} \tag{1.6}$$

where $\exp(G^{ab}) = p^e$ is the exponent of G^{ab} (e is a positive integer). If G^{ab} is an elementary abelian p-group, then one can see that (1.4), (1.5), and (1.6) are all equal. This is to testify that (1.6) is sharper than (1.5) and (1.4) for p-groups that do not have elementary abelianization.

We inform the reader that numerical inequalities of a similar form were recently addressed in [1, 14, 13] and have motivated us to write the present paper. The purpose of the present paper is in fact a further investigation of the order of the tensor square of nonabelian p-groups and its relations with algebraic topology. We concentrate on nonabelian p-groups, because the nonabelian tensor square of abelian groups is the usual abelian tensor square for which all is already known.

Our first main theorem generalizes (1.6) and consequently (1.5) and (1.4).

Theorem 1.1 *Let p be a prime and n, m be positive integers. A nonabelian p-group G of $|G| = p^n$ and $|G'| = p^m$ has*

$$|G \otimes G| \leq p^{(n-1)(n-m)+2}.$$

Let us denote with D_8, Q_8, E_1, and E_2 the dihedral group of order 8, the quaternion group of order 8, the extraspecial p-group of order p^3 and exponent p, and the extraspecial p-group of order p^3 and exponent p^2, respectively.

In case $m = 1$, Theorem 1.1 may be still improved, providing information on the explicit structure of the group. This is the purpose of our second main theorem.

Theorem 1.2 *Let p be a prime and n a positive integer. If G is a nonabelian p-group of $|G| = p^n$ and $|G'| = p$, then*

$$|G \otimes G| \leq p^{(n-1)^2+2},$$

and the equality holds if and only if G is isomorphic to $H \times E$, where $H \cong E_1$ or $H \cong Q_8$ and E is an elementary abelian p-group of order p^{n-3}.

It easy to see that the bound of Theorem 1.1 is sharper than (1.4), unless G is isomorphic to Q_8 or E_1. This indirectly gives a new criterion for detecting the size of the third homotopy groups of a suspension of an Eilenberg–MacLane space.

Corollary 1.3 *Let p be a prime and nandm positive integers. A nonabelian p-group G of $|G| = p^n$ and $|G'| = p^m$ has*

$$|\pi_3(SK(G,1))| \leq p^{n(n-m-1)+2}.$$

In particular, if $m = 1$, then

$$|\pi_3(SK(G,1))| \leq p^{n(n-2)+2},$$

and the equality holds if and only if $G \simeq H \times E$, where $H \simeq E_1$ or $H \cong Q_8$ and E is an elementary abelian p-group.

Our final result compares the bound of Theorem 1.1 with (1.6).

Corollary 1.4 *Let p be a prime, nandm positive integers, and G a nonabelian p-group of $|G| = p^n$ and $|G'| = p^m$. If G^{ab} is elementary abelian, then the bound of Theorem 1.1 is better than (1.6) and they are equal only for $d(G^{ab}) = 2$.*

Terminology and notations are standard and follow [2, 3, 4, 8, 11, 16, 17].

2. Proofs of the main results

The following 2 propositions recall some homological conditions about the existence of exact sequences for the nonabelian tensor products of groups. In some of these propositions and in some of the successive results, it may be useful to denote with $C_{p^s}^{(k)}$ the direct product of k-copies of the cyclic group C_{p^s} of order p^s (here s and k are positive integers and p a prime).

Proposition 2.1 (See [2], Proposition 9) *Given a central extension of groups*

$$1 \to Z \longrightarrow H \longrightarrow G \longrightarrow 1,$$

there exists a group homomorphism φ *and an exact sequence*

$$(Z \otimes H) \times (H \otimes Z) \xrightarrow{\varphi} H \otimes H \longrightarrow G \otimes G \longrightarrow 1$$

such that $\mathrm{im}\ \varphi$ *is central in* $H \otimes H$.

Proposition 2.2 (See [2], Propositions 13, 14) *We have that* $D_8 \otimes D_8 \simeq C_2^{(3)} \times C_4$ *and that* $Q_8 \otimes Q_8 \simeq C_2^{(2)} \times C_4^{(2)}$.

It is shown in [1, 2, 3, 13] that it is possible to combine (1.1) and (1.2), getting

$$|G \otimes G| = |\nabla(G)||M(G)||G'|. \tag{2.1}$$

We also recall that $M(G)$ is exactly the second homology group $H_2(G, \mathbb{Z})$ over G with integral coefficients.

The next lemma provides analogies when we have an extraspecial p-group of order p^3 with $p \neq 2$.

Lemma 2.3 *For an odd prime* p, *we have* $E_1 \otimes E_1 \simeq C_p^{(6)}$ *and* $E_2 \otimes E_2 \simeq C_p^{(4)}$.

Proof From [5, Theorem 2] , we deduce that $E_1 \otimes E_1$ is elementary abelian. Now, invoking [1, Proposition 2.2 (iii)], we deduce $\nabla(E_1) \cong \nabla(E_1^{ab})$ and hence $|\nabla(E_1)| = p^3$. On the other hand, [11, Theorem 3.3.6] implies $|M(E_1)| = p^2$, and the formula (2.1) implies $|E_1 \otimes E_1| = p^6$.

In case of E_2, we can prove analogously that $E_2 \otimes E_2 \simeq E_2^{ab} \otimes E_2^{ab}$. □

The next result can be found in [12], so its proof is omitted.

Lemma 2.4 (See [12], Corollary 2.3) *Let* H *be an extraspecial* p*-group of order* p^{2h+1} *for some positive integer* h. *Then* $H \otimes H$ *is elementary abelian of order* p^{4h^2} *for* $h \geq 2$.

We may proceed to show a fundamental step of our investigations.

Proposition 2.5 *Let* G *be a nonabelian* p*-group of* $|G| = p^n$ *and* $|G'| = p$ *with* $n \geq 1$. *If one of the following conditions holds:*

(i) G^{ab} *is not elementary abelian;*

(ii) G^{ab} *is elementary abelian and* $Z(G)$ *is not elementary abelian;*

then $|G \otimes G| \leq p^{(n-1)^2+2}$.

Proof (i). The proof is an upstanding result of Proposition 2.1 while $Z = G'$.

Let $G^{ab} = C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}}$ where $\sum_{i=1}^{k} m_i = n-1$ and $m_i \leq m_{i+1}$ for all $1 \leq i \leq k-1$. From [11], we recall that

$$|C_{p^u} \otimes C_{p^v}| = |C_{\gcd(p^u,p^v)}| = \gcd(p^u, p^v)$$

for 2 positive integers u and v and that

$$(A \times B) \otimes (C \times D) = (A \otimes C) \times (A \otimes D) \times (B \otimes C) \times (B \otimes D)$$

for (the usual abelian tensor product of) given abelian groups A, B, C, and D. These rules will be often applied in the next lines.

Since $G' \subseteq Z(G)$, we have $G' \otimes G \cong G' \otimes G^{ab}$. On the other hand, [2, Proposition 3 (10)] implies that $G \otimes G'$ and $G' \otimes G$ have identical images in $G \otimes G$, so that

$$|G \otimes G| \leq |G' \otimes G^{ab}||G^{ab} \otimes G^{ab}|$$

$$= |C_p \otimes \big(C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}}\big)| \cdot$$

$$\cdot |\big(C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}}\big) \otimes \big(C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}}\big)|$$

$$= |\big(C_p \otimes C_{p^{m_1}}\big) \times \big(C_p \otimes C_{p^{m_2}}\big) \times \ldots \times \big(C_p \otimes C_{p^{m_k}}\big)| \cdot$$

$$\cdot |\big(C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}}\big) \otimes C_{p^{m_1}}| \cdot \ldots \cdot |\big(C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}}\big) \otimes C_{p^{m_k}}|$$

$$= |C_p^{(k)}| \cdot |\big(C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}}\big) \otimes C_{p^{m_1}}| \cdot \ldots \cdot |\big(C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}}\big) \otimes C_{p^{m_k}}|$$

$$= p^k \cdot |\big(C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}}\big) \otimes C_{p^{m_1}}| \cdot \ldots \cdot |\big(C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}}\big) \otimes C_{p^{m_k}}|$$

$$= p^k \cdot |(C_{p^{m_1}} \otimes C_{p^{m_1}}) \times (C_{p^{m_2}} \otimes C_{p^{m_1}}) \ldots \times (C_{p^{m_k}} \otimes C_{p^{m_1}})| \cdot \ldots$$

$$\ldots \cdot |(C_{p^{m_1}} \otimes C_{p^{m_k}}) \times (C_{p^{m_2}} \otimes C_{p^{m_k}}) \ldots \times (C_{p^{m_k}} \otimes C_{p^{m_k}})|$$

$$= p^k \cdot p^{m_1+\ldots+m_k+2(m_1+\ldots+m_{k-1}+m_1+\ldots+m_{k-2}+\ldots+m_1)}$$

$$= p^{m_1+\ldots+m_k+2(m_1+\ldots+m_{k-1}+m_1+\ldots+m_{k-2}+\ldots+m_1)+k}$$

$$\leq p^{n-1+2(n-3+n-4+\ldots+n-k-1)+k} < p^{(n-1)^2+2}.$$

In order to justify the last inequality, it is enough to compare the polynomial $(n-1)+2((n-3)+(n-4)+\ldots+(n-k-1))+k$ in the variable n (and fixed k) with the polynomial $(n-1)^2+2$ in the variable n and this can be done in the following way. Since G^{ab} is not elementary and $m_k \geq 2$, we have $k \leq n-2$. Then

$$(n-1)+2((n-3)+(n-4)+\ldots+(n-k-1))+k \leq (n-1)+2((n-3)+(n-4)+\ldots+1)+k$$

$$\leq (n-1)+2((n-3)+(n-4)+\ldots+1)+(n-2) = (n-1)+2\frac{(n-3)(n-2)}{2}+(n-2),$$

where in the last step we use the Gauss formula,

$$1+\ldots+(n-4)+(n-3) = \frac{(n-3)(n-2)}{2},$$

for the sum of the first $(n-3)$ integers. Now

$$(n-1)+2\frac{(n-3)(n-2)}{2}+(n-2) = (n-1)+(n-3)(n-2)+(n-2)$$

$$= n - 1 + n^2 - 5n + 6 + n - 2 = n^2 - 3n + 3 < n^2 - 2n + 3 = (n-1)^2 + 2,$$

as claimed.

(ii). Since G^{ab} is a vector space over C_p, we may consider the complement H/G' of $Z(G)/G'$ in G^{ab}. Moreover, H is extraspecial and $G = HZ(G)$. This implies the existence of an epimorphism

$$(a,b) \otimes (c,d) \in (H \times Z(G)) \otimes (H \times Z(G)) \longmapsto ab \otimes cd \in G \otimes G$$

such that

$$|G \otimes G| \leq |(H \times Z(G)) \otimes (H \times Z(G))|.$$

Letting $|Z(G)| = p^k$ and $|H| = p^{2h+1}$, we can suppose that $k \geq 2$ by using Proposition 2.2. Now the following 2 cases can be considered.

Case (ii).1. First suppose that $h \geq 2$.

Let $Z(G) \cong C_{p^{k_1}} \times \ldots \times C_{p^{k_t}}$ and $\sum_{i=1}^{t} k_i = n - 2h$. Applying Lemma 2.4 and [2, Proposition 11], we have

$$|G \otimes G| \leq |H \otimes H| \ \ |H \otimes Z(G)|^2 \ \ |Z(G) \otimes Z(G)|$$

$$= p^{4h^2} \ |C_p^{(h)} \otimes (C_{p^{k_1}} \times \ldots \times C_{p^{k_t}})|^2 \ \ |(C_{p^{k_1}} \times \ldots \times C_{p^{k_t}}) \otimes (C_{p^{k_1}} \times \ldots \times C_{p^{k_t}})|$$

$$= p^{4h^2} p^{2ht} p^{(2t-1)k_1 + (2t-3)k_2 + \ldots + k_t}$$

$$\leq p^{4h^2} p^{2ht} p^{n-2h+2(n-2h-2+\ldots+n-2h-t)} < p^{(n-1)^2+2}.$$

In order to justify the last inequality, it is enough to compare the polynomial $4h^2 + 2ht + n - 2h + 2((n-2h-2) + \ldots + (n-2h-t))$ in the variable n (and fixed h, t) with the polynomial $(n-1)^2 + 2$ in the variable n. One way is the following. Since G^{ab} is not elementary and $\sum_{i=1}^{t} k_i = n - 2h$, we have $t \leq n - 2h - 1$. Then

$$4h^2 + 2ht + (n-2h) + 2((n-2h-2) + \ldots + (n-2h-t))$$

$$\leq 4h^2 + 2ht + (n-2h) + 2((n-2h-2) + \ldots + (n-2h-t) + \ldots + 1))$$

$$= 4h^2 + 2ht + (n-2h) + 2\frac{(n-h-2)(n-2h-1)}{2},$$

where we use the Gauss formula,

$$1 + \ldots + (n-h-2) = \frac{(n-h-2)(n-h-1)}{2},$$

for the sum of the first $(n-h-2)$ integers, and so we continue

$$= 4h^2 + 2ht + (n-2h) + (n-h-2)(n-2h-1) = 4h^2 + 2ht + (n-2h-1+1) + (n-h-2)(n-2h-1)$$

$$= 4h^2 + 2ht + 1 + (n-2h-1)(n-h-2+1) = 4h^2 + 2ht + (n-2h-1)^2 + 1$$

$$\leq 4h^2 + 2h(n-2h-1) + (n-2h-1)^2 + 1$$

$$= 4h^2 + 2h(n-1) - 4h^2 + (n-1)^2 + 4h^2 - 4h(n-1) + 1$$

$$= (n-1)^2 - 2h(n-1) + 4h^2 + 1 = (n-1)^2 - 2h(n-2h-1) + 1 < (n-1)^2 + 2,$$

as claimed.

Case (ii).2. Without loss of generality, we can suppose that $Z(G) \cong C_{p^2}$. Now the result is obtained by using Proposition 2.1 and the fact that $|\mathrm{im}\ \varphi| \geq p$. □

Now we may prove the main results. We begin with Theorem 1.2.

Proof [Proof of Theorem 1.2] Assume that G^{ab} and $Z(G)$ are elementary abelian and $|Z(G)| \geq p^2$ by Lemma 2.3. Let E be the complement of G' in $Z(G)$. Thus, there exists an extraspecial p-group H of order p^{2h+1} such that $G \cong H \times E$.

In case $h \geq 2$, it is easy to check $|G \otimes G| < p^{(n-1)^2+2}$. For $h = 1$,

$$|G \otimes G| = |H \otimes H||E \otimes E||E \otimes H|^2$$

where $|E \otimes E||E \otimes H|^2 = p^{(n-1)(n-3)}$.

Now Proposition 2.2 and Lemma 2.3 imply that $|G \otimes G| = p^{(n-1)^2+2}$ when $H \cong Q_8$ or H has exponent p. The result follows. □

The following result is the general situation that we may have.

Proof [Proof of Theorem 1.1] We argue by induction on m. For $m = 1$, the result follows by Theorem 1.2.

Let $m \geq 2$ and K be a central subgroup of order p contained in G'. The induction hypothesis and Proposition 2.1 yield

$$\begin{aligned} |G \otimes G| &\leq |K \otimes G^{ab}||G/K \otimes G/K| \\ &\leq p^{n-m} p^{(n-m)(n-2)+2} = p^{(n-1)(n-m)+2}. \end{aligned}$$

□

The next corollary is an interesting consequence of Theorem 1.2.

Corollary 2.6 *Let G be a nonabelian p-group such that $|G \otimes G| = p^{(n-1)^2+2}$.*

(i) If $p > 2$, then $G \otimes G \cong {C_p}^{((n-1)^2+2)}$.

(ii) If $p = 2$, then $G \otimes G \cong C_4^{(2)} \times C_2^{((n-1)^2-2)}$.

The proof of the third main result is the following.

Proof [Proof of Corollary 1.3] It follows from Theorems 1.1, 1.2 and (1.2), (1.3). □

We end by comparing the bound, which we have found in Theorem 1.1, with previous contributions on the topic, namely (1.4), (1.5), and (1.6).

Proof [Proof of Corollary 1.4] Since G^{ab} is elementary abelian, $d = d(G) = d(G^{ab}) = n-m$ and $\exp(G^{ab}) = p$. Then (1.4), (1.5), and (1.6) give the same inequality

$$|G \otimes G| \leq p^{n(n-m)}.$$

Since $d(G^{ab}) \geq 2$, we have

$$n - m \geq 2 \Leftrightarrow -n + m + 2 \leq 0 \Leftrightarrow n^2 - nm - n + m + 2 \leq n^2 - nm$$

$$\Leftrightarrow (n-1)(n-m)+2 \leq n(n-m)$$

and the result follows. □

Acknowledgments

We thank the referee for comments and suggestions that have strongly improved the presentation of our results. The second author is also grateful to CAPES for Project 061/2013.

References

[1] Blyth RD, Fumagalli F, Morigi M. Some structural results on the non-abelian tensor square of groups. J Group Theory 2010; 13: 83–94.

[2] Brown R, Johnson DL, Robertson EF. Some computations of nonabelian tensor products of groups. J Algebra 1987; 111: 177–202.

[3] Brown R, Loday JL. Van Kampen theorems for diagrams of spaces. Topology 1987; 26: 311–335.

[4] Brown R, Higgins PJ, Sivera R. Nonabelian Algebraic Topology. Zürich, Switzerland: EMS Publishing House, 2011.

[5] Ellis G. On the tensor square of a prime power group. Arch Math (Basel) 1996: 66: 467469.

[6] Ellis G. On the relation between upper central quotients and lower central series of a group. Trans Amer Math Soc 2001; 353: 4219–4234.

[7] Ellis G, McDermott A. Tensor products of prime power groups. J Pure Appl Algebra 1998; 132: 119–128.

[8] Hatcher A. Algebraic Topology. Cambridge, UK: Cambridge University Press, 2002.

[9] Jafari SH, Moghaddam MRR, Niroomand P. Some properties of the tensor centre of groups. J Korean Math Soc 2009; 46: 249–256.

[10] Jafari SH. A bound on the order of nonabelian tensor square of a prime power group. Comm Algebra 2012; 40: 528–530.

[11] Karpilovsky G. The Schur Multiplier. Oxford, UK: Clarendon Press, 1987.

[12] Moghaddam MRR, Niroomand P. Some properties of certain subgroups of tensor squares of p-groups. Comm Algebra 2012; 40: 1188–1193.

[13] Niroomand P, Russo FG. A note on the exterior centralizer. Arch Math (Basel) 2009; 93: 505–512.

[14] Niroomand P, Russo FG. An improvement of a bound of Green. J Algebra Appl 2012; 11: 1250116.

[15] Rocco NR. On a construction related to the nonabelian tensor square of a group. Bol Soc Brasil Mat 1991; 22: 63–79.

[16] Rocco NR. A presentation for crossed embedding of finite solvable groups. Comm Algebra 1994; 22: 1975–1998.

[17] Rotman J. An Introduction to Algebraic Topology. Berlin, Germany: Springer, 1988.