

A note on closed G_2 -structures and 3-manifolds

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Abstract: This article shows that given any orientable 3-manifold X , the 7-manifold $T^*X \times \mathbb{R}$ admits a closed G_2 -structure $\varphi = \operatorname{Re} \Omega - \omega \wedge dt$ where Ω is a certain complex-valued 3-form on T^*X ; next, given any 2-dimensional submanifold S of X , the conormal bundle N^*S of S is a 3-dimensional submanifold of $T^*X \times \mathbb{R}$ such that $\varphi|_{N^*S} \equiv 0$. A corollary of the proof of this result is that $N^*S \times \mathbb{R}$ is a 4-dimensional submanifold of $T^*X \times \mathbb{R}$ such that $\varphi|_{N^*S \times \mathbb{R}} \equiv 0$.

1. Introduction

Berger's classification of the possible holonomy groups for a given Riemannian manifold includes the exceptional Lie group G_2 as the holonomy group of a 7-dimensional manifold. On a given 7-dimensional manifold with holonomy group a subgroup of G_2 , there is a nondegenerate differential 3-form φ that is torsion-free; that is, $\nabla\varphi = 0$. This torsion-free condition is equivalent to φ being closed and coclosed. Much work has been done to study manifolds with G_2 -holonomy, e.g., [5, 16], but the condition φ being coclosed is a very rigid condition. If we drop the coclosed condition, then we are studying manifolds with a closed G_2 -structure. In particular, manifolds with closed G_2 -structures were studied in [6, 8]; these papers focused predominantly on the metric defined by the nondegenerate closed 3-form φ . We shift our focus to the form φ itself, and in particular to the results that depend on φ being nondegenerate and closed.

Links between Calabi–Yau geometry and G_2 geometry were explored in the context of mirror symmetry by Akbulut and Salur [2]. Of course, the connections between symplectic geometry and Calabi–Yau geometry are obvious; moreover, connections between symplectic and contact geometry have been explored for centuries. Thus, it seems completely natural to try to find connections between contact geometry and G_2 geometry. The study of these relationships is an ongoing project that begins with the work by Arikan et al. [3], and the purpose of the current article (and many upcoming articles) is to continue this study by examining the geometry of closed G_2 -structures as an analogue of symplectic geometry.

Treating symplectic geometry and G_2 geometry as being analogues of one another is not new. In [4, 14], vector cross products (on manifolds) are studied in a general setting; in particular, symplectic geometry is the geometry of 1-fold vector cross products, better known as almost complex structures, and G_2 geometry is the geometry of 2-fold vector cross products in dimension 7. Furthermore, in all cases, one can show that using the metric, there is a nondegenerate differential form of degree $k + 1$ associated to a k -fold vector cross product.

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This yields the symplectic form associated to almost complex structures and the G_2 3-form φ associated to 2-fold vector cross products in dimension 7. Examples of manifolds with G_2 structures satisfying various conditions (including closed G_2 structures) are studied and classified in [7, 10, 11, 12, 13].

This article is based on 2 elementary results from symplectic geometry: 1) the cotangent bundle T^*X of any n -dimensional manifold X admits a symplectic form, and 2) the conormal bundle N^*S of any k -dimensional submanifold S of X with $k < n$ is a Lagrangian submanifold of T^*X . It is a well-known result that any 7-manifold that is spin admits a G_2 -structure [6]. We show that for an orientable 3-manifold X , $T^*X \times \mathbb{R}$ is spin and hence admits a G_2 -structure. Next, we calculate an explicit formula in coordinates for a G_2 -structure φ on $T^*X \times \mathbb{R}$. Using this, we prove:

Theorem 1 *Let S be a 2-dimensional submanifold of X , and let N^*S denote the conormal bundle of S in T^*X . Let $i : N^*S \hookrightarrow (T^*X \times \mathbb{R}, \varphi)$ be the inclusion map. Then $i^*\varphi = 0$.*

2. G_2 geometry

We begin with a description of G_2 geometry in flat space, and then we consider this geometry on 7-manifolds.

Consider the octonions \mathbb{O} as an 8-dimensional real vector space. This becomes a normed algebra when equipped with the standard Euclidean inner product on \mathbb{R}^8 . Furthermore, there is a cross product operation given by $u \times v = \text{Im}(\bar{v}u)$ where \bar{v} is the conjugate of v for $u, v \in \mathbb{O}$. This is an alternating form on $\text{Im } \mathbb{O}$ since, for any $u \in \text{Im } \mathbb{O}$, $u^2 \in \text{Re } \mathbb{O}$. We now define a 3-form on $\text{Im } \mathbb{O}$ by $\varphi(u, v, w) = \langle u \times v, w \rangle$. In terms of the standard orthonormal basis $\{e_1, \dots, e_7\}$ of $\text{Im } \mathbb{O}$, $\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$, where $e^{ijk} = e^i \wedge e^j \wedge e^k$. Under the isomorphism $\mathbb{R}^7 \simeq \text{Im } \mathbb{O}$, with coordinates on \mathbb{R}^7 given by (x^1, \dots, x^7) , we have $\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356}$.

Definition 2.1 *Let M be a 7-dimensional manifold. M has a G_2 -structure if there is a smooth 3-form $\varphi \in \Omega^3(M)$ such that at each $x \in M$, the pair $(T_x(M), \varphi(x))$ is isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$. A 7-manifold M has a closed G_2 -structure if the 3-form φ is also closed, $d\varphi = 0$.*

Equivalently, a smooth 7-dimensional manifold M has a G_2 -structure if its tangent frame bundle reduces to a G_2 -bundle. For a manifold with G_2 -structure φ , there is a natural Riemannian metric and orientation induced by φ given by $(Y \lrcorner \varphi) \wedge (\tilde{Y} \lrcorner \varphi) \wedge \varphi = \langle Y, \tilde{Y} \rangle_\varphi \text{dvol}_M$. In particular, the 3-form φ is nondegenerate.

Remark 1 *One defines a G_2 -manifold as a smooth 7-manifold with torsion-free G_2 -structure, i.e. $\nabla\varphi = 0$ where ∇ is the Levi-Civita connection of the metric $\langle \cdot, \cdot \rangle_\varphi$. This means that (M, φ) has a holonomy group contained in G_2 and that $d\varphi = d*\varphi = 0$. For our purposes, we do not assume that $d*\varphi = 0$, so that (M, φ) will be a manifold with closed G_2 -structure as defined above.*

3. The cotangent bundle

This material can be found in most introductions to symplectic geometry or topology, e.g., [9, 18].

Let X be any n -dimensional manifold (all manifolds under consideration are assumed to be C^∞); let $(U; x_1, \dots, x_n)$ be a coordinate chart for X , so that for each $1 \leq i \leq n$, $x_i : U \rightarrow \mathbb{R}$. For any point $x \in X$, the differentials $(dx_1)_x, \dots, (dx_n)_x$ form a basis for the cotangent space T_x^*X at x ; hence, for any covector

$\xi \in T_x^*X$, $\xi = \sum_i \xi_i(dx_i)_x$ with $\xi_i \in \mathbb{R}$. This yields a coordinate chart $(T^*U; x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ for the cotangent bundle of X associated to the coordinates x_1, \dots, x_n on X .

Using these coordinates, the so-called tautological 1-form on T^*U is defined by $\alpha := \sum_i \xi_i dx_i$. This definition is invariant under changes of coordinates: let $(T^*V; y_1, \dots, y_n, \eta_1, \dots, \eta_n)$ be an overlapping coordinate chart; then:

$$\eta_j = \sum_i \xi_i \frac{\partial x_i}{\partial y_j}$$

and

$$dx_i = \sum_j \frac{\partial x_i}{\partial y_j} dy_j.$$

Therefore, in the overlap,

$$\sum_i \xi_i dx_i = \sum_i \xi_i \left(\sum_j \frac{\partial x_i}{\partial y_j} dy_j \right) = \sum_{i,j} \xi_i \frac{\partial x_i}{\partial y_j} dy_j = \sum_j \eta_j dy_j.$$

Now define a 2-form by $\omega := -d\alpha = \sum_i dx_i \wedge d\xi_i$; ω is also independent of the choice of coordinates, so ω is a symplectic form on the cotangent bundle T^*X , called the canonical symplectic form.

Now let S be any k -dimensional submanifold of X with $k < n$. Recall that the conormal space N_x^*S at $x \in S$ is given by

$$N_x^*S := \{ \xi \in T_x^*X : \xi(v) = 0 \text{ for all } v \in T_x S \},$$

and the conormal bundle N^*S of S by

$$N^*S = \{ (x, \xi) \in T^*X : x \in S, \xi \in N_x^*S \}.$$

Let $(U; x_1, \dots, x_n)$ be a coordinate chart on X such that $S \cap U$ is given by the equations

$$x_{k+1} = \dots = x_n = 0.$$

In the associated cotangent bundle coordinate chart $(T^*U; x_1, \dots, x_n, \xi_1, \dots, \xi_n)$, any $\xi \in N^*S \cap T^*U$ is given by

$$\xi = \sum_{i=1}^k \xi_i dx_i$$

since $x_{k+1} = \dots = x_n = 0$ on S ; furthermore, since $\xi \in N_x^*S$ and $T_x S$ is spanned by

$$\left(\frac{\partial}{\partial x_1} \right)_x, \dots, \left(\frac{\partial}{\partial x_k} \right)_x,$$

we find that

$$0 = \xi \left(\left(\frac{\partial}{\partial x_i} \right)_x \right) = \xi_i, \text{ for all } 1 \leq i \leq k.$$

Hence, $N^*S \cap T^*U$ is described by the equations $x_{k+1} = \dots = x_n = \xi_1 = \dots = \xi_k = 0$, so N^*S is an n -dimensional submanifold of T^*X ; furthermore, $\alpha = \sum_{i=1}^n \xi_i dx_i$ when restricted to N^*S is zero, so $\omega|_{N^*S} \equiv 0$. Thus, N^*S is a Lagrangian submanifold of T^*X .

4. G_2 -structures on $T^*X \times \mathbb{R}$

References for this section are [1, 6, 17, 19].

Recall that for each $n \geq 3$, the Lie group $SO(n)$ is connected, and it has a double-covering map $\iota : Spin(n) \rightarrow SO(n)$ where the Lie group $Spin(n)$ is a compact, connected, simply connected Lie group. An oriented Riemannian manifold (X, g) has $SO(n)$ as its structure group on the tangent bundle. A spin structure on (X, g) is a $Spin(n)$ -principal bundle over X , together with a bundle map $\pi : P_{Spin(n)}X \rightarrow P_{SO(n)}X$ such that $\pi(pg) = \pi(p)\iota(g)$ for $p \in P_{Spin(n)}X, g \in Spin(n)$. A spin manifold is an oriented Riemannian manifold with a spin structure on its tangent bundle.

Theorem 4.1 *For any oriented Riemannian 3-manifold X , $T^*X \times \mathbb{R}$ has a G_2 -structure.*

Proof Let X be an oriented Riemannian 3-manifold. We know that every orientable 3-manifold is parallelizable, and since a framing on a bundle gives a spin structure, X is a spin manifold.

Now we check that T^*X is itself a spin manifold for a spin manifold X . Recall that T^*X carries a canonical 1-form $\sum p_i dx^i$, where p_i are coordinates in T^*X and x^i are coordinates on X . One can define the map $T^*X \times Spin(3) \rightarrow T^*X$ by $(p_i, g) \mapsto p_i g|_{x^i} = p_i(v_i g)$ for $v_i \in T_{x^i}X$; it is well defined because of the spin structure on X . This implies that T^*X has an induced spin structure from X . For another way to see this, one can find a bundle map of the principal bundles $b : P_{Spin}T^*X \rightarrow P_{Spin}TX$ over X because the map $\mathfrak{F} : \mathcal{E}(T^*X) \rightarrow \mathcal{E}(TX)$ is linear, where \mathcal{E} is the set of sections of the bundle. This gives a map $P_{Spin(3)}X \rightarrow P_{SO(3)}X$ for which the following diagram commutes:

$$\begin{array}{ccc} P_{Spin(3)}T^*X & \rightarrow & P_{SO(3)}T^*X \\ \downarrow & & \downarrow \\ P_{Spin(3)}TX & \rightarrow & P_{SO(3)}TX \end{array} .$$

Let $E(\xi)$ be the total space of ξ with base space X . A principal $Spin(3)$ -bundle $E(\xi) \times Spin(3) \rightarrow X$ induces a principal $Spin(4)$ -bundle $E(\xi) \times Spin(4) \rightarrow X$, which is itself induced from $Spin(3) (\simeq SU(2) \simeq S^3) \rightarrow Spin(4) (\simeq SU(2) \times SU(2))$; hence, a spin structure on ξ gives a spin structure on $\xi \oplus \varepsilon$, so $T^*X \times \mathbb{R}$ admits a spin structure. By [1, p. 4], we conclude that $T^*X \times \mathbb{R}$ is a smooth 7-dimensional manifold with a G_2 -structure. □

Let X be an orientable 3-dimensional manifold with symplectic cotangent bundle $(M = T^*X, \omega := -d\alpha)$ where α is the tautological 1-form on M . If $x_1, x_2, x_3, \xi_1, \xi_2, \xi_3$ are the standard cotangent bundle coordinates associated to the coordinates x_1, x_2, x_3 on X , define a complex-valued $(3, 0)$ -form on M by

$$\Omega = (dx_1 + id\xi_1) \wedge (dx_2 + id\xi_2) \wedge (dx_3 + id\xi_3).$$

Consider $M \times \mathbb{R}$. This is a 7-manifold with coordinates $x_1, x_2, x_3, \xi_1, \xi_2, \xi_3, t$ where t is the \mathbb{R} coordinate. Finally, define $\varphi = \text{Re}(\Omega) + \omega \wedge dt$.

We show that this defines a G_2 structure on $M \times \mathbb{R}$; that is, we exhibit an isomorphism of $(T_{(p,t)}(M \times \mathbb{R}), \varphi_{(p,t)})$ with $(\mathbb{R}^7, \varphi_0)$. We first calculate

$$\begin{aligned} \Omega &= (dx_1 + id\xi_1) \wedge ((dx_2 \wedge dx_3 - d\xi_2 \wedge d\xi_3) + i(dx_2 \wedge d\xi_3 - dx_3 \wedge d\xi_2)) \\ &= (dx_1 \wedge dx_2 \wedge dx_3 - dx_1 \wedge d\xi_2 \wedge d\xi_3 + dx_2 \wedge d\xi_1 \wedge d\xi_3 - dx_3 \wedge d\xi_1 \wedge d\xi_2) \end{aligned}$$

$$+i(dx_1 \wedge dx_2 \wedge d\xi_3 - dx_1 \wedge dx_3 \wedge d\xi_2 + dx_2 \wedge dx_3 \wedge d\xi_1 - d\xi_1 \wedge d\xi_2 \wedge d\xi_3),$$

so we find that

$$\begin{aligned} \varphi &= \operatorname{Re} \Omega + \omega \wedge dt \\ &= dx_1 \wedge dx_2 \wedge dx_3 - dx_1 \wedge d\xi_2 \wedge d\xi_3 + dx_2 \wedge d\xi_1 \wedge d\xi_3 - dx_3 \wedge d\xi_1 \wedge d\xi_2 \\ &\quad + dx_1 \wedge d\xi_1 \wedge dt + dx_2 \wedge d\xi_2 \wedge dt + dx_3 \wedge d\xi_3 \wedge dt. \end{aligned}$$

Now, for $p = (x, \xi) \in T^*U$, let

$$\left\{ \left(\frac{\partial}{\partial x_1} \right)_p, \left(\frac{\partial}{\partial x_2} \right)_p, \left(\frac{\partial}{\partial x_3} \right)_p, \left(\frac{\partial}{\partial \xi_1} \right)_p, \left(\frac{\partial}{\partial \xi_2} \right)_p, \left(\frac{\partial}{\partial \xi_3} \right)_p \right\}$$

be the basis for the tangent space T_pM at p with respect to the cotangent coordinates on M ; let $\left(\frac{\partial}{\partial t} \right)_q$ be the basis for $T_q\mathbb{R}$ with respect to the coordinate t on \mathbb{R} . Let (x_1, \dots, x_7) be the standard Euclidean coordinates on \mathbb{R}^7 . Define an isomorphism of the tangent vector spaces by $\Phi : T_0\mathbb{R}^7 \rightarrow T_{(p,q)}(M \times \mathbb{R})$ by

$$\begin{aligned} \Phi\left(\frac{\partial}{\partial x_1}\right)_0 &= -\left(\frac{\partial}{\partial x_3}\right)_p, \quad \Phi\left(\frac{\partial}{\partial x_2}\right)_0 = \left(\frac{\partial}{\partial x_2}\right)_p, \quad \Phi\left(\frac{\partial}{\partial x_3}\right)_0 = \left(\frac{\partial}{\partial x_1}\right)_p \\ \Phi\left(\frac{\partial}{\partial x_4}\right)_0 &= \left(\frac{\partial}{\partial \xi_1}\right)_p, \quad \Phi\left(\frac{\partial}{\partial x_5}\right)_0 = \left(\frac{\partial}{\partial \xi_2}\right)_p, \quad \Phi\left(\frac{\partial}{\partial x_6}\right)_0 = \left(\frac{\partial}{\partial \xi_3}\right)_p \\ \Phi\left(\frac{\partial}{\partial x_7}\right)_0 &= -\left(\frac{\partial}{\partial t}\right)_q. \end{aligned}$$

This induces an isomorphism of the cotangent vector spaces $\Phi^* : T_{(p,q)}^*(M \times \mathbb{R}) \rightarrow T_0^*\mathbb{R}^7$ where

$$\begin{aligned} \Phi^*(dx_1)_p &= (dx_3)_0, \quad \Phi^*(dx_2)_p = (dx_2)_0, \quad \Phi^*(dx_3)_p = -(dx_1)_0 \\ \Phi^*(d\xi_1)_p &= (dx_4)_0, \quad \Phi^*(d\xi_2)_p = (dx_5)_0, \quad \Phi^*(d\xi_3)_p = (dx_6)_0 \\ \Phi^*(dt)_t &= -(dx_7)_0. \end{aligned}$$

Then

$$\begin{aligned} \Phi^*\varphi &= \Phi^*dx_1 \wedge \Phi^*dx_2 \wedge \Phi^*dx_3 - \Phi^*dx_1 \wedge \Phi^*d\xi_2 \wedge \Phi^*d\xi_3 + \Phi^*dx_2 \wedge \Phi^*d\xi_1 \wedge \Phi^*d\xi_3 - \Phi^*dx_3 \wedge \Phi^*d\xi_1 \wedge \Phi^*d\xi_2 \\ &\quad + \Phi^*dx_1 \wedge \Phi^*d\xi_1 \wedge \Phi^*dt + \Phi^*dx_2 \wedge \Phi^*d\xi_2 \wedge \Phi^*dt + \Phi^*dx_3 \wedge \Phi^*d\xi_3 \wedge \Phi^*dt \\ &= dx_3 \wedge dx_2 \wedge (-dx_1) - dx_3 \wedge dx_5 \wedge dx_6 + dx_2 \wedge dx_4 \wedge dx_6 - (-dx_1) \wedge dx_4 \wedge dx_5 \\ &\quad + dx_3 \wedge dx_4 \wedge -(dx_7) + dx_2 \wedge dx_5 \wedge (-dx_7) + (-dx_1) \wedge dx_6 \wedge (-dx_7) \\ &= dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dx_4 \wedge dx_5 + dx_1 \wedge dx_6 \wedge dx_7 + dx_2 \wedge dx_4 \wedge dx_6 \\ &\quad - dx_2 \wedge dx_5 \wedge dx_7 - dx_3 \wedge dx_4 \wedge dx_7 - dx_3 \wedge dx_5 \wedge dx_6 = \varphi_0. \end{aligned}$$

In order to show that φ is independent of the choice of coordinates, it is enough prove that our coordinate definition of Ω on T^*X is independent of the choice of coordinates on T^*X . This calculation follows as in [15]. Let L be an oriented 3-dimensional subspace of T^*X . Let $\{f_1, f_2, f_3\}$ be any oriented linearly independent subset of L , let $\{e_1, e_2, e_3, Je_1, Je_2, Je_3\}$ denote the standard basis for \mathbb{R}^6 , and let A be the map defined by

$e_j \mapsto f_j$ and $Je_j \mapsto Jf_j$. Now $A \in GL(3; \mathbb{C})$; that is, A is complex linear and $A(e_1 \wedge e_2 \wedge e_3) = f_1 \wedge f_2 \wedge f_3$. Let $\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3\}$ be an oriented orthonormal basis for L , and let B be the map defined by $\tilde{f}_j \mapsto f_j$. Then $f_1 \wedge f_2 \wedge f_3 = (\det B)\tilde{f}_1 \wedge \tilde{f}_2 \wedge \tilde{f}_3$. Since $\Omega(A(e_1 \wedge e_2 \wedge e_3)) = \det_{\mathbb{C}} A$ and $\Omega = \operatorname{Re} \Omega + i \operatorname{Im} \Omega$, we have

$$(\det B)[\operatorname{Re} \Omega(\tilde{f}_1 \wedge \tilde{f}_2 \wedge \tilde{f}_3)] = \operatorname{Re}(\det_{\mathbb{C}} A)$$

and

$$(\det B)[\operatorname{Im} \Omega(\tilde{f}_1 \wedge \tilde{f}_2 \wedge \tilde{f}_3)] = \operatorname{Im}(\det_{\mathbb{C}} A).$$

Hence,

$$\begin{aligned} & (\operatorname{Re} \Omega(f_1 \wedge f_2 \wedge f_3))^2 + (\operatorname{Im} \Omega(f_1 \wedge f_2 \wedge f_3))^2 \\ &= |\det_{\mathbb{C}} A|^2 = \det_{\mathbb{R}} A = \operatorname{vol}(A(e_1 \wedge Je_1 \wedge e_2 \wedge Je_2 \wedge e_3 \wedge Je_3)) \\ &= \operatorname{vol}(A(e_1 \wedge e_2 \wedge e_3 \wedge Je_1 \wedge Je_2 \wedge Je_3)) = \operatorname{vol}(f_1 \wedge f_2 \wedge f_3 \wedge Jf_1 \wedge Jf_2 \wedge Jf_3) \\ &= (\det B)^2 \operatorname{vol}(\tilde{f}_1 \wedge \tilde{f}_2 \wedge \tilde{f}_3 \wedge J\tilde{f}_1 \wedge J\tilde{f}_2 \wedge J\tilde{f}_3). \end{aligned}$$

5. The conormal bundle

Let X be a 3-dimensional manifold as in the previous section. Then $T^*X \times \mathbb{R}$ admits the G_2 -structure $\varphi = \operatorname{Re} \Omega + \omega \wedge dt$. Let S be a 2-dimensional submanifold of X , and let (U, x_1, x_2, x_3) be a coordinate chart on X such that $S \cap U$ is given by the equation $x_3 = 0$. For the associated cotangent coordinate chart $(T^*U, x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)$, any $\xi \in N^*S \cap T^*U$ is given by

$$\xi = \xi_1 dx_1 + \xi_2 dx_2$$

since $x_3 = 0$ on S ; furthermore, since $\xi \in N_x^*S$ and $T_x S$ is spanned by

$$\left(\frac{\partial}{\partial x_1}\right)_x, \left(\frac{\partial}{\partial x_2}\right)_x,$$

we find that

$$0 = \xi\left(\left(\frac{\partial}{\partial x_1}\right)_x\right) = \xi_1$$

and

$$0 = \xi\left(\left(\frac{\partial}{\partial x_2}\right)_x\right) = \xi_2.$$

Hence, $N^*S \cap T^*U$ is described by the equations $x_3 = \xi_1 = \xi_2 = 0$. Thus, N^*S is a 3-dimensional submanifold of T^*X .

Proposition 5.1 *Let $i : N^*S \hookrightarrow (T^*X \times \mathbb{R}, \varphi)$ be the inclusion. Then $i^*\varphi = 0$.*

Proof In this case, $N^*S \cap (T^*U \times \mathbb{R})$ is given by the equations $x_3 = \xi_1 = \xi_2 = t = 0$ where t is the \mathbb{R} coordinate. Recall that in this coordinate system, we have

$$\begin{aligned} \varphi &= dx_1 \wedge dx_2 \wedge dx_3 - dx_1 \wedge d\xi_2 \wedge d\xi_3 + dx_2 \wedge d\xi_1 \wedge d\xi_3 - dx_3 \wedge d\xi_1 \wedge d\xi_2 \\ &\quad + dx_1 \wedge d\xi_1 \wedge dt + dx_2 \wedge d\xi_2 \wedge dt + dx_3 \wedge d\xi_3 \wedge dt. \end{aligned}$$

Hence, for $p \in N^*S$, we have $(i^*\varphi)_p = \varphi_p|_{T_p(N^*S)} = 0$ since every term contains a factor that on N^*S is zero. \square

Corollary 5.2 *Let $i : N^*S \times \mathbb{R} \hookrightarrow (T^*X \times \mathbb{R}, \varphi)$ be the inclusion. Then $i^*\varphi = 0$.*

Proof Note that $(N^*S \times \mathbb{R})$ is a 4-dimensional submanifold of T^*X , given by the equations $x_3 = \xi_1 = \xi_2 = 0$ on $(N^*S \times \mathbb{R}) \cap (T^*U \times \mathbb{R})$. \square

Note that in contrast to the symplectic case, S must be 2-dimensional; if S is 1-dimensional, then $\varphi|_{N^*S} \neq 0$.

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