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Almost contact metric submersions and symplectic manifolds

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Abstract: In this paper, we discuss some geometric properties of almost contact metric submersions involving symplectic manifolds. We show that this is obtained if the total space is an *b-almost Kenmotsu manifold*.

Key words: Riemannian submersions, almost Hermitian manifolds, almost contact metric manifolds, almost contact metric submersions, symplectic manifolds

1. Introduction

The theory of almost contact submersions intertwines contact geometry with the almost Hermitian one. For instance, the fibers of an almost contact metric submersion of type I, in the sense of Watson [12], are almost Hermitian manifolds. However, certain classes of almost Hermitian manifolds are closely related to symplectic manifolds. Specifically, almost Kähler manifolds are endowed with symplectic structure while quasi-Kählerian manifolds are related to $(1,2)$-symplectic ones. Almost contact metric and almost symplectic manifolds were developed in [1], but in [4], the concept of k-symplectic manifolds was extensively studied.

In this paper, we study almost contact metric submersions involving symplectic structures. It is organized in the following way. In Section 2, devoted to the preliminaries on manifolds, we review the main classes of almost Hermitian manifolds that have some relation with almost symplectic structures; almost contact metric manifolds that can be used as total space of fibration are also reviewed.

Section 3 deals with almost contact metric submersions. Here, after recalling some fundamental properties, we determine the structure of the fibers according to that of the total space. It is shown that some manifolds have a common property that forces the fibers of an almost contact metric submersion of type I to lie in a fixed class of almost Hermitian manifolds. We show that quasi-K-cosymplectic and quasi-Kenmotsu manifolds, which have a common relation, are related to (1,2)-symplectic manifolds.

The Chinea structure equations of an almost contact metric submersion are used to establish some relationships between Lee 1-forms of the total, the base space, or the fiber submanifolds.

We shall end the paper with the following problem. *Suppose that the base space of an almost contact metric submersion of type II admits a symplectic structure. What is the corresponding structure on the total space?*

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2. Preliminaries on manifolds

2.1. Almost Hermitian manifolds

An almost Hermitian manifold is a Riemannian manifold, (M, g), furnished with a tensor field, J, of type $(1,1)$ satisfying the following 2 conditions:

(i) $J^2D = -D$, and

(ii) $g(JD, JE) = g(D, E)$, for all $D, E \in \mathfrak{X}(M)$.

It is known that any almost Hermitian manifold, (M, g, J), is of even dimensions, say $2m$, and possesses a fundamental 2-form, Ω, defined by $\Omega(D, E) = g(D, JE)$.

Noting by ∇ the Levi-Civita connection of M, we recall some remarkable identities:

$(\nabla_D J)E = \nabla_D JE - J\nabla_D E$;

$(\nabla_D \Omega)(E, G) = g(G, (\nabla_D J)E) = -g((\nabla_D J)G, E)$.

Let $\{E_1, ..., E_m, JE_1, ..., JE_m\}$ be a local J-basis of an open subset of M, and then the codifferential δ of Ω is defined by

$$\delta\Omega(D) = -\sum_{i=1}^{m} \{(\nabla_{E_i}\Omega)(E_i, D) + (\nabla_{JE_i}\Omega)(JE_i, D)\}.$$

The Lee form θ, of M is a 1-form defined by

$$\theta(D) = \frac{1}{m-1}\delta\Omega(JD).$$

From the classification of almost Hermitian structures, obtained by Gray and Hervella [5], we shall be interested in the following classes of manifolds:

(a) *Kählerian* if $d\Omega(D, E, G) = 0$ and $N_J = 0$, where N_J denotes the Nijenhuis tensor of J;

(b) *almost Kählerian* if $d\Omega(D, E, G) = 0$;

(c) *locally conformal almost Kähler* if $d\Omega = \Omega\Lambda\theta$ or $\mathcal{G}\left\{(\nabla_D\Omega)(E, G) - \frac{1}{m-1}\Omega(D, E)\delta\Omega(JG)\right\} = 0$,

where $\mathcal{G}$ denotes the cyclic sum over D, E, and G;

(d) *locally conformal Kähler* if $(\nabla_D\Omega)(E, G) = \frac{-1}{2(m-1)}\{g(D, E)\delta\Omega(G) - g(D, G)\delta\Omega(JG)\}$.

2.2. Almost symplectic manifolds

By an almost symplectic manifold, one understands a differentiable manifold M^{2m} equipped with a nondegenerate 2-form Ω.

Suppose that every point $x \in M$ possesses an open neighborhood $\mathcal{U}$, and that $\sigma : \mathcal{U} \longrightarrow \mathbb{R}$ is a function on $\mathcal{U}$.

From Vaisman [11], we have the following classes:

(i) If $d(e^{-\sigma}\Omega/\mathcal{U}) = 0$, then (M^{2m}, Ω) is said to be *locally conformal symplectic;*

(ii) If $\mathcal{U} = M$ and $d(e^{-\sigma}\Omega) = 0$, then (M^{2m}, Ω) is called *globally conformal symplectic;*

(iii) If in (i), σ is a constant, then (M^{2m}, Ω) is called *symplectic*.

In other words, a symplectic manifold is an almost symplectic manifold (M^{2m}, Ω) for which Ω is closed and $\Omega^m \neq 0$.

Locally and globally conformal symplectic manifolds were characterized by Lee [6] in:

Theorem 2.1 ([6]) *Let (M^{2m}, Ω) be an almost symplectic manifold. Then it is:*

(a) locally conformal symplectic if and only if there exists a differential 1-form θ such that $d\Omega = \theta \wedge \Omega$, and $d\theta = 0$;

(b) globally conformal symplectic if and only if moreover θ is also exact.

2.3. Almost contact metric manifolds

Let M be a differentiable manifold of odd dimensions $2m+1$. An almost contact structure on M is a triple (φ, ξ, η) where:

(1) ξ is a characteristic vector field,

(2) η is a 1-form such that $\eta(\xi) = 1$, and

(3) φ is a tensor field of type $(1,1)$ satisfying

$$\varphi^2 = -I + \eta \otimes \xi,$$

where I is the identity transformtion.

If M is equipped with a Riemannian metric g such that

$$g(\varphi D, \varphi E) = g(D, E) - \eta(D)\eta(E),$$

then (g, φ, ξ, η) is called an almost contact metric structure. Thus, the quintuple $(M^{2m+1}, g, \varphi, \xi, \eta)$ is an almost contact metric manifold. As in the case of almost Hermitian manifolds, any almost contact metric manifold admits a fundamental 2-form, ϕ, defined by

$$\phi(D, E) = g(D, \varphi E).$$

Let us recall some of the important identities.

$$(\nabla_D \eta) E = g(E, \nabla_D \xi) = (\nabla_D \phi)(\xi, \varphi E); \tag{1}$$

$$2d\eta(D, E) = D\eta(E) - E\eta(D) - \eta([D, E]). \tag{2}$$

Let $\{E_1, \ldots, E_m, \varphi E_1, \ldots, \varphi E_m, \xi\}$ be a local φ-basis of an open subset of M; then the coderivative, δ, is given by:

$$\delta\phi(D) = -\sum_{i=1}^{m} \{(\nabla_{E_i}\phi)(E_i, D) + (\nabla_{\varphi E_i}\phi)(\varphi E_i, D)\} - (\nabla_\xi \phi)(\xi, D);$$

$$\delta\eta = -\sum_{i=1}^{m} \{(\nabla_{E_i}\eta)E_i + (\nabla_{\varphi E_i}\eta)\varphi E_i\}\,.$$

The analogue of the Lee form is the 1-form, ω, defined in [9] by setting

$$\omega(D) = \frac{1}{m}(\delta\phi(\varphi D) - \eta(D)\delta\eta).$$

In [10], Sasaki and Hatakeyama defined tensor fields $N^{(1)}$ of type $(0,2)$ by setting

$$N^{(1)}(D,E) = N_{\varphi}(D,E) + 2d\eta(D,E)\xi,$$

where N_{φ} is the Nijenhuis tensor of φ.

Let us recall the defining relations of almost contact metric manifolds to be used in this paper.

An almost contact metric manifold is said to be:

(1) *almost cosymplectic* if $d\phi = 0$ and $d\eta = 0$;

(2) *almost Kenmotsu* if $d\phi(D,E,G) = \frac{2}{3}\mathcal{G}\{\eta(D)\phi(E,G)\}$;

(3) *locally conformal almost cosymplectic* if $d\phi = -2\phi\wedge\omega$ and $d\eta = \eta\wedge\omega$;

(4) *locally conformal cosymplectic* if $d\phi = -2\phi\wedge\omega$, $d\eta = \eta\wedge\omega$ and $N^{(1)} = 0$.

(5) *trans-Sasakian* if $(\nabla_D\phi)(E,G) = \frac{-1}{2m}\{g(D,E)\eta(G) - g(D,G)\eta(E)\delta\phi(E)\} + \frac{-1}{2m}\{(g(D,\varphi E)\eta(G) - g(D,\varphi G)\eta(E))\delta\eta\}$;

(6) *almost trans-Sasakian* if $d\phi = \phi\wedge\omega$ and $d\eta = \frac{1}{2m}\{\delta\phi(\xi)\phi - 2\eta\wedge\varphi^*(\delta\phi)\}$;

(7) *C_4-manifold* if $(\nabla_D\phi)(E,G) = \frac{-1}{2(m-1)}\{g(\varphi D,\varphi E)\delta\phi(G) - g(\varphi D,\varphi G)\delta\phi(E)\} + \frac{-1}{2(m-1)}\{\phi(D,G)\delta\phi(\varphi E) - \phi(D,E)\delta\phi(\varphi G)\}$ and $\delta\phi(\xi) = 0$.

3. Almost contact metric submersions

In [8], O'Neill defined a Riemannian submersion as a surjective mapping $\pi : M \longrightarrow B$ between 2 Riemannian manifolds such that:

(i) π is of maximal rank;

(ii) $\pi_*|(ker\pi_*)^{\perp}$ is a linear isometry.

The tangent bundle $T(M)$ of the total space M admits an orthogonal decomposition

$$T(M) = V(M) \oplus H(M),$$

where $V(M)$ and $H(M)$ denote respectively the vertical and horizontal distributions. We denote by $\mathcal{V}$ and $\mathcal{H}$ the vertical and horizontal projections, respectively. A vector field X of the horizontal distribution is called a basic vector field if it is π-related to a vector field X_* of the base space B. That is, $\pi_*X = X_*$.

On the base space, tensors and other objects will be denoted with a prime symbol, while those tangent to the fibers will be specified by a caret symbol. Herein, vector fields tangent to the fibers will be denoted by U, V, and W.

Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ and $(M'^{2m'+1}, g', \varphi', \xi', \eta')$ be 2 almost contact metric manifolds. By an almost contact metric submersion of type I, in the sense of Watson [12], one understands a Riemannian submersion

$$\pi : M^{2m+1} \to M'^{2m'+1}$$

satisfying

(i) $\pi_*\varphi = \varphi'\pi_*$,

(ii) $\pi_*\xi = \xi'$.

Referring to Watson [12], we point out that when the base space is an almost Hermitian manifold, $(B^{2m'}, g', J')$, the Riemannian submersion

$$\pi : M^{2m+1} \to B^{2m'}$$

is called an almost contact metric submersion of type II if $\pi_*\varphi = J'\pi_*$.

Now, we give an overview some of the fundamental properties of these submersions.

Proposition 3.1 *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. Then*

(a) $\pi^*\phi' = \phi$;

(b) $\pi^*\eta' = \eta$;

(c) *the horizontal and vertical distributions are φ-invariant;*

(d) $\eta(U) = 0$ *for all* $U \in V(M)$;

(e) $\mathcal{H}(\nabla_X\varphi)Y$ *is the basic vector field associated to* $(\nabla'_{X_*}\varphi')Y_*$ *if* X *and* Y *are basic.*

Proof See Watson [12]. □

Proposition 3.2 *Let $\pi : M^{2m+1} \longrightarrow B^{2m'}$ be an almost contact metric submersion of type II. Then:*

(a) $\pi^*\Omega' = \phi$;

(b) *the horizontal and vertical distributions are φ-invariant;*

(c) $\eta(X) = 0$ *for all* $X \in H(M)$;

(d) $\mathcal{H}(\nabla_X\varphi)Y$ *is the basic vector field associated to* $(\nabla'_{X_*}J')Y_*$ *if* X *and* Y *are basic.*

Proof See again Watson [12]. □

Proposition 3.3 *The fibers of an almost contact metric submersion of type II are almost contact metric manifolds.*

Proof Since the total space is of dimension $2m+1$ and the base space has dimension $2m'$, the fibers have dimension $2(m-m')+1$. This shows that the dimension of the fibers is odd.

Let $(\hat{g}, \hat{\varphi}, \hat{\xi}, \hat{\eta})$ be the restriction of the almost contact metric structure (g, φ, ξ, η) of the total space on the fibers. We have to show that $(\hat{g}, \hat{\varphi}, \hat{\xi}, \hat{\eta})$ is an almost contact metric structure. That is,

(i) $\hat{\varphi}^2 U = -U + \hat{\eta}(U)\hat{\xi}$;

(ii) $\hat{(\eta)} = \hat{g}(\hat{\xi}, \hat{\xi}) = g(\xi, \xi) = 1$;

(iii) $\hat{g}(\hat{\varphi}U, \hat{\varphi}V) = -\hat{g}(U, \hat{\varphi}^2 V) = \hat{g}(U, V) - \hat{g}(U, \hat{\eta}(V)\hat{\xi})$;

but $\hat{g}(U, \hat{\eta}(V)\hat{\xi}) = \hat{g}(U, \hat{\xi})\hat{\eta}(V) = \hat{\eta}(U)\hat{\eta}(V)$; thus, $\hat{g}(\hat{\varphi U}, \hat{\varphi V}) = \hat{g}(U, V) - \hat{\eta}(U)\hat{\eta}(V)$.

□

Proposition 3.4 *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. If the total space is almost cosymplectic, a C_2-manifold, or an almost Kenmotsu manifold, then the fibers are almost Kählerian.*

Proof All these manifolds have in common the following defining relation:

$$d\phi(U, V, W) = \frac{\alpha}{3}\mathcal{G}\left\{\eta(U)\phi(V, W)\right\},$$

which becomes $d\phi(U, V, W) = 0$ because of the vanishing of η on vertical vector fields. Thus, on the fibers, we have $d\hat{\phi}(U, V, W) = 0$, which defines the almost Kähler structure. □

In the case of an almost contact metric submersion of type II, the analogue of this proposition is the following:

Proposition 3.5 *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. If the total space is almost cosymplectic, a C_2-manifold or an almost Kenmotsu manifold, then the base space is almost Kählerian.*

Proof Let X, Y, and Z be 3 basic vector fields. The case of almost cosymplectic and C_2-manifolds is obvious since these manifolds are defined by $d\phi = 0$, from which $d\Omega' = 0$ follows, because $\pi_* d\Omega' = 0$ implies that $d(\pi^*\Omega') = d\phi$ and then $\pi_* d\Omega' = d\phi = 0$, which implies that $d\Omega' = 0$ since π_* is a linear isometry.

Let us consider the case of almost Kenmotsu structure on the total space. Since η vanishes on the horizontal vector fields, we have

$$d\phi(X, Y, Z) = \frac{2}{3}\mathcal{G}\left\{\eta(X)\phi(Y, Z)\right\} = 0,$$

which gives $d\Omega' = 0$. □

In light of Proposition 3.4 and Proposition 3.5, it appears that almost cosymplectic, almost Kenmotsu, and C_2-manifolds have a common property that forces the fibers (resp. the base space) to lie in the same class of almost Kähler manifolds.

The common property is their defining relation, which is

$$d\phi(D,E,G) = \frac{\alpha}{3}\mathcal{G}\left\{\eta(D)\phi(E,G)\right\}$$

where α is a real number. This defining relation is a generalization of the class of almost Kenmotsu and we call it an *almost α-Kenmotsu*.

Taking $\alpha = 0$, we get one of the main defining relations of an almost cosymplectic and a C_2-manifold. If $\alpha = 2$, we have one of the defining relations of an almost Kenmotsu manifold.

According to this common property, Propositions 3.4 and 3.5 can be reformulated in the following way:

Proposition 3.6 *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. If the total space is almost α-Kenmotsu manifold, then the fibers are almost Kählerian.*

Proof Let U, V, and W be 3 vector fields tangent to the fibers. The defining relation of the total space then becomes

$$d\hat{\phi}(U,V,W) = \frac{\alpha}{3}\mathcal{G}\left\{\hat{\eta}(U)\hat{\phi}(V,W)\right\}.$$

With the vanishing of η on vertical vector fields, according to Proposition 3.1(d), we get $d\hat{\phi} = 0$, giving $d\hat{\Omega} = 0$, which is the defining relation of an almost Kähler structure. □

Proposition 3.7 *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. If the total space is almost α-Kenmotsu manifold, then the base space is an almost Kählerian manifold.*

Proof As in the preceding proposition, let X, Y, and Z be 3 basic vector fields. The defining relation of the total space then becomes

$$d\phi(X,Y,Z) = \frac{\alpha}{3}\mathcal{G}\left\{\eta(X)\phi(Y,Z)\right\}.$$

With the vanishing of η on horizontal vector fields, according to Proposition 3.2(c), we get $d\phi = 0$, from which $d\Omega' = 0$ is the defining relation of an almost Kähler structure. □

Since, in the light of Liberman and Marle [7], an almost Kähler is symplectic, we claim that any b-almost Kenmotsu manifold possesses a symplectic submanifold.

Now, let us turn to the manifolds defined by the codifferential of the fundmental 2-form, ϕ, or the 1-form, η. In this case, the Chinea structure equations of a submersion play an important role.

Recall that the O'Neill configuration tensor A, of the total space of a Riemannian submersion, is defined in [8] by setting

$$A_D E = \mathcal{V}\nabla_{\mathcal{H}D}\mathcal{H}E + \mathcal{H}\nabla_{\mathcal{H}D}\mathcal{V}E.$$

Using this tensor, Chinea [3] defined an associated tensor A^* on horizontal vector fields by setting

$$A^*(X,Y) = A_X\varphi Y - A_{\varphi X}Y$$

and established the following structure equations:

$$\delta\phi(U) = \delta\hat{\phi}(U) + \frac{1}{2}g(trA^*, U), \tag{3}$$

$$\delta\phi(X) = \delta\phi'(X_*) + g(H, \varphi X), \tag{4}$$

$$\delta\eta = \delta\eta' \circ \pi - g(H, \xi), \tag{5}$$

where trA^* is the trace of A^* and H is the mean curvature vector field of the fibers.

Concerning the Lee forms ω and θ, we have the following

Proposition 3.8 *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. Then we have:*

(a) $\pi^*\omega' = \omega$ if and only if the fibers are minimal;

(b) $\theta(U) = \hat{\omega}(U)$ if and only if $trA^* = 0$.

Proof (a) Let X be a basic vector field; then we have

$$\begin{aligned}(\pi^*\omega')(X) &= \omega'(\pi_* X)\\ &= \omega'(X_*)\\ &= \frac{1}{m}(\delta\phi'(\varphi' X_*)) - (\delta\eta')(\eta'(X_*)).\end{aligned}$$

According to equation (4), it is clear that

$$\delta\phi(X) = \delta\phi'(X_*)$$

if and only if $H = 0$, which is the required condition for the minimality of the fibers.

Similarly, by equation (5), it follows that $\delta\eta = \delta\eta' \circ \pi$ if and only if $H = 0$, which completes the proof of (a).

(b) The vanishing of η on the vertical distribution leads to

$$\hat{\omega}(U) = \frac{1}{m-1}\delta\hat{\phi}(\hat{\varphi U}).$$

But, by equation (3), we have

$$\delta\hat{\phi}(\hat{\varphi U}) = \delta\phi(\varphi U)$$

if and only if $trA^* = 0$; that is to say, $\hat{\omega}(U) = \omega(U)$ if and only if $trA^* = 0$.

Since the fibers of an almost contact metric submersion of type I are almost Hermitian manifolds, we then have $\hat{\varphi} U = JU$ and $\delta\hat{\phi} = \delta\Omega$ so that $\hat{\omega}(U) = \theta(U)$ as required. □

Proposition 3.9 *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. Then we have:*

(a) $\pi^*\theta' = \omega$ if and only if the fibers are minimal;

(b) $\hat{\omega}(U) = \omega(U)$ if and only if $trA^* = 0$.

Proof The proof of (b) follows from equation (3). By the vanishing of η on the horizontal vector fields, the proof of assertion (a) is obtained as in the preceding Proposition 3.8. $\square$

Proposition 3.10 *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. If the total space is trans-Sasakian, locally conformal cosymplectic, or a C_4-manifold, then the fibers are locally conformal Kählerian if and only if $trA^* = 0$.*

Proof Let U, V, and W be 3 vector fields tangent to the fibers. The manifolds under consideration have in common the following defining relation:

$$d\phi(D, E, G) = \frac{\alpha}{3m} \mathcal{G} \left\{ \phi(D, E)C \right\},$$

where the factor C is defined by the codifferential of the fundamental 2-form.

In the case of vertical vector fields, the defining relation becomes

$$d\hat{\phi}(U, V, W) = \frac{\alpha}{3(m-1)} \mathcal{G} \left\{ \hat{\phi}(D, E)\hat{C} \right\},$$

in which $\hat{C}$ is defined by the codifferential if, and only if, using equation (3), we have $trA^* = 0$. $\square$

Proposition 3.11 *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. If the total space is almost trans-Sasakian or a locally conformal almost cosymplectic manifold, then the fibers are locally conformal almost Kählerian if and only if $trA^* = 0$.*

Proof Note that the common defining relation of these manifolds is

$$d\phi = \alpha.\phi \wedge \omega.$$

By Proposition 3.8, we have $\hat{\omega}(U) = \omega(U)$ if and only if $trA^* = 0$. Since $\hat{\phi}(U, V) = \phi(U, V)$, we deduce that $d\hat{\phi} = \alpha.\hat{\phi} \wedge \hat{\omega}$ if and only if $trA^* = 0$. The proof follows from the fact that, in this case, $\hat{\omega}(U) = \theta(U)$ according to Proposition 3.8 (b). $\square$

Proposition 3.12 *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. If the total space is almost trans-Sasakian or a locally conformal almost cosymplectic manifold, then the base space is a locally conformal almost Kählerian manifold if and only if the fibers are minimal.*

Proof By equation (4), we see that $\delta\Omega' = 0$ if and only if the fibers are minimal. $\square$

Proposition 3.13 *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. If the total space is quasi-K-cosymplectic or quasi-Kenmotsu, then the base space is a (1,2)-symplectic manifold.*

Proof Note that all these manifolds have in common the following relation:

$$(\nabla_D \phi)(E, G) + (\nabla_{\varphi D} \phi)(\varphi E, G) = \alpha.\eta(D)C$$

where C is a factor dertermined by the class of the manifold. For instance, if $\alpha = 1$ and $C = \eta(E)(\nabla_{\varphi D}\xi)$, we get the defining relation of a quasi-K-cosymplectic strucure. If $\alpha = 1$ and $C = \eta(E)\phi(G,D) + 2\eta(G)\phi(D,E)$, we obtain the principal defining relation of a quasi-Kenmotsu structure.

Let X, Y, and Z be 3 basic vector fields. Since η vanishes on horizontal vector fields, the common relation becomes

$$(\nabla_X\phi)(Y,Z) + (\nabla_{\varphi X}\phi)(\varphi Y, Z) = 0.$$

As $\pi^*\Omega' = \phi$, we get

$$(\nabla'_{X_*}\Omega')(Y_*, Z_*) + (\nabla'_{J'X_*}\Omega')(J'Y_*, Z_*) = 0.$$

This last relation is the defining relation of a quasi-Kählerian structure on the base space.

Recalling that

$$(\nabla'_{X_*}\Omega')(Y_*, Z_*) = g'((\nabla'_{X_*}J')Y_*, Z_*)$$

and

$$(\nabla'_{J'X_*}\Omega')(J'Y_*, Z_*) = g'((\nabla'_{J'X_*}J')J'Y_*, Z_*),$$

we then get

$$g'((\nabla'_{X_*}J')Y_*, Z_*) + g'((\nabla'_{J'X_*}J')J'Y_*, Z_*) = 0,$$

which is equivalent to

$$g'((\nabla'_{X_*}J')Y_*) + (\nabla'_{J'X_*}J')J'Y_*, Z_*) = 0,$$

from which

$$(\nabla'_{X_*}J')Y_* + (\nabla'_{J'X_*}J')J'Y_* = 0$$

follows.

This is the defining relation of a (1,2)-symplectic manifold, as noted in [2]. □

Proposition 3.14 *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. If the base space is a (1,2)-symplectic manifold, then the horizontal distribution of the total space looks like a quasi-K-cosymplectic or a quasi-Kenmotsu manifold.*

Proof Let X, Y, and Z be basic vector fields. It is known that on the base space $\pi_*X = X_*$, $\pi_*Y = Y_*$, and $\pi_*Z = Z_*$. Consider that the base space is defined by

$$(\nabla'_{X_*}J')Y_* + (\nabla'_{J'X_*}J')J'Y_* = 0.$$

This is to say that

$$(\nabla'_{X_*}\Omega')(Y_*, Z_*) + (\nabla'_{J'X_*}\Omega')(J'Y_*, Z_*) = 0.$$

Since $\pi^*\Omega' = \phi$, we have

$\pi^*(\nabla'_{X_*}\Omega')(Y_*, Z_*) = (\nabla_X\phi)(Y,Z)$ and

$\pi^*(\nabla'_{J'X_*}\Omega')(J'Y_*, Z_*) = (\nabla_{\varphi X}\phi)(\varphi Y, Z)$,

which lead to

$$(\nabla_X\phi)(Y,Z) + (\nabla_{\varphi X}\phi)(\varphi Y, Z) = 0.$$

Taking into account that η vanishes on the horizontal distribution, the last relation means that this distribution is of the kind

$$(\nabla_X \phi)(Y,Z) + (\nabla_{\varphi X} \phi)(\varphi Y, Z) = \eta(Z)C.$$

□

Proposition 3.15 *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. Assume that the base space admits a symplectic structure. Then the total space is an almost α-Kenmotsu manifold.*

Proof If $(M'^{2m'}, g', J')$ admits a symplectic structure, we have $d\Omega' = 0$ on horizontal vector fields. Referring to Proposition 3.2, $\pi^*\Omega' = \phi$, which implies that $d(\pi^*\Omega') = d\phi$. On the other hand, taking $d\Omega' = 0$ implies that $d\phi = 0$. To get $d\phi = 0$ on horizontal vector fields, we turn to Proposition 3.2(c), where η vanishes on horizontal distribution. Thus, we claim that the total space is an almost α-Kenmotsu manifold. □

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