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Relaxed elastic line in a Riemannian manifold

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Abstract: We obtain a differential equation with 2 boundary conditions for a relaxed elastic line in a Riemannian manifold. This differential equation, which is found with respect to constant sectional curvature G, geodesic curvature κ, and 2 boundary conditions, gives a more direct and more geometric approach to questions concerning a relaxed elastic line in a Riemannian manifold. We give various theorems and results in terms of a relaxed elastic line. Consequently, we examine the concept of a relaxed elastic line in $2-$ and $3-$dimensional space forms.

Key words: Relaxed elastic line, Riemannian manifold, geodesic curvature, space forms

1. Introduction

An elastic curve (or elastica), as proposed by Daniel Bernoulli to Leonhard Euler in 1744, is the solution to a variational problem of minimizing the integral of the squared curvature

$$\int_0^\ell \kappa^2(s)\, ds \tag{1.1}$$

for curves of a fixed length ℓ satisfying given first-order boundary conditions, where s, $0 \leq s \leq \ell$, is arc length. The elastic curve was studied by David Singer in $3-$dimensional Euclidean space in [7]. He used the classical techniques of the calculus of variations to derive the equations of the elastic curve. He also formulated a generalized variational problem, that of the elastic curve in a Riemannian manifold.

If no boundary conditions are imposed at $s = \ell$, and if no external forces act at any s, the elastic curve is relaxed. Thus, a relaxed elastic line (or curve) with fewer boundary conditions than an elastic curve is a more general solution to variational problem of elastic curve. The relaxed elastic line problem was first defined by Gerald S Manning in [5] as a critical point of the functional (1.1) among all curves of length ℓ having the same initial point and initial direction. He showed in [5] that the arc of a geodesic for a plane or sphere (a straight line or a great circle, respectively) is a relaxed elastic line. However, the geodesics of a cylinder and a torus provide solutions for the relaxed elastic line only in special cases. Nickerson and Manning in [6] then derived the intrinsic equations for a relaxed elastic line on an oriented surface and gave alternate proofs of some results of [5].

In this paper, by using the variation problem that was mentioned in the paper about elastic curves by Singer, we put forward a problem of a relaxed elastic line. After that, we formulate the relaxed elastic line in

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a Riemannian manifold. Thus, we obtain the differential equation with 2 boundary conditions for the relaxed elastic line in a Riemannian manifold with constant sectional curvature G. Moreover, we show that geodesics are always relaxed elastic lines in an $n-$dimensional Riemannian manifold with constant sectional curvature G. We also examine the relaxed elastic line in 2 and $3-$space forms.

2. Variation formulas

Let M be a smooth Riemannian manifold with Riemannian metric $<,>$, that is, a positive definite symmetric bilinear form on tangent space at each point. The ordinary derivative is replaced by the covariant derivative $\nabla_X Y$, which measures the derivative of a vector field Y in the direction of a vector X.

For vector fields X, Y, and Z on M, we write the structural equations

$$[X,Y] = \nabla_X Y - \nabla_Y X$$

and

$$R\left(X,Y\right)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where $[,]$ is the Lie bracket and R is the Riemannian curvature tensor of M.

Let $\gamma = \gamma(t)$ be a regular curve in an $n-$dimensional Riemannian manifold M. $V\left(t\right)$ will denote the tangent vector to γ, $T = T(t)$ the unit tangent vector, and v the speed $v\left(t\right) = \left\|V\left(t\right)\right\| = \sqrt{< V\left(t\right), V\left(t\right) >}$. $\kappa^2\left(t\right) = \left\|\nabla_T T\right\|^2$ is the squared geodesic curvature of γ. γ has curvatures $\kappa_1 = \kappa > 0$, $\kappa_2 = \tau$, κ_3, $\kappa_4, ..., \kappa_{n-1}$ and Frenet frame $N_0 = T$, $N_1 = N$, $N_2 = B$, N_3, $N_4, ..., N_{n-1}$. The Frenet equations are then:

$$\nabla_T N_i = -\kappa_i N_{i-1} + \kappa_{i+1} N_{i+1}, \qquad 0 \leq i \leq n-1 \tag{2.2}$$

(defining $\kappa_0 = \kappa_n = 0$) [4].

We will also denote by γ a variation

$$\begin{array}{llll} \gamma: & (-\varepsilon,\varepsilon)\times I & \rightarrow & M \\ & (w,t) & \rightarrow & \gamma\left(w,t\right) = \gamma_w\left(t\right) \end{array}$$

with $\gamma(0,t) = \gamma(t)$. Associated with such a variation is the variation vector field $W = W(t) = (\frac{\partial \gamma}{\partial w})(0,t)$ along the curve $\gamma(t)$. We will also write $V = V\left(w,t\right)$, $W = W\left(w,t\right)$, $T = T\left(w,t\right)$, $v = v\left(w,t\right)$, etc., with the obvious meanings.

Let s denote the arc length parameter. We then write $\gamma(s)$, $\kappa^2(w,s)$, $V(s)$, etc., for the corresponding reparametrizations, where $s \in [0,\ell]$ and ℓ is the arc length of γ. In order for the critical curve of functional (1.1) to be the relaxed elastic line, the variation vector field must satisfy the following conditions:

$$W(0) = 0, \;\; \nabla_T W(0) = 0 \tag{2.3}$$

and

$$W(\ell) \neq 0, \;\; \nabla_T W(\ell) \neq 0. \tag{2.4}$$

If $W(\ell) = 0$ and $\nabla_T W(\ell) = 0$, then the minimum curve of the functional (1.1) is an elastic curve.

By direct computation, we have the following lemma.

Lemma 2.1 *(see [4] and [7]). Let M be an $n-$dimensional Riemannian manifold and $\gamma(w,t)$ be a variation of γ. Then the following formulas are satisfied:*

i) $[V,W]=0,$

ii) $W(v)=<\nabla_T W,T>v=-gv,\ g=-<\nabla_T W,T>,$

iii) $[W,T]=gT,$

iv) $[[W,T],T]=-T(g)T=-g_sT,$ *where g_s denotes the derivative of g with regard to s.*

v) $W(\kappa^2)=2<\nabla_T\nabla_T W,\nabla_T T>+4g\kappa^2+2<R(W,T)T,\nabla_T T>.$

3. Integration of a relaxed elastic line

Let M be an n-dimensional Riemannian manifold and γ be a regular curve of length ℓ in M. The Frenet equations of γ are given by (2.2).

Now we will get a critical point of the functional

$$\mathcal{F}(\gamma)=\frac{1}{2}\int_0^\ell \kappa^2 ds=\frac{1}{2}\int_0^1 \kappa^2 v dt. \tag{3.5}$$

Assume that γ is a critical point of the functional $\mathcal{F}$. Then for a variation γ_w associated with a variation vector field W along γ, we compute

$$\begin{aligned}\frac{d}{dw}\mathcal{F}(\gamma_w) &= \frac{1}{2}\int_0^1 (W(\kappa^2)v+\kappa^2 W(v))dt,\\ &= \frac{1}{2}\int_0^\ell (W(\kappa^2)-\kappa^2 g)ds,\\ &= \frac{1}{2}\int_0^\ell (<\nabla_T\nabla_T W,\nabla_T T>+2g\kappa^2+<R(W,T)T,W>-\frac{1}{2}\kappa^2 g)ds.\end{aligned}$$

The Riemannian curvature tensor R has property

$$<R(W,T)T,\nabla_T T>=<R(\nabla_T T,T)T,W>.$$

Using $g=-<\nabla_T W,T>$ and integrating by parts, we obtain

$$\begin{aligned}\frac{d}{dw}\mathcal{F}(\gamma_w) &= \int_0^\ell (\langle\nabla_T\nabla_T W,\nabla_T T\rangle-\langle\nabla_T W,2\kappa^2 T\rangle\\ &\quad +\langle R(\nabla_T T,T)T,W\rangle+\frac{1}{2}\langle\nabla_T W,\kappa^2 T\rangle)ds\\ &= \int_0^\ell <\nabla_T^3 T-\Lambda\nabla_T T+R(\nabla_T T,T)T,W>ds\\ &\quad +(\langle\nabla_T T,\nabla_T W\rangle+<-\nabla_T^2 T+\Lambda T,W>)\Big|_0^\ell,\end{aligned} \tag{3.6}$$

where

$$\Lambda = -\frac{3\kappa^2}{2}.$$

When M is a Riemannian manifold with constant sectional curvature G, formula (3.6) can be simplified to

$$\frac{d}{dw}\mathcal{F}(\gamma_w) = \int_0^\ell \langle E, W\rangle \, ds + \left(\langle \nabla_T T, \nabla_T W\rangle + \left\langle -\nabla_T^2 T + \Lambda T, W\right\rangle\right)\Big|_0^\ell, \tag{3.7}$$

where

$$E = \nabla_T^3 T - \nabla_T \Lambda_G T$$

such that

$$\Lambda_G = \frac{-2G - 3\kappa^2}{2}.$$

Then by using the Frenet equations (2.2), we compute

$$E = (\kappa_{ss} - \kappa^3 - \kappa\tau^2 + \frac{2G\kappa + 3\kappa^3}{2})N + (2\kappa_s\tau + \kappa\tau_s)B + \kappa\tau\kappa_3 N_3. \tag{3.8}$$

Equation (3.7) for a relaxed elastic line must be zero (see [2] and [8]), that is,

$$(\kappa_{ss} - \kappa^3 - \kappa\tau^2 + \frac{2G\kappa + 3\kappa^3}{2})N + (2\kappa_s\tau + \kappa\tau_s)B + \kappa\tau\kappa_3 N_3 = 0 \tag{3.9}$$

and

$$\left(< \nabla_T T, \nabla_T W > + < -\nabla_T^2 T - \frac{3\kappa^2}{2} T, W >\right)\Big|_0^\ell = 0. \tag{3.10}$$

Since N, B, and N_3 in equation (3.9) are linear independent, we obtain

$$\begin{aligned} 2\kappa_{ss} + \kappa^3 - 2\kappa\tau^2 + 2G\kappa &= 0, \\ 2\kappa_s\tau + \kappa\tau_s &= 0, \\ \kappa\tau\kappa_3 &= 0. \end{aligned} \tag{3.11}$$

The second equation in (3.11) integrates to

$$\kappa^2\tau = c.$$

Eliminating τ from the first equation in (3.11) and integrating:

$$\begin{gathered} \kappa_s^2 + \tfrac{\kappa^4}{4} + G\kappa^2 - \tfrac{c^2}{\kappa^2} = A, \quad A = const \\ \kappa_i = 0, \quad i \geq 3. \end{gathered} \tag{3.12}$$

Now we find the boundary conditions that belong to differential equation (3.12) for a relaxed elastic line. Because of $E = \nabla_T^3 T - \nabla_T \Lambda_G T = 0$, the expression $-\nabla_T^2 T - \frac{3\kappa^2}{2} T$ is a constant. So, we get $< -\nabla_T^2 T - \frac{3\kappa^2}{2} T, W >\Big|_0^\ell = 0$ in equation (3.10). Equation (3.10) is then reduced to $< \nabla_T T, \nabla_T W >|_0^\ell = 0$. By taking into account conditions (2.3) and (2.4), and the Frenet equations (2.2), we obtain

$$< \kappa N, \nabla_T W >|_0^\ell = \kappa(\ell) < N(\ell), \nabla_T W(\ell) >= 0$$

and so

$$\kappa(\ell) = 0.$$

On the other hand, substituting $\kappa(\ell) = 0$, the Frenet equations (2.2), and conditions (2.3) and (2.4) into the equation $< -\nabla_T^2 T - \frac{3\kappa^2}{2} T, W > \Big|_0^\ell = 0$, we find

$$\begin{aligned} < -\kappa_s(\ell) N(\ell) + \kappa(\ell) \nabla_T N(\ell) - \frac{3\kappa^2(\ell)}{2} T(\ell), W(\ell) > &= 0 \\ \kappa_s(\ell) &= 0. \end{aligned}$$

Consequently, boundary conditions belonging to differential equation (3.12) are calculated as

$$\begin{aligned} \kappa(\ell) = 0, \\ \kappa_s(\ell) = 0. \end{aligned} \tag{3.13}$$

Then we can give the following theorem.

Theorem 3.1 *Let M be an $n-$dimensional Riemannian manifold with constant sectional curvature G. A relaxed elastic line in M can be determined by the differential equation* (3.12) *with 2 boundary conditions* (3.13).

The following corollary can easily be observed in Theorem 3.1.

Corollary 3.2 *If γ has a constant curvature in an $n-$dimensional Riemannian manifold with constant sectional curvature G, then γ is a relaxed elastic line.*

Thus, the following theorem provides the relationship between geodesics and relaxed elastic lines.

Theorem 3.3 *Each geodesic of an $n-$dimensional Riemannian manifold with constant sectional curvature G is a relaxed elastic line.*

Proof Let γ be a geodesic of M. Then the geodesic curvature of γ is zero. Thus, the geodesic curvature of γ satisfies the differential equation (3.12) with 2 boundary conditions (3.13). This explanation says that γ is a relaxed elastic line. □

4. Relaxed elastic line in space forms

A Riemannian manifold of constant sectional curvature is called elliptic, hyperbolic, or locally Euclidean if the sectional curvature is respectively positive, negative or zero. If this Riemannian manifold is complete, then it is called space form (see [1] and [3]).

Since $\kappa_i = 0$, $i \geq 3$, for the differential equation of a relaxed elastic line in an $n-$dimensional Riemannian manifold with constant sectional G, there is no essential loss of generality in assuming M has dimensions 2 and 3.

Now we characterize a relaxed elastic line in $2-$dimensional space forms.

Theorem 4.1 *A relaxed elastic line in* $2-$ *dimensional space form* M *is given by the differential equation*

$$\kappa_s^2+\frac{\kappa^4}{4}+G\kappa^2-\frac{c^2}{\kappa^2}=A,\quad A=const \tag{4.14}$$

together with the boundary conditions

$$\kappa(\ell)=0 \text{ and } \kappa_s(\ell)=0. \tag{4.15}$$

According to the constant sectional curvature G *of* $2-$ *dimensional space form* M *being positive, zero, or negative,* M *is called* $2-$ *sphere* $\mathbb{S}^2$, *Euclidean plane* $\mathbb{R}^2$, *or a hyperbolic plane* $\mathbb{H}^2$, *respectively.*

Now we examine the relaxed elastic line on $\mathbb{S}^2$, $\mathbb{R}^2$, and $\mathbb{H}^2$, respectively.

$i)$ A relaxed elastic line on $\mathbb{S}^2$ with constant sectional curvature $G=1$ is given by the differential equation

$$\kappa_s^2+\frac{\kappa^4}{4}+\kappa^2-\frac{c^2}{\kappa^2}=A,\quad A=const$$

together with the boundary conditions

$$\kappa(\ell)=0 \text{ and } \kappa_s(\ell)=0.$$

We can easily see that the geodesic of $\mathbb{S}^2$ is a relaxed elastic line.

$ii)$ The torsion of all curves is zero in $\mathbb{R}^2$ with constant sectional curvature $G=0$. Then a relaxed elastic line in $\mathbb{R}^2$ is given by the differential equation

$$\kappa_s^2+\frac{\kappa^4}{4}=A,\quad A=const$$

together with the boundary conditions

$$\kappa(\ell)=0 \text{ and } \kappa_s(\ell)=0.$$

We can easily see that the geodesic of $\mathbb{R}^2$ is a relaxed elastic line.

$iii)$ A relaxed elastic line on $\mathbb{H}^2$ with constant sectional curvature $G=-1$ is given by the differential equation

$$\kappa_s^2+\frac{\kappa^4}{4}-\kappa^2-\frac{c^2}{\kappa^2}=A,\quad A=const$$

together with the boundary conditions

$$\kappa(\ell)=0 \text{ and } \kappa_s(\ell)=0.$$

We can easily see that the geodesic of $\mathbb{H}^2$ is a relaxed elastic line.

Now we give a relaxed elastic line in $3-$dimensional space forms:

Theorem 4.2 *A relaxed elastic line in* $3-$ *dimensional space form* M *is given by the differential equation*

$$\kappa_s^2+\frac{\kappa^4}{4}+G\kappa^2-\frac{c^2}{\kappa^2}=A,\quad A=const,$$

together with the boundary conditions

$$\kappa(\ell)=0 \text{ and } \kappa_s(\ell)=0.$$

According to the constant sectional curvature G *of* $3-$ *dimensional space form* M *being positive, zero, or negative,* M *is called* $3-$ *sphere* $\mathbb{S}^3$, *Euclidean* $3-$ *space* $\mathbb{R}^3$, *or hyperbolic* $3-$ *space* $\mathbb{H}^3$, *respectively.*

Corollary 4.3 *Each geodesic of* $\mathbb{S}^3$, $\mathbb{R}^3$ *and* $\mathbb{H}^3$ *is a relaxed elastic line.*

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References

[1] do Carmo, M.P.: Riemannian Geometry. Boston. Birkhäuser 1992.

[2] Jurdjevic, V.: Geometric Control Theory. New York. Cambridge University Press 1997.

[3] Kühnel, W.: Differential Geometry: Curves - Surfaces - Manifolds. Providence. American Mathematical Society 2006.

[4] Langer, J., Singer, D.A.: The total squared curvature of closed curves. J. Diff. Geom. 1984; 20, 1–22.

[5] Manning, G.S.: Relaxed elastic line on a curved surface. Q. Appl. Math. 1987; 45, 515–527.

[6] Nickerson, H.K., Manning, G.S.: Intrinsic equations for a relaxed elastic line on an oriented surface. Geometriae Dedicata 1988; 27, 127–136.

[7] Singer, D.: Lectures on elastic curves and rods. AIP Conf. Proc. 1002. Melville, New York. Amer. Inst. Phys. 2008.

[8] Weinstock, R.: Calculus of Variations. New York. Dover Publications 1974.