

Continuity of Wigner-type operators on Lorentz spaces and Lorentz mixed normed modulation spaces

Ayşe SANDIKÇI*

Department of Mathematics, Faculty of Arts and Sciences, Ondokuz Mayıs University, Kurupelit, Samsun, Turkey

Received: 19.11.2013 • Accepted: 02.01.2014 • Published Online: 25.04.2014 • Printed: 23.05.2014

Abstract: We study various continuity properties for τ -Wigner transform on Lorentz spaces and τ -Weyl operators W_τ^a with symbols belonging to appropriate Lorentz spaces. We also study the action of τ -Wigner transform on Lorentz mixed normed modulation spaces.

Key words: τ -Wigner transform, τ -Weyl operators, Lorentz spaces, Lorentz mixed normed spaces, Lorentz mixed normed modulation spaces

1. Introduction

In this paper we will work on \mathbb{R}^d with Lebesgue measure dx . We denote $\mathcal{S}(\mathbb{R}^d)$ as the space of complex-valued continuous functions on \mathbb{R}^d rapidly decreasing at infinity. Let f be a complex valued measurable function on \mathbb{R}^d . The operators $T_x f(t) = f(t-x)$ and $M_w f(t) = e^{2\pi i w t} f(t)$ are called translation and modulation operators for $x, w \in \mathbb{R}^d$, respectively. The compositions

$$T_x M_w f(t) = e^{2\pi i w(t-x)} f(t-x) \quad \text{or} \quad M_w T_x f(t) = e^{2\pi i w t} f(t-x)$$

are called time-frequency shifts (see [9]). We write $(L^p(\mathbb{R}^d), \|\cdot\|_p)$ as the Lebesgue spaces for $1 \leq p \leq \infty$.

For $f \in L^1(\mathbb{R}^d)$ the Fourier transform \hat{f} (or $\mathcal{F}f$) is defined as

$$\hat{f}(t) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x t} dx,$$

where $xt = \sum_{i=1}^d x_i t_i$ is the usual scalar product on \mathbb{R}^d .

Fix a function $g \neq 0$ (called the window function). The short-time Fourier transform (STFT) of a function f with respect to g is given by

$$V_g f(x, w) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i t w} dt,$$

*Correspondence: ayses@omu.edu.tr

2010 AMS Mathematics Subject Classification: 42B10; 47B38.

for $x, w \in \mathbb{R}^d$. It is known that if $f, g \in L^2(\mathbb{R}^d)$ then $V_g f \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $V_g f$ is uniformly continuous (see [9]).

Let define $V_g^\tau f$ as the function

$$V_g^\tau f(x, w) = V_g f\left(\frac{1}{1-\tau}x, \frac{1}{\tau}w\right)$$

for $\tau \in (0, 1)$ and $(x, w) \in \mathbb{R}^{2d}$. The generalized spectrogram depending on 2 windows ϕ, ψ is also defined as

$$Sp_{\phi\psi}(f, g)(x, w) = V_\phi f(x, w) \overline{V_\psi g(x, w)}.$$

The cross-Wigner distribution of $f, g \in L^2(\mathbb{R}^d)$ is defined to be

$$W(f, g)(x, w) = \int_{\mathbb{R}^d} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i t w} dt.$$

If $f = g$, then $W(f, f) = Wf$ is called the Wigner distribution of $f \in L^2(\mathbb{R}^d)$.

For $\tau \in [0, 1]$ and $f, g \in \mathcal{S}(\mathbb{R}^d)$, the τ -Wigner transform is defined as

$$W_\tau(f, g)(x, w) = \int_{\mathbb{R}^d} f(x + \tau t) \overline{g(x - (1-\tau)t)} e^{-2\pi i t w} dt.$$

If $\tau = \frac{1}{2}$, then the τ -Wigner transform is the cross-Wigner distribution. Moreover, for $\tau = 0$, W_0 is the Rihaczek transform,

$$W_0(f, g)(x, w) = R(f, g)(x, w) = e^{-2\pi i x w} f(x) \overline{\hat{g}(w)},$$

and for $\tau = 1$, W_1 is the conjugate Rihaczek transform,

$$W_1(f, g)(x, w) = \overline{R(g, f)}(x, w) = e^{2\pi i x w} \overline{\hat{g}(x)} f(w).$$

For $\tau \in (0, 1)$, the τ -Wigner transform can be rewritten as

$$W_\tau(f, g)(x, w) = \frac{1}{|\tau|^d} e^{2\pi i \frac{1}{\tau} x w} V_{A_\tau g} f\left(\frac{1}{1-\tau}x, \frac{1}{\tau}w\right), \quad (1.1)$$

where the operator A_τ is defined by

$$A_\tau : h(t) \rightarrow \tilde{h}\left(\frac{1-\tau}{\tau}t\right)$$

with $\tilde{h}(t) = h(-t)$ (see [4, 5]).

Let $a \in \mathcal{S}(\mathbb{R}^{2d})$, and then for $\tau \in [0, 1]$, the τ -Weyl pseudo-differential operators with τ -symbol a

$$W_\tau^a : f \rightarrow W_\tau^a f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)w} a((1-\tau)x + \tau y, w) f(y) dy dw$$

are defined as a continuous map from $\mathcal{S}(\mathbb{R}^d)$ to itself (see [5]).

Fix a nonzero window $g \in \mathcal{S}(\mathbb{R}^d)$ and $1 \leq p, q \leq \infty$. Then the modulation space $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the short-time Fourier transform $V_g f$ is in the mixed-norm space $L^{p,q}(\mathbb{R}^{2d})$. The norm on $M^{p,q}(\mathbb{R}^d)$ is $\|f\|_{M^{p,q}} = \|V_g f\|_{L^{p,q}}$. If $p = q$, then we write $M^p(\mathbb{R}^d)$ instead of $M^{p,p}(\mathbb{R}^d)$. Modulation spaces are Banach spaces whose definitions are independent of the choice of the window g (see [7, 9]).

$L(p, q)$ spaces are function spaces that are closely related to L^p spaces. We consider complex valued measurable functions f defined on a measure space (X, μ) . The measure μ is assumed to be nonnegative. We assume that the functions f are finite valued a.e. and some $y > 0$, $\mu(E_y) < \infty$, where $E_y = E_y[f] = \{x \in X \mid |f(x)| > y\}$. Then, for $y > 0$,

$$\lambda_f(y) = \mu(E_y) = \mu(\{x \in X \mid |f(x)| > y\})$$

is the distribution function of f . The rearrangement of f is given by

$$f^*(t) = \inf \{y > 0 \mid \lambda_f(y) \leq t\} = \sup \{y > 0 \mid \lambda_f(y) > t\}$$

for $t > 0$. The average function of f is also defined by

$$f^{**}(x) = \frac{1}{x} \int_0^x f^*(t) dt.$$

Note that λ_f , f^* , and f^{**} are nonincreasing and right continuous functions on $(0, \infty)$. If $\lambda_f(y)$ is continuous and strictly decreasing then $f^*(t)$ is the inverse function of $\lambda_f(y)$. The most important property of f^* is that it has the same distribution function as f . It follows that

$$\left(\int_X |f(x)|^p d\mu(x) \right)^{\frac{1}{p}} = \left(\int_0^\infty [f^*(t)]^p dt \right)^{\frac{1}{p}}. \quad (1.2)$$

The Lorentz space denoted by $L(p, q)(X, \mu)$ (shortly $L(p, q)$) is defined to be vector space of all (equivalence classes) of measurable functions f such that $\|f\|_{pq}^* < \infty$, where

$$\|f\|_{pq}^* = \begin{cases} \left(\frac{q}{p} \int_0^\infty t^{\frac{q}{p}-1} [f^*(t)]^q dt \right)^{\frac{1}{q}}, & 0 < p, q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), & 0 < p \leq q = \infty. \end{cases}$$

By (1.2), it follows that $\|f\|_{pp}^* = \|f\|_p$ and so $L(p, p) = L^p$. Also, $L(p, q)(X, \mu)$ is a normed space with the norm

$$\|f\|_{pq} = \begin{cases} \left(\frac{q}{p} \int_0^\infty t^{\frac{q}{p}-1} [f^{**}(t)]^q dt \right)^{\frac{1}{q}}, & 0 < p, q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t), & 0 < p \leq q = \infty. \end{cases}$$

For any one of the cases $p = q = 1$; $p = q = \infty$ or $1 < p < \infty$ and $1 \leq q \leq \infty$, the Lorentz space $L(p, q)(X, \mu)$ is a Banach space with respect to the norm $\|\cdot\|_{pq}$. It is also known that if $1 < p < \infty$, $1 \leq q \leq \infty$ we have

$$\|\cdot\|_{pq}^* \leq \|\cdot\|_{pq} \leq \frac{p}{p-1} \|\cdot\|_{pq}^*,$$

(see [11, 12]).

It is known from [11] that $L(\infty, q) = \{0\}$ if $q \neq \infty$ and $L(\infty, q) = L^\infty$ if $q = \infty$. However, in [1, 2], $L(\infty, q)$ are defined as the class of all measurable functions f for which $f^*(t) < \infty$ for all $t > 0$ and for which $f^{**}(t) - f^*(t)$ is a bounded function of t such that

$$\|f\|_{\infty q} = \left(\int_0^\infty [f^{**}(t) - f^*(t)]^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty, \quad 0 < q < \infty.$$

Moreover, if $q = 1$, $L(\infty, 1) = L^\infty$ and the norms coincide.

Let X and Y be 2 measure spaces with σ -finite measures μ and ν , respectively, and let f be a complex-valued measurable function on $(X \times Y, \mu \times \nu)$, $1 < P = (p_1, p_2) < \infty$ and $1 \leq Q = (q_1, q_2) \leq \infty$. The Lorentz mixed norm space $L(P, Q) = L(P, Q)(X \times Y)$ is defined by

$$L(P, Q) = L(p_2, q_2)[L(p_1, q_1)] = \left\{ f : \|f\|_{PQ} = \|f\|_{L(p_2, q_2)(L(p_1, q_1))} = \left\| \|f\|_{p_1 q_1} \right\|_{p_2 q_2} < \infty \right\}.$$

Thus, $L(P, Q)$ occurs by taking an $L(p_1, q_1)$ -norm with respect to the first variable and an $L(p_2, q_2)$ -norm with respect to the second variable. The $L(P, Q)$ space is a Banach space under the norm $\|\cdot\|_{PQ}$ (see [3, 8]).

Fix a window function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, $1 \leq P = (p_1, p_2) < \infty$ and $1 \leq Q = (q_1, q_2) \leq \infty$. We let $M(P, Q)(\mathbb{R}^d)$ denote the subspace of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ consisting of $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the Gabor transform $V_g f$ of f is in the Lorentz mixed norm space $L(P, Q)(\mathbb{R}^{2d})$. We endow it with the norm $\|f\|_{M(P, Q)} = \|V_g f\|_{PQ}$, where $\|\cdot\|_{PQ}$ is the norm of the Lorentz mixed norm space. It is known that $M(P, Q)(\mathbb{R}^d)$ is a Banach space and different windows yield equivalent norms. If $p_1 = q_1 = p$ and $p_2 = q_2 = q$, then the space $M(P, Q)(\mathbb{R}^d)$ is the standard modulation space $M^{p, q}(\mathbb{R}^d)$, and if $P = p$ and $Q = q$, in this case $M(P, Q)(\mathbb{R}^d) = M(p, q)(\mathbb{R}^d)$ (see [14]), where the space $M(p, q)(\mathbb{R}^d)$ is Lorentz type modulation space (see [10]). Furthermore, the space $M(p, q)(\mathbb{R}^d)$ was generalized to $M(p, q, w)(\mathbb{R}^d)$ by taking weighted Lorentz space rather than Lorentz space (see [15, 16]).

In this paper, we will denote the Lorentz space by $L(p, q)$, the Lorentz mixed norm space by $L(P, Q)$, the standard modulation space by $M^{p, q}$, the Lorentz type modulation space by $M(p, q)$, and the Lorentz mixed normed modulation space by $M(P, Q)$.

In Section 2, we consider continuity for generalized spectrogram, τ -Wigner transform, and τ -Weyl pseudo-differential operators acting on Lorentz spaces. We extend the results in [4, 5] to the Lorentz spaces. In Section 3, we also study continuity properties of τ -Wigner transform on Lorentz mixed normed modulation spaces. This result extends Proposition 2.5 in [6] and Proposition 15 in [14] since the τ -Wigner transform is the cross-Wigner transform for $\tau = \frac{1}{2}$ and the similar sufficient conditions provide boundedness on both classical and Lorentz mixed normed modulation spaces.

2. Continuity of some operators on Lorentz spaces

In this section, we present $L(p, q)$ -boundedness of generalized spectrogram, τ -Wigner transform, and τ -Weyl pseudo-differential operators with τ -symbol a .

We begin with the following 2 Lemmas, will be used later on.

Lemma 1 *If $\tau \in (0, 1)$, $1 < p < \infty$, $1 \leq q \leq \infty$ and $f \in L(p, q)(\mathbb{R}^d, \mu)$, then we have*

$$\|A_\tau f\|_{pq} = \left(\frac{|\tau|}{|1-\tau|} \right)^{\frac{d}{p}} \|f\|_{pq}.$$

Proof Let $\tau \in (0, 1)$ and $f \in L(p, q)(\mathbb{R}^d, \mu)$. Then we have

$$\begin{aligned} \lambda_{A_\tau f}(y) &= \mu \left\{ x \in \mathbb{R}^d \mid |A_\tau f(x)| > y \right\} = \mu \left\{ x \in \mathbb{R}^d \mid \left| \tilde{f} \left(\frac{1-\tau}{\tau} x \right) \right| > y \right\} \\ &= \mu \left\{ x \in \mathbb{R}^d \mid \left| f \left(\frac{\tau-1}{\tau} x \right) \right| > y \right\} \\ &= \mu \left\{ \frac{\tau}{\tau-1} u \in \mathbb{R}^d \mid |f(u)| > y \right\} \\ &= \left| \frac{\tau}{\tau-1} \right|^d \mu \left\{ u \in \mathbb{R}^d \mid |f(u)| > y \right\} = \left| \frac{\tau}{1-\tau} \right|^d \lambda_f(y) \end{aligned}$$

for $y > 0$. Thus, the rearrangement of $A_\tau f$ is

$$\begin{aligned} (A_\tau f)^*(t) &= \inf \{ y > 0 \mid \lambda_{A_\tau f}(y) \leq t \} = \inf \left\{ y > 0 \mid \left| \frac{\tau}{\tau-1} \right|^d \lambda_f(y) \leq t \right\} \\ &= \inf \left\{ y > 0 \mid \lambda_f(y) \leq \left| \frac{1-\tau}{\tau} \right|^d t \right\} = f^* \left(\left| \frac{1-\tau}{\tau} \right|^d t \right) \end{aligned}$$

for $t > 0$. Additionally, the average function of $A_\tau f$ is

$$\begin{aligned} (A_\tau f)^{**}(x) &= \frac{1}{x} \int_0^x (A_\tau f)^*(t) dt = \frac{1}{x} \int_0^x f^* \left(\left| \frac{1-\tau}{\tau} \right|^d t \right) dt \\ &= \frac{1}{\left| \frac{1-\tau}{\tau} \right|^d x} \int_0^{\left| \frac{1-\tau}{\tau} \right|^d x} f^*(u) du = f^{**} \left(\left| \frac{1-\tau}{\tau} \right|^d x \right). \end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\|A_\tau f\|_{pq} &= \left(\frac{q}{p} \int_0^\infty x^{\frac{q}{p}-1} [(A_\tau f)^{**}(x)]^q dx \right)^{\frac{1}{q}} = \left(\frac{q}{p} \int_0^\infty x^{\frac{q}{p}-1} \left[f^{**} \left(\left| \frac{1-\tau}{\tau} \right|^d x \right) \right]^q dx \right)^{\frac{1}{q}} \\
&= \left(\frac{q}{p} \int_0^\infty \left| \frac{\tau}{1-\tau} \right|^{d(\frac{q}{p}-1)} t^{\frac{q}{p}-1} [f^{**}(t)]^q \left| \frac{\tau}{1-\tau} \right|^d dt \right)^{\frac{1}{q}} \\
&= \left| \frac{\tau}{1-\tau} \right|^{\frac{d}{p}} \left(\frac{q}{p} \int_0^\infty t^{\frac{q}{p}-1} [f^{**}(t)]^q dt \right)^{\frac{1}{q}} \\
&= \left| \frac{\tau}{1-\tau} \right|^{\frac{d}{p}} \|f\|_{pq}.
\end{aligned}$$

□

Lemma 2 For $\tau \in (0, 1)$ and $1 < p < \infty$, $1 \leq q \leq \infty$, and then

$$\|V_g^\tau f\|_{pq} = (|1-\tau| \cdot |\tau|)^{\frac{d}{p}} \|V_g f\|_{pq},$$

when $V_g f \in L(p, q) (\mathbb{R}^{2d})$.

Proof Let ν is a measure on \mathbb{R}^d . Then $\mu = \nu \times \nu$ is a measure on \mathbb{R}^{2d} . Thus, the distribution function of $V_g^\tau f$ is

$$\begin{aligned}
\lambda_{V_g^\tau f}(y) &= \mu \{ (x, w) \in \mathbb{R}^{2d} \mid |V_g^\tau f(x, w)| > y \} \\
&= \mu \left\{ (x, w) \in \mathbb{R}^{2d} \mid \left| V_g f \left(\frac{1}{1-\tau} x, \frac{1}{\tau} w \right) \right| > y \right\} \\
&= \mu \left[\left\{ x \in \mathbb{R}^d \mid \left| V_g f \left(\frac{1}{1-\tau} x, \cdot \right) \right| > y \right\} \times \left\{ w \in \mathbb{R}^d \mid \left| V_g f \left(\cdot, \frac{1}{\tau} w \right) \right| > y \right\} \right] \\
&= \nu \left\{ x \in \mathbb{R}^d \mid \left| V_g f \left(\frac{1}{1-\tau} x, \cdot \right) \right| > y \right\} \nu \left\{ w \in \mathbb{R}^d \mid \left| V_g f \left(\cdot, \frac{1}{\tau} w \right) \right| > y \right\} \\
&= (|1-\tau| \cdot |\tau|)^d \nu \{ u \in \mathbb{R}^d \mid |V_g f(u, \cdot)| > y \} \nu \{ v \in \mathbb{R}^d \mid |V_g f(\cdot, v)| > y \} \\
&= (|1-\tau| \cdot |\tau|)^d \mu \{ (u, v) \in \mathbb{R}^{2d} \mid |V_g f(u, v)| > y \} \\
&= (|1-\tau| \cdot |\tau|)^d \lambda_{V_g f}(y)
\end{aligned}$$

for $y > 0$. Then the rearrangement function of $V_g^\tau f$ is

$$\begin{aligned}
(V_g^\tau f)^*(t) &= \inf \{ y > 0 \mid \lambda_{V_g^\tau f}(y) \leq t \} \\
&= \inf \{ y > 0 \mid (|1-\tau| \cdot |\tau|)^d \lambda_{V_g f}(y) \leq t \} \\
&= \inf \left\{ y > 0 \mid \lambda_{V_g f}(y) \leq \frac{t}{(|1-\tau| \cdot |\tau|)^d} \right\} = (V_g f)^* \left(\frac{t}{(|1-\tau| \cdot |\tau|)^d} \right)
\end{aligned}$$

for $t > 0$. Also, the average function of $V_g^\tau f$ is

$$\begin{aligned} (V_g^\tau f)^{**}(x) &= \frac{1}{x} \int_0^x (V_g^\tau f)^*(t) dt = \frac{1}{x} \int_0^x (V_g f)^* \left(\frac{t}{(|1-\tau| \cdot |\tau|)^d} \right) dt \\ &= \frac{(|1-\tau| \cdot |\tau|)^d}{x} \int_0^{\frac{x}{(|1-\tau| \cdot |\tau|)^d}} (V_g f)^*(u) du = (V_g f)^{**} \left(\frac{x}{(|1-\tau| \cdot |\tau|)^d} \right). \end{aligned}$$

Thus, we have

$$\begin{aligned} \|V_g^\tau f\|_{pq} &= \left(\frac{q}{p} \int_0^\infty x^{\frac{q}{p}-1} [(V_g^\tau f)^{**}(x)]^q dx \right)^{\frac{1}{q}} \\ &= \left(\frac{q}{p} \int_0^\infty x^{\frac{q}{p}-1} \left[(V_g f)^{**} \left(\frac{x}{(|1-\tau| \cdot |\tau|)^d} \right) \right]^q dx \right)^{\frac{1}{q}} \\ &= (|1-\tau| \cdot |\tau|)^{\frac{d}{p}} \left(\frac{q}{p} \int_0^\infty t^{\frac{q}{p}-1} [(V_g f)^{**}(t)]^q dt \right)^{\frac{1}{q}} \\ &= (|1-\tau| \cdot |\tau|)^{\frac{d}{p}} \|V_g f\|_{pq}. \end{aligned}$$

□

We shall need the following continuity property of the short-time Fourier transform on Lorentz spaces in order to prove the continuity properties concerning the generalized spectrogram and τ -Wigner transform.

Proposition 3 Let $1 < p < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{p_1} + \frac{1}{p_2} < 1$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p'}$, and $q \geq 1$ be any number such that $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{q}$. Then the Gabor transform

$$V : (f, g) \in L(p_1, q_1)(\mathbb{R}^d) \times L(p_2, q_2)(\mathbb{R}^d) \rightarrow V_g f \in L(p, q)(\mathbb{R}^{2d})$$

is bounded. In particular,

$$\|V_g f\|_{pq} \leq C \|f\|_{p_1 q_1} \|g\|_{p_2 q_2}.$$

Proof Let $f \in L(p_1, q_1)(\mathbb{R}^d)$ and $g \in L(p_2, q_2)(\mathbb{R}^d)$. Using the equality $V_g f(x, w) = (f \cdot T_x g)^\wedge(w)$, Theorem 4.3. in [11], and a generalization of Hölder's inequality for Lorentz spaces (see [12]), we obtain

$$\begin{aligned} \|V_g f\|_{pq} &= \|(f \cdot T_x g)^\wedge\|_{pq} \leq \|f \cdot T_x g\|_{p'q} \\ &\leq C \|f\|_{p_1 q_1} \|T_x g\|_{p_2 q_2} = C \|f\|_{p_1 q_1} \|g\|_{p_2 q_2}. \end{aligned}$$

This is the desired result. □

Now we will state the continuity of $Sp_{\phi\psi}$ on the Lorentz spaces.

Theorem 4 Let $1 < p, p_3, p_4 < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{p_3} + \frac{1}{p'_3} = 1$, $\frac{1}{p_4} + \frac{1}{p'_4} = 1$, $\frac{1}{p_i} + \frac{1}{p'_i} = 1$ ($i = 1, 2$), $\frac{1}{p_1} + \frac{1}{p'_1} = \frac{1}{p'_3}$, $\frac{1}{p_2} + \frac{1}{p'_2} = \frac{1}{p'_4}$, and $q, q_3, q_4 \geq 1$ be numbers such that $\frac{1}{q_3} + \frac{1}{q_4} \geq \frac{1}{q}$, $\frac{1}{q_1} + \frac{1}{q'_1} \geq \frac{1}{q_3}$, and $\frac{1}{q_2} + \frac{1}{q'_2} \geq \frac{1}{q_4}$. Then $(f, \phi, g, \psi) \rightarrow Sp_{\phi\psi}(f, g) = V_\phi f \overline{V_\psi g}$ is continuous from $L(p_1, q_1)(\mathbb{R}^d) \times L(p'_1, q'_1)(\mathbb{R}^d) \times L(p_2, q_2)(\mathbb{R}^d) \times L(p'_2, q'_2)(\mathbb{R}^d)$ into $L(p, q)(\mathbb{R}^{2d})$. In particular,

$$\|Sp_{\phi\psi}(f, g)\|_{pq} = \|V_\phi f \overline{V_\psi g}\|_{pq} \leq C \|f\|_{p_1 q_1} \|\phi\|_{p'_1 q'_1} \|g\|_{p_2 q_2} \|\psi\|_{p'_2 q'_2}.$$

Proof By using Proposition 3, we write that

$$V_\phi f : L(p_1, q_1)(\mathbb{R}^d) \times L(p'_1, q'_1)(\mathbb{R}^d) \rightarrow L(p_3, q_3)(\mathbb{R}^{2d}),$$

with

$$\|V_\phi f\|_{p_3 q_3} \leq C \|f\|_{p_1 q_1} \|\phi\|_{p'_1 q'_1}$$

and

$$V_\psi g : L(p_2, q_2)(\mathbb{R}^d) \times L(p'_2, q'_2)(\mathbb{R}^d) \rightarrow L(p_4, q_4)(\mathbb{R}^{2d})$$

with

$$\|\overline{V_\psi g}\|_{p_4 q_4} \leq C \|g\|_{p_2 q_2} \|\psi\|_{p'_2 q'_2}$$

being continuous. Hence, we get that $Sp_{\phi\psi}(f, g) = V_\phi f \overline{V_\psi g}$ is continuous from $L(p_1, q_1)(\mathbb{R}^d) \times L(p'_1, q'_1)(\mathbb{R}^d) \times L(p_2, q_2)(\mathbb{R}^d) \times L(p'_2, q'_2)(\mathbb{R}^d)$ into $L(p_3, q_3)(\mathbb{R}^{2d}) \cdot L(p_4, q_4)(\mathbb{R}^{2d})$ with

$$\|V_\phi f\|_{p_3 q_3} \|\overline{V_\psi g}\|_{p_4 q_4} \leq C \|f\|_{p_1 q_1} \|\phi\|_{p'_1 q'_1} \|g\|_{p_2 q_2} \|\psi\|_{p'_2 q'_2}. \tag{2.3}$$

We thus obtain that

$$\|V_\phi f \overline{V_\psi g}\|_{pq} \leq \|V_\phi f\|_{p_3 q_3} \|\overline{V_\psi g}\|_{p_4 q_4} \tag{2.4}$$

by the generalized Hölder inequality for Lorentz spaces. Moreover, (2.4) means that $L(p_3, q_3)(\mathbb{R}^{2d}) \cdot L(p_4, q_4)(\mathbb{R}^{2d})$ is continuously embedded into $L(p, q)(\mathbb{R}^{2d})$. Then by (2.3) and (2.4), we have the desired result. \square

Theorem 5 i. Assume that $1 < p < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{p_1} + \frac{1}{p_2} < 1$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p'}$, and $q \geq 1$ is any number such that $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{q}$. Then for $\tau \in (0, 1)$,

$$W_\tau : L(p_1, q_1)(\mathbb{R}^d) \times L(p_2, q_2)(\mathbb{R}^d) \rightarrow L(p, q)(\mathbb{R}^{2d})$$

is continuous.

ii. Let $1 < p' < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < r \leq \infty$, $0 < q, s \leq \infty$ and let the following 2 inequalities be satisfied:

- 1) $\max(q, r) \leq s$
- 2) $\frac{1}{p} + \frac{1}{s} \leq \frac{1}{q} + \frac{1}{r}$.

Then for $\tau = 0$,

$$W_0 : L(p, q)(\mathbb{R}^d) \times L(p', r)(\mathbb{R}^d) \rightarrow L(p, s)(\mathbb{R}^{2d})$$

is continuous and holds

$$\|W_0(f, g)\|_{ps} \leq C \|f\|_{pq} \|g\|_{p'r}.$$

iii. Let $1 < p' < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < q \leq \infty$, $0 < r, s \leq \infty$ and let the following 2 inequalities be satisfied:

$$\begin{aligned} 1) \quad & \max(q, r) \leq s \\ 2) \quad & \frac{1}{p} + \frac{1}{s} \leq \frac{1}{q} + \frac{1}{r}. \end{aligned}$$

Then for $\tau = 1$,

$$W_1 : L(p', q)(\mathbb{R}^d) \times L(p, r)(\mathbb{R}^d) \rightarrow L(p, s)(\mathbb{R}^{2d})$$

is continuous. In particular,

$$\|W_1(f, g)\|_{ps} \leq C \|f\|_{p'q} \|g\|_{pr}.$$

Proof i. Using Lemma 1, Lemma 2, and Proposition 3, we have

$$\begin{aligned} \|W_\tau(f, g)\|_{pq}^q &= \left\| \frac{1}{|\tau|^d} e^{2\pi i \frac{1}{\tau} xw} V_{A_\tau g}^\tau f \right\|_{pq}^q = \frac{1}{|\tau|^{dq}} \|V_{A_\tau g}^\tau f\|_{pq}^q \\ &= \frac{1}{|\tau|^{dq}} (|1 - \tau| \cdot |\tau|)^{\frac{dq}{p}} \|V_{A_\tau g} f\|_{pq}^q \\ &\leq \frac{1}{|\tau|^{dq}} (|1 - \tau| \cdot |\tau|)^{\frac{dq}{p}} C \|f\|_{p_1 q_1}^q \|A_\tau g\|_{p_2 q_2}^q \\ &= \frac{1}{|\tau|^{dq}} (|1 - \tau| \cdot |\tau|)^{\frac{dq}{p}} C \|f\|_{p_1 q_1}^q \left| \frac{\tau}{1 - \tau} \right|^{\frac{dq}{p_2}} \|g\|_{p_2 q_2}^q \\ &= |\tau|^{dq(\frac{1}{p} + \frac{1}{p_2} - 1)} |1 - \tau|^{dq(\frac{1}{p} - \frac{1}{p_2})} C \|f\|_{p_1 q_1}^q \|g\|_{p_2 q_2}^q. \end{aligned}$$

This completes the proof.

ii. Let $f \in L(p, q)(\mathbb{R}^d)$ and $g \in L(p', r)(\mathbb{R}^d)$. Then $\hat{g} \in L(p, r)(\mathbb{R}^d)$ and

$$\left\| \hat{g} \right\|_{pr} \leq B \|g\|_{p'r} \quad (2.5)$$

by Theorem 4.3 in [11]. By using the equality $W_0(f, g)(x, w) = e^{-2\pi i xw} f(x) \overline{\hat{g}(w)} = R(f, g)(x, w)$, inequality (2.5), and Theorem 7.7 in [13], we get

$$\begin{aligned} \|W_0(f, g)\|_{ps} &= \|R(f, g)\|_{ps} \leq K \|f\|_{pq} \left\| \hat{g} \right\|_{pr} \\ &\leq C \|f\|_{pq} \|g\|_{p'r}. \end{aligned}$$

This is the desired result.

iii. Let $f \in L(p', q)(\mathbb{R}^d)$ and $g \in L(p, r)(\mathbb{R}^d)$. Then $\hat{f} \in L(p, q)(\mathbb{R}^d)$ and

$$\left\| \hat{f} \right\|_{pq} \leq B \|f\|_{p'q} \quad (2.6)$$

by Theorem 4.3 in [11]. By using the equality $W_1(f, g)(x, w) = e^{2\pi i x w} \overline{g(x)} \hat{f}(w) = \overline{R(g, f)}(x, w)$, inequality (2.6), and Theorem 7.7 in [13], we have

$$\begin{aligned} \|W_1(f, g)\|_{ps} &= \left\| \overline{R(g, f)} \right\|_{ps} \leq K \left\| \hat{f} \right\|_{pq} \|g\|_{pr} \\ &\leq C \|f\|_{p'q} \|g\|_{pr}. \end{aligned}$$

□

If $(0, \infty)$ is taken instead of \mathbb{R}^d in Theorem 5 (ii) and (iii), then the boundedness of $W_0(f, g)$ and $W_1(f, g)$ is equivalent to conditions (1) and (2) by Theorem 7.7 in [13].

In the next theorem the Lorentz mixed normed space $L(P, Q)(\mathbb{R}^{2d})$ is taken, where $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, instead of the Lorentz space $L(p, s)(\mathbb{R}^{2d})$ as the range of W_0 and W_1 .

Proposition 6 *Let $1 < p_1 < \infty$, $1 < p'_2 < 2$, $\frac{1}{p_2} + \frac{1}{p'_2} = 1$, $P = (p_1, p_2)$, $1 \leq Q = (q_1, q_2) \leq \infty$. For $\tau = 0, 1$,*

$$W_0 : L(p_1, q_1)(\mathbb{R}^d) \times L(p'_2, q_2)(\mathbb{R}^d) \rightarrow L(P, Q)(\mathbb{R}^{2d})$$

and

$$W_1 : L(p'_2, q_2)(\mathbb{R}^d) \times L(p_1, q_1)(\mathbb{R}^d) \rightarrow L(P, Q)(\mathbb{R}^{2d})$$

are continuous. In particular,

$$\|W_0(f, g)\|_{PQ} \leq B \|f\|_{p_1 q_1} \|g\|_{p'_2 q_2}$$

and

$$\|W_1(f, g)\|_{PQ} \leq B \|g\|_{p_1 q_1} \|f\|_{p'_2 q_2}.$$

Proof If $g \in L(p'_2, q_2)(\mathbb{R}^d)$, then $\hat{g} \in L(p_2, q_2)(\mathbb{R}^d)$ and $\left\| \hat{g} \right\|_{p_2 q_2} \leq \|g\|_{p'_2 q_2}$. By using the equality

$W_0(f, g)(x, w) = e^{-2\pi i x w} f(x) \overline{\hat{g}}(w) = R(f, g)(x, w)$, we have

$$\begin{aligned} \|W_0(f, g)\|_{PQ} &= \|R(f, g)\|_{PQ} = \left\| \|f\|_{p_1 q_1(\mathbb{R}_x^d)} \overline{\hat{g}} \right\|_{p_2 q_2(\mathbb{R}_w^d)} \\ &= \|f\|_{p_1 q_1} \left\| \overline{\hat{g}} \right\|_{p_2 q_2} \\ &\leq B \|f\|_{p_1 q_1} \|g\|_{p'_2 q_2}, \end{aligned}$$

which proves the continuity of W_0 . The continuity of W_1 is proved in a similar way to the continuity of W_0 . □

The following Theorem is proven from Proposition 5.1 in [5] and Theorem 5.

Theorem 7 *i. Let $\tau \in (0, 1)$. A necessary and sufficient condition that*

$$a \in L(p', q')(\mathbb{R}^{2d}) \rightarrow W_\tau^a \in B(L(p_2, q_2)(\mathbb{R}^d), L(p'_1, q'_1)(\mathbb{R}^d))$$

is continuous is that $1 < p < 2$, $\frac{1}{p_1} + \frac{1}{p_2} < 1$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p'}$, and $q \geq 1$ be any number such that $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{q}$, where p', q', p'_1 , and q'_1 are the conjugates of p, q, p_1 , and q_1 .

ii. Let $\tau = 0$. A necessary and sufficient condition that

$$a \in L(p', s')(\mathbb{R}^{2d}) \rightarrow W_0^a \in B(L(p', r)(\mathbb{R}^d), L(p', q')(\mathbb{R}^d))$$

is continuous is that $1 < p' < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{s} + \frac{1}{s'} = 1$, $1 < r \leq \infty$, $0 < q, s \leq \infty$, and also that the following 2 inequalities be satisfied:

$$\begin{aligned} 1) \quad & \max(q, r) \leq s \\ 2) \quad & \frac{1}{p} + \frac{1}{s} \leq \frac{1}{q} + \frac{1}{r}. \end{aligned}$$

iii. Let $\tau = 1$. A necessary and sufficient condition that

$$a \in L(p', s')(\mathbb{R}^{2d}) \rightarrow W_0^a \in B(L(p, r)(\mathbb{R}^d), L(p, q')(\mathbb{R}^d))$$

is continuous is that $1 < p' < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{s} + \frac{1}{s'} = 1$, $1 < q \leq \infty$, $0 < r, s \leq \infty$ and also that the following 2 inequalities be satisfied:

$$\begin{aligned} 1) \quad & \max(q, r) \leq s \\ 2) \quad & \frac{1}{p} + \frac{1}{s} \leq \frac{1}{q} + \frac{1}{r}. \end{aligned}$$

3. Boundedness of τ -Wigner transform on Lorentz mixed normed modulation spaces

The aim of this section is to study continuity properties of the τ -Wigner transform when acting on the Lorentz mixed normed modulation spaces.

In Proposition 8-10 below we have listed some properties for τ -Wigner transform. From these results we then prove the continuity of the τ -Wigner transform.

Proposition 8 For $f, g \in \mathcal{S}(\mathbb{R}^d)$, $\tau \in [0, 1]$ and $u, v, \eta, \gamma \in \mathbb{R}^d$, we have

$$\begin{aligned} W_\tau(T_u M_\eta f, T_v M_\gamma g)(x, w) &= e^{2\pi i x(\eta - \gamma)} e^{2\pi i w(v - u)} e^{2\pi i(\gamma - \eta)(\tau v + (1 - \tau)u)} \\ &W_\tau(f, g)(x - (\tau v + (1 - \tau)u), w - (\tau\eta + (1 - \tau)\gamma)). \end{aligned}$$

In particular,

$$W_\tau(T_u M_\eta f)(x, w) = W_\tau f(x - u, w - \eta). \quad (3.7)$$

Proof For $\tau \in (0, 1)$ and $u, v, \eta, \gamma \in \mathbb{R}^d$, we have

$$\begin{aligned} &W_\tau(T_u M_\eta f, T_v M_\gamma g)(x, w) \\ &= \int_{\mathbb{R}^d} T_u M_\eta f(x + \tau t) \overline{T_v M_\gamma g(x - (1 - \tau)t)} e^{-2\pi i t w} dt \\ &= \int_{\mathbb{R}^d} f((x - u) + \tau t) \overline{g((x - v) - (1 - \tau)t)} e^{2\pi i \eta((x - u) + \tau t)} e^{-2\pi i \gamma((x - v) - (1 - \tau)t)} e^{-2\pi i t w} dt. \end{aligned}$$

We make the substitution $z = x - u + \tau t$ and obtain

$$\begin{aligned} &W_\tau (T_u M_\eta f, T_v M_\gamma g) (x, w) \\ &= \frac{1}{|\tau|^d} \int_{\mathbb{R}^d} f(z) g \left(- \left(\frac{1-\tau}{\tau} z - \left(\frac{x}{\tau} - v - \frac{1-\tau}{\tau} u \right) \right) \right) e^{2\pi i \eta z} e^{-2\pi i \gamma \left(-\frac{1-\tau}{\tau} z + \frac{x}{\tau} - v - \frac{1-\tau}{\tau} u \right)} \\ &\quad e^{-2\pi i \frac{w}{\tau} (z-x+u)} dz \\ &= \frac{1}{|\tau|^d} e^{-2\pi i \gamma \left(\frac{x}{\tau} - v - \frac{1-\tau}{\tau} u \right) + 2\pi i \frac{w}{\tau} (x-u)} \int_{\mathbb{R}^d} f(z) g^\sim \left(\frac{1-\tau}{\tau} \left(z - \left(\frac{x}{1-\tau} - \frac{\tau}{1-\tau} v - u \right) \right) \right) \\ &\quad e^{-2\pi i z \left(\frac{1}{\tau} w - \eta - \frac{1-\tau}{\tau} \gamma \right)} dz \\ &= \frac{1}{|\tau|^d} e^{-2\pi i \gamma \left(\frac{x}{\tau} - v - \frac{1-\tau}{\tau} u \right) + 2\pi i \frac{w}{\tau} (x-u)} \int_{\mathbb{R}^d} f(z) A_\tau g \left(z - \left(\frac{x}{1-\tau} - \frac{\tau}{1-\tau} v - u \right) \right) \\ &\quad e^{-2\pi i z \left(\frac{1}{\tau} w - \eta - \frac{1-\tau}{\tau} \gamma \right)} dz \\ &= \frac{1}{|\tau|^d} e^{-2\pi i \gamma \left(\frac{x}{\tau} - v - \frac{1-\tau}{\tau} u \right) + 2\pi i \frac{w}{\tau} (x-u)} V_{A_\tau g} f \left(\frac{1}{1-\tau} (x - \tau v - (1-\tau) u), \frac{1}{\tau} (w - \tau \eta - (1-\tau) \gamma) \right). \end{aligned}$$

Now, equality (1.1) is applied, we have the desired equality for $\tau \in (0, 1)$.

Let $\tau = 0$. For $u, v, \eta, \gamma \in \mathbb{R}^d$, by using the equality $(T_v M_\gamma g)^\wedge = M_{-v} T_\gamma \hat{g}$, we get

$$\begin{aligned} W_0 (T_u M_\eta f, T_v M_\gamma g) (x, w) &= R (T_u M_\eta f, T_v M_\gamma g) (x, w) = e^{-2\pi i x w} (T_u M_\eta f) (x) \overline{(T_v M_\gamma g)^\wedge (w)} \\ &= e^{-2\pi i x w} e^{2\pi i \eta (x-u)} f(x-u) \overline{(M_{-v} T_\gamma \hat{g}) (w)} \\ &= e^{2\pi i x (\eta-\gamma)} e^{2\pi i w (v-u)} e^{2\pi i u (\gamma-\eta)} e^{-2\pi i (x-u)(w-\gamma)} f(x-u) \overline{\hat{g}(w-\gamma)} \\ &= e^{2\pi i x (\eta-\gamma)} e^{2\pi i w (v-u)} e^{2\pi i u (\gamma-\eta)} W_0 (f, g) (x-u, w-\gamma). \end{aligned}$$

Similarly, if $\tau = 1$, for $u, v, \eta, \gamma \in \mathbb{R}^d$, we obtain

$$\begin{aligned} W_1 (T_u M_\eta f, T_v M_\gamma g) (x, w) &= R^* (T_u M_\eta f, T_v M_\gamma g) (x, w) = e^{2\pi i x w} \overline{(T_v M_\gamma g) (x)} (T_u M_\eta f)^\wedge (w) \\ &= e^{2\pi i x w} e^{2\pi i \gamma (x-v)} \overline{g(x-v)} \left(M_{-u} T_\eta \hat{f} \right) (w) \\ &= e^{2\pi i x (\eta-\gamma)} e^{2\pi i w (v-u)} e^{2\pi i v (\gamma-\eta)} e^{2\pi i (x-v)(w-\eta)} \overline{g(x-v)} \hat{f}(w-\eta) \\ &= e^{2\pi i x (\eta-\gamma)} e^{2\pi i w (v-u)} e^{2\pi i v (\gamma-\eta)} W_1 (f, g) (x-v, w-\eta). \end{aligned}$$

□

Proposition 9 Let $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $\tau \in [0, 1]$. Then we have

$$V_{T_\tau \xi_2 M_{-(1-\tau)\xi_1} A_\tau g T_\tau \xi_2} M_{-(1-\tau)\xi_1} f(x, w) = e^{-2\pi i (x(1-\tau)\xi_1 + w\tau\xi_2)} V_{A_\tau g} f(x, w), \tag{3.8}$$

for $x, w, \xi_1, \xi_2 \in \mathbb{R}^d$.

Proof Assume that $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $\tau \in [0, 1]$. Then we write

$$\begin{aligned} V_{T_{\tau\xi_2}M_{-(1-\tau)\xi_1}A_\tau g T_{\tau\xi_2}M_{-(1-\tau)\xi_1}f}(x, w) &= \langle T_{\tau\xi_2}M_{-(1-\tau)\xi_1}f, M_w T_x T_{\tau\xi_2}M_{-(1-\tau)\xi_1}A_\tau g \rangle \\ &= \langle f, M_{(1-\tau)\xi_1}T_{-\tau\xi_2}M_w T_x T_{\tau\xi_2}M_{-(1-\tau)\xi_1}A_\tau g \rangle. \end{aligned}$$

Since

$$\begin{aligned} M_{(1-\tau)\xi_1}T_{-\tau\xi_2}M_w T_x T_{\tau\xi_2}M_{-(1-\tau)\xi_1}A_\tau g(t) &= e^{2\pi i t(1-\tau)\xi_1} M_w T_x T_{\tau\xi_2}M_{-(1-\tau)\xi_1}A_\tau g(t + \tau\xi_2) \\ &= e^{2\pi i t(1-\tau)\xi_1} e^{2\pi i w(t + \tau\xi_2)} M_{-(1-\tau)\xi_1}A_\tau g(t - x) \\ &= e^{2\pi i t(1-\tau)\xi_1} e^{2\pi i w(t + \tau\xi_2)} e^{-2\pi i(1-\tau)\xi_1(t-x)} A_\tau g(t - x) \\ &= e^{2\pi i(x(1-\tau)\xi_1 + w\tau\xi_2)} e^{2\pi i w t} A_\tau g(t - x) \\ &= e^{2\pi i(x(1-\tau)\xi_1 + w\tau\xi_2)} M_w T_x A_\tau g(t), \end{aligned}$$

we have

$$\begin{aligned} V_{T_{\tau\xi_2}M_{-(1-\tau)\xi_1}A_\tau g T_{\tau\xi_2}M_{-(1-\tau)\xi_1}f}(x, w) &= \langle f, e^{2\pi i(x(1-\tau)\xi_1 + w\tau\xi_2)} M_w T_x A_\tau g \rangle \\ &= e^{-2\pi i(x(1-\tau)\xi_1 + w\tau\xi_2)} \langle f, M_w T_x A_\tau g \rangle \\ &= e^{-2\pi i(x(1-\tau)\xi_1 + w\tau\xi_2)} V_{A_\tau g} f(x, w). \end{aligned}$$

Proposition 10 *i) If $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $\tau \in [0, 1]$, then $W_\tau(f, g) \in \mathcal{S}(\mathbb{R}^{2d})$.* □

ii) Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and set $\Phi = W_\tau(\varphi, \varphi) = W_\tau(\varphi) \in \mathcal{S}(\mathbb{R}^{2d})$. For $\tau \in (0, 1)$, we have

$$V_\Phi(W_\tau(f, g))(z, \xi) = e^{-4\pi i z_2 \tau \xi_2} V_\varphi f(z_1 - \tau\xi_2, z_2 + (1-\tau)\xi_1) \overline{V_\varphi g(z_1 + \tau\xi_2, z_2 - (1-\tau)\xi_1)},$$

where $z = (z_1, z_2)$ and $\xi = (\xi_1, \xi_2)$.

iii) For $\tau = 0$, $\Phi = W_0(\varphi, \varphi) = W_0(\varphi) = R(\varphi) \in \mathcal{S}(\mathbb{R}^{2d})$, and $W_0(f, g) = R(f, g)$, we have

$$\begin{aligned} V_\Phi(W_0(f, g))(z, \xi) &= V_{W_0(\varphi)}(W_0(f, g))(z, \xi) = V_{R(\varphi)}(R(f, g))(z, \xi) \\ &= e^{-2\pi i z_2 \xi_2} V_\varphi f(z_1, z_2 + \xi_1) \overline{V_\varphi g(z_1 + \xi_2, z_2)}. \end{aligned}$$

iv) For $\tau = 1$, $\Phi = W_1(\varphi, \varphi) = W_1(\varphi) = \overline{R(\varphi)} \in \mathcal{S}(\mathbb{R}^{2d})$, and $W_1(f, g) = \overline{R(f, g)}$, we have

$$\begin{aligned} V_\Phi(W_1(f, g))(z, \xi) &= V_{W_1(\varphi)}(W_1(f, g))(z, \xi) = V_{\overline{R(\varphi)}}(\overline{R(f, g)})(z, \xi) \\ &= e^{-2\pi i z_2 \xi_2} V_\varphi f(z_1 - \xi_2, z_2) \overline{V_\varphi g(z_1, z_2 - \xi_1)}. \end{aligned}$$

Proof i) Since

$$W_\tau(f, g)(x, w) = \frac{1}{|\tau|^d} e^{2\pi i \frac{1}{\tau} x w} V_{A_\tau g} f\left(\frac{1}{1-\tau}x, \frac{1}{\tau}w\right),$$

for $\tau \in (0, 1)$, we obtain $W_\tau(f, g) \in \mathcal{S}(\mathbb{R}^{2d})$ by using Theorem 11.2.5 in [9].

If $\tau = 0$, let $f \otimes g$ be the tensor product $(f \otimes g)(x, t) = f(x)g(t)$, and set $\mathcal{T}_a F(x, t) = F(x, x - t)$. Then we write

$$\begin{aligned} W_0(f, g)(x, w) &= R(f, g)(x, w) = \int_{\mathbb{R}^d} f(x) \overline{g(x-t)} e^{-2\pi i t w} dt \\ &= \int_{\mathbb{R}^d} (f \otimes \bar{g})(x, x-t) e^{-2\pi i t w} dt \\ &= \int_{\mathbb{R}^d} \mathcal{T}_a(f \otimes \bar{g})(x, t) e^{-2\pi i t w} dt \\ &= \mathcal{F}_2 \mathcal{T}_a(f \otimes \bar{g})(x, w), \end{aligned}$$

where \mathcal{F}_2 is the Fourier transform with respect to the second variable. So, since $f, g \in \mathcal{S}(\mathbb{R}^d)$, then $f \otimes \bar{g} \in \mathcal{S}(\mathbb{R}^{2d})$. Also, since $\mathcal{S}(\mathbb{R}^{2d})$ is invariant under the transformation and the Fourier transform, then $W_0(f, g) \in \mathcal{S}(\mathbb{R}^{2d})$.

For $\tau = 1$, if we set $\mathcal{T}_b F(x, t) = F(x + t, x)$, we get

$$W_1(f, g)(x, w) = R^*(f, g)(x, w) = \mathcal{F}_2 \mathcal{T}_b(f \otimes \bar{g})(x, w).$$

Then, similarly to the case $\tau = 0$, we have $W_1(f, g) \in \mathcal{S}(\mathbb{R}^{2d})$.

ii) If we use the equalities (1.1) and (3.7), then we write

$$\begin{aligned} &V_{\Phi}(W_{\tau}(f, g))(z, \xi) \\ &= \iint_{\mathbb{R}^{2d}} W_{\tau}(f, g)(x, w) \overline{W_{\tau}(\varphi)(x - z_1, w - z_2)} e^{-2\pi i(x\xi_1 + w\xi_2)} dx dw \\ &= \frac{1}{|\tau|^d} \iint_{\mathbb{R}^{2d}} e^{2\pi i \frac{1}{\tau} x w} V_{A_{\tau} g} f\left(\frac{1}{1-\tau} x, \frac{1}{\tau} w\right) \overline{W_{\tau}(T_{z_1} M_{z_2} \varphi)(x, w)} e^{-2\pi i(x\xi_1 + w\xi_2)} dx dw \\ &= \frac{1}{|\tau|^{2d}} \iint_{\mathbb{R}^{2d}} V_{A_{\tau} g} f\left(\frac{1}{1-\tau} x, \frac{1}{\tau} w\right) e^{2\pi i(-x\xi_1 - w\xi_2)} \overline{V_{A_{\tau}(T_{z_1} M_{z_2} \varphi)}(T_{z_1} M_{z_2} \varphi)\left(\frac{1}{1-\tau} x, \frac{1}{\tau} w\right)} dx dw \\ &= \frac{|1-\tau|^d}{|\tau|^d} \iint_{\mathbb{R}^{2d}} V_{A_{\tau} g} f(x, w) e^{2\pi i(-x(1-\tau)\xi_1 - w\tau\xi_2)} \overline{V_{A_{\tau}(T_{z_1} M_{z_2} \varphi)}(T_{z_1} M_{z_2} \varphi)(x, w)} dx dw. \end{aligned}$$

Additionally, if equality (3.8) and orthogonality relations (see Theorem 3.2.1 in [9]) are applied, then we get

$$\begin{aligned} &V_{\Phi}(W_{\tau}(f, g))(z, \xi) \\ &= \frac{|1-\tau|^d}{|\tau|^d} \iint_{\mathbb{R}^{2d}} V_{T_{\tau\xi_2} M_{-(1-\tau)\xi_1} A_{\tau} g} T_{\tau\xi_2} M_{-(1-\tau)\xi_1} f(x, w) \overline{V_{A_{\tau}(T_{z_1} M_{z_2} \varphi)}(T_{z_1} M_{z_2} \varphi)(x, w)} dx dw \\ &= \frac{|1-\tau|^d}{|\tau|^d} \langle T_{\tau\xi_2} M_{-(1-\tau)\xi_1} f, T_{z_1} M_{z_2} \varphi \rangle \overline{\langle T_{\tau\xi_2} M_{-(1-\tau)\xi_1} A_{\tau} g, A_{\tau}(T_{z_1} M_{z_2} \varphi) \rangle}. \end{aligned}$$

The first factor on the right side of the equality is

$$\begin{aligned}
& \langle T_{\tau\xi_2} M_{-(1-\tau)\xi_1} f, T_{z_1} M_{z_2} \varphi \rangle \\
&= \langle f, M_{(1-\tau)\xi_1} T_{z_1 - \tau\xi_2} M_{z_2} \varphi \rangle \\
&= \int_{\mathbb{R}^d} f(x) e^{-2\pi i(1-\tau)\xi_1 x} e^{-2\pi i z_2(x - z_1 + \tau\xi_2)} \overline{\varphi(x - z_1 + \tau\xi_2)} dx \\
&= e^{2\pi i z_2(z_1 - \tau\xi_2)} \int_{\mathbb{R}^d} f(x) \overline{\varphi(x - (z_1 - \tau\xi_2))} e^{-2\pi i x((1-\tau)\xi_1 + z_2)} dx \\
&= e^{2\pi i z_2(z_1 - \tau\xi_2)} V_\varphi f(z_1 - \tau\xi_2, z_2 + (1-\tau)\xi_1).
\end{aligned}$$

Also, since

$$\begin{aligned}
A_\tau(T_u M_\eta g)(x) &= (T_u M_\eta g) \sim \left(\frac{1-\tau}{\tau} x\right) = (T_u M_\eta g) \left(-\frac{1-\tau}{\tau} x\right) \\
&= e^{-2\pi i \eta \left(\frac{1-\tau}{\tau} x + u\right)} g \sim \left(\frac{1-\tau}{\tau} x + u\right) \\
&= T_{-u} M_{-\eta} g \sim \left(\frac{1-\tau}{\tau} x\right) = T_{-u} M_{-\eta} A_\tau g(x),
\end{aligned}$$

the second factor is

$$\begin{aligned}
& \langle T_{\tau\xi_2} M_{-(1-\tau)\xi_1} A_\tau g, A_\tau(T_{z_1} M_{z_2} \varphi) \rangle \\
&= \langle A_\tau(T_{-\tau\xi_2} M_{(1-\tau)\xi_1} g), A_\tau(T_{z_1} M_{z_2} \varphi) \rangle \\
&= \int_{\mathbb{R}^d} (T_{-\tau\xi_2} M_{(1-\tau)\xi_1} g) \sim \left(\frac{1-\tau}{\tau} x\right) \overline{(T_{z_1} M_{z_2} \varphi) \sim \left(\frac{1-\tau}{\tau} x\right)} dx \\
&= \frac{|\tau|^d}{|1-\tau|^d} \int_{\mathbb{R}^d} M_{(1-\tau)\xi_1} g(-u + \tau\xi_2) \overline{M_{z_2} \varphi(-u - z_1)} du \\
&= \frac{|\tau|^d}{|1-\tau|^d} \int_{\mathbb{R}^d} e^{2\pi i(1-\tau)\xi_1(-u + \tau\xi_2)} g(-u + \tau\xi_2) \overline{\varphi(-u - z_1)} e^{-2\pi i z_2(-u - z_1)} du \\
&= \frac{|\tau|^d}{|1-\tau|^d} e^{2\pi i z_2(z_1 + \tau\xi_2)} \int_{\mathbb{R}^d} g(x) \overline{\varphi(x - (z_1 + \tau\xi_2))} e^{-2\pi i x(z_2 - (1-\tau)\xi_1)} dx \\
&= \frac{|\tau|^d}{|1-\tau|^d} e^{2\pi i z_2(z_1 + \tau\xi_2)} V_\varphi g(z_1 + \tau\xi_2, z_2 - (1-\tau)\xi_1).
\end{aligned}$$

Hence, we have

$$\begin{aligned} & V_{\Phi}(W_{\tau}(f, g))(z, \xi) \\ &= \frac{|1 - \tau|^d}{|\tau|^d} e^{2\pi i z_2(z_1 - \tau \xi_2)} V_{\varphi} f(z_1 - \tau \xi_2, z_2 + (1 - \tau) \xi_1) \cdot \\ & \quad \overline{\frac{|\tau|^d}{|1 - \tau|^d} e^{2\pi i z_2(z_1 + \tau \xi_2)} V_{\varphi} g(z_1 + \tau \xi_2, z_2 - (1 - \tau) \xi_1)} \\ &= e^{-4\pi i z_2 \tau \xi_2} V_{\varphi} f(z_1 - \tau \xi_2, z_2 + (1 - \tau) \xi_1) \overline{V_{\varphi} g(z_1 + \tau \xi_2, z_2 - (1 - \tau) \xi_1)}. \end{aligned}$$

iii) For $\tau = 0$, by using the equality $V_g f(x, w) = e^{-2\pi i x w} V_{\hat{g}} f(w, -x)$, we get

$$\begin{aligned} & V_{W_0 \varphi}(W_0(f, g))(z, \xi) \\ &= \langle W_0(f, g), M_{\xi} T_z W_0 \varphi \rangle \\ &= \iint_{\mathbb{R}^{2d}} W_0(f, g)(x, w) \overline{M_{\xi} T_z W_0 \varphi(x, w)} dx dw \\ &= \iint_{\mathbb{R}^{2d}} f(x) \overline{\hat{g}(w)} e^{-2\pi i x w} \overline{W_0 \varphi(x - z_1, w - z_2)} e^{-2\pi i(x \xi_1 + w \xi_2)} dx dw \\ &= \iint_{\mathbb{R}^{2d}} f(x) \overline{\hat{g}(w)} \overline{\varphi(x - z_1)} \overline{\hat{\varphi}(w - z_2)} e^{-2\pi i(xw + x \xi_1 + w \xi_2 - (x - z_1)(w - z_2))} dx dw \\ &= e^{2\pi i z_1 z_2} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x) \overline{\varphi(x - z_1)} e^{-2\pi i x(\xi_1 + z_2)} dx \right) \overline{\hat{g}(w)} \overline{\hat{\varphi}(w - z_2)} e^{-2\pi i w(\xi_2 + z_1)} dw \\ &= e^{2\pi i z_1 z_2} V_{\varphi} f(z_1, \xi_1 + z_2) \overline{\int_{\mathbb{R}^d} \hat{g}(w) \hat{\varphi}(w - z_2) e^{2\pi i w(\xi_2 + z_1)} dw} \\ &= e^{2\pi i z_1 z_2} V_{\varphi} f(z_1, \xi_1 + z_2) V_{\hat{\varphi}} \hat{g}(z_2, -z_1 - \xi_2) \\ &= e^{-2\pi i z_2 \xi_2} V_{\varphi} f(z_1, \xi_1 + z_2) \overline{V_{\varphi} g(z_1 + \xi_2, z_2)}. \end{aligned}$$

iv) It is proven by using the same proof technique as in iii. □

We can now prove the continuity of the τ -Wigner transform for Lorentz mixed normed modulation spaces.

Proposition 11 *Let $P = (1, p_2)$, $Q = (q_1, q_2)$, $1 \leq Q < \infty$ and $1 < p_2 < \infty$. If $\varphi_1 \in M^1(\mathbb{R}^d)$, and $\varphi_2 \in M(p_2, q_2)(\mathbb{R}^d)$; then $W_{\tau}(\varphi_2, \varphi_1) \in M(P, Q)(\mathbb{R}^{2d})$ and satisfies*

$$\|W_{\tau}(\varphi_2, \varphi_1)\|_{M(P, Q)} \leq \|\varphi_1\|_{M^1} \|\varphi_2\|_{M(p_2, q_2)}$$

for $\tau \in [0, 1]$.

Proof Let $\varphi_1, \varphi_2, g \in \mathcal{S}(\mathbb{R}^d)$, $\tau \in [0, 1]$, and $\Phi = W_\tau g \in \mathcal{S}(\mathbb{R}^{2d})$. Then $W_\tau(\varphi_2, \varphi_1) \in \mathcal{S}(\mathbb{R}^{2d})$ and so $V_\Phi(W_\tau(\varphi_2, \varphi_1)) \in \mathcal{S}(\mathbb{R}^{4d})$ by Proposition 10 (i) and Theorem 11.2.5 in [9], respectively. On the other hand, if $\varphi_1 \in \mathcal{S}(\mathbb{R}^d)$, then it is known that φ_1 is in the standard modulation space $M^1(\mathbb{R}^d)$, and if $\varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then $\varphi_2 \in M(p_2, q_2)(\mathbb{R}^d)$ by Proposition 2.1 in [10].

For $\tau \in (0, 1)$, Proposition 10 (ii) says that

$$|V_\Phi W_\tau(\varphi_2, \varphi_1)(z, \xi)| = |V_g \varphi_1(z_1 + \tau \xi_2, z_2 - (1 - \tau)\xi_1)| |V_g \varphi_2(z_1 - \tau \xi_2, z_2 + (1 - \tau)\xi_1)|.$$

Write $\tilde{\xi} = (\tau \xi_2, -(1 - \tau)\xi_1)$ and

$$|V_\Phi W_\tau(\varphi_2, \varphi_1)(z, \xi)| = \left| V_g \varphi_1(z + \tilde{\xi}) \right| \left| V_g \varphi_2(z - \tilde{\xi}) \right|.$$

Thus, by using the inequality $\|\cdot\|_{1q_1} \leq \|\cdot\|_{11} = \|\cdot\|_1$ when $1 \leq q_1$ and changing variables $z \rightarrow z - \tilde{\xi}$, we have

$$\begin{aligned} \|V_\Phi W_\tau(\varphi_2, \varphi_1)\|_{1q_1}(\xi) &\leq \|V_\Phi W_\tau(\varphi_2, \varphi_1)\|_1(\xi) \\ &= \int_{\mathbb{R}^{2d}} |V_g \varphi_1(z)| |V_g \varphi_2(z - 2\tilde{\xi})| dz \\ &= (|V_g \varphi_1| * |V_g \varphi_2^\sim|)(2\tilde{\xi}). \end{aligned}$$

Again using the fact that the Lorentz space $L(p_2, q_2)(\mathbb{R}^{2d})$ is an essential Banach convolution module over $L^1(\mathbb{R}^{2d})$, we obtain

$$\begin{aligned} \|W_\tau(\varphi_2, \varphi_1)\|_{M(p, q)} &= \left\| \|V_\Phi W_\tau(\varphi_2, \varphi_1)\|_{1q_1} \right\|_{p_2 q_2} \\ &\leq \| |V_g \varphi_1| * |V_g \varphi_2^\sim| \|_{p_2 q_2} \leq \|V_g \varphi_1\|_1 \|V_g \varphi_2\|_{p_2 q_2} \\ &= \|\varphi_1\|_{M^1} \|\varphi_2\|_{p_2 q_2} \end{aligned}$$

for $\tau \in (0, 1)$.

If $\tau = 0$, then we write

$$|V_\Phi W_0(\varphi_2, \varphi_1)(z, \xi)| = |V_\Phi R(\varphi_2, \varphi_1)(z, \xi)| = |V_g \varphi_1(z_1 + \xi_2, z_2)| |V_g \varphi_2(z_1, z_2 + \xi_1)|$$

from Proposition 10 (iii), where $\Phi = W_0 g = R(g) \in \mathcal{S}(\mathbb{R}^{2d})$. Changing variable $z_1 \rightarrow z_1 - \xi_2$ and writing $\tilde{\xi} = (\xi_2, -\xi_1)$, we get

$$\begin{aligned} \|V_\Phi W_0(\varphi_2, \varphi_1)\|_{1q_1}(\xi) &\leq \|V_\Phi W_0(\varphi_2, \varphi_1)\|_1(\xi) \\ &= \int_{\mathbb{R}^{2d}} |V_g \varphi_1(z_1 + \xi_2, z_2)| |V_g \varphi_2(z_1, z_2 + \xi_1)| dz_1 dz_2 \\ &= \int_{\mathbb{R}^{2d}} |V_g \varphi_1(z)| |V_g \varphi_2(z - \tilde{\xi})| dz \\ &= (|V_g \varphi_1| * |V_g \varphi_2^\sim|)(\tilde{\xi}) \end{aligned}$$

and

$$\|W_0(\varphi_2, \varphi_1)\|_{M(P,Q)} = \|R(\varphi_2, \varphi_1)\|_{M(P,Q)} \leq \|\varphi_1\|_{M^1} \|\varphi_2\|_{M(p_2, q_2)}.$$

If we apply the same proof technique above for $\tau = 1$, by using Proposition 10 (iv), we have

$$\|W_1(\varphi_2, \varphi_1)\|_{M(P,Q)} = \|R^*(\varphi_2, \varphi_1)\|_{M(P,Q)} \leq \|\varphi_1\|_{M^1} \|\varphi_2\|_{M(p_2, q_2)}.$$

□

References

- [1] Bastero J, Milman M, Ruiz F. A note on $L(\infty, q)$ spaces and Sobolev embeddings. *Indiana Univ Math J* 2003; 52: 1215–1230.
- [2] Bennet C, DeVore RA, Sharpley R. Weak- L^∞ and BMO. *Annals Math* 1981; 3: 601–611.
- [3] Blozinski AP. Multivariate rearrangements and Banach function spaces with mixed norms. *Trans Amer Math Soc* 1981; 1: 149–167.
- [4] Boggiatto P, De Donno G, Oliaro A. A class of quadratic time-frequency representations based on the short-time Fourier transform. *Oper Theor* 2007; 172: 235–249.
- [5] Boggiatto P, De Donno G, Oliaro A. Time-frequency representations of Wigner type and pseudo-differential operators. *Trans Amer Math Soc* 2010; 362: 4955–4981.
- [6] Cordero E, Gröchenig K. Time-frequency analysis of localization operators. *J Funct Anal* 2003; 205: 107–131.
- [7] Feichtinger HG. Modulation spaces on locally compact Abelian groups. Technical Report, University of Vienna, 1983.
- [8] Fernandez DL. Lorentz spaces, with mixed norms. *J Funct Anal* 1977; 25: 128–146.
- [9] Gröchenig K. *Foundation of Time-Frequency Analysis*. Boston: Birkhäuser, 2001.
- [10] Gürkanlı AT. Time-frequency analysis and multipliers of the spaces $M(p, q)(\mathbb{R}^d)$ and $S(p, q)(\mathbb{R}^d)$. *J Math Kyoto Univ* 2006; 46-3: 595–616.
- [11] Hunt RA. On $L(p, q)$ spaces. *Extrait de L'Enseignement Mathématique* 1966; 12: 249–276.
- [12] O'Neil R. Convolution operators and $L(p, q)$ spaces. *Duke Math J* 1963; 30: 129–142.
- [13] O'Neil R. Integral transforms and tensor products on Orlicz spaces and $L(p, q)$ spaces. *J d'Analyse Math* 1968; 21: 1–276.
- [14] Sandıkçı A. On Lorentz mixed normed modulation spaces. *J Pseudo-Differ Oper Appl* 2012; 3: 263–281.
- [15] Sandıkçı A, Gürkanlı AT. Gabor analysis of the spaces $M(p, q, w)(\mathbb{R}^d)$ and $S(p, q, r, w, \omega)(\mathbb{R}^d)$. *Acta Math Sci* 2011; 31B: 141–158.
- [16] Sandıkçı A, Gürkanlı AT. Generalized Sobolev-Shubin spaces, boundedness and Schatten class properties of Toeplitz operators. *Turk J Math* 2013; 37: 676–692.